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Applied Mechanics. — *The circular ring under the combined action of compressive and bending loads.* By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of December 29, 1945.)

1. *Introduction.* In a previous paper [1]), (in these lines to be quoted as "I") it was stated that every equilibrium-system (q, t) of radial and tangential forces, acting on a circular ring can be decomposed into a "compressive" system $A(q, t)$ giving rise only to a normal force N in any cross-section of the ring (so that both the bending moment M and the shearing force D are zero) and a "bending" system $B(q, t)$ characterized by $N = 0$. The first system A, if suitably magnified, leads to elastic instability of the ring. The required factors of magnification λ_1, λ_2, arranged after their order of magnitude have been introduced in "I" as the "characteristic numbers" of the buckling problem connected with the A-system and their numerical computation has been the object of that paper. Here it will be supposed, that the A-system is sufficiently small as to guarantee the elastic stability of the construction. It is obvious, that the deflections (and internal stresses) of the ring, due to the *single* action of the B-system, will be affected in a rather complicated way by the simultaneous action of the A-system, which — alone — would produce no deflections at all.

In this paper it will be shown how the deflection and internal stresses of the ring, under the *combined* action of the A and B-system can be derived from the corresponding quantities occurring with the B-system alone, with the aid of the characteristic numbers λ and the corresponding characteristic functions U (comp. "I") of the A-system.

To prevent lengthy repetitions, the reader is supposed to be fully acquainted with paper "I".

2. *The method.* If the ring is subjected to the k—*th* characteristic (or critical) load $\lambda_k A$, the corresponding characteristic mode of distortion U_k is determined except for a factor of proportionality. To remove this ambiguity we restict our attention to that deflection u_k, which corresponds to the "normalized" characteristic function $U_k = u_k + u_k = -\frac{r^2}{EI} M_k$ (comp. "I", 3, and "I", 4, 36). The same deflection u_k and consequently the same bending moment $M_k = -\frac{EI}{r^2} U_k$ can likewise be sustained by a "bending"

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[1]) C. B. BIEZENO and J. J. KOCH. The generalized buckling problem of the circular ring. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 48, 447 (1945).

loadsystem $B_k(q,t)$, which readily can be derived from the equations of equilibrium

$$\left.\begin{aligned} -D' &= qr \\ D &= -tr \\ M' &= -tr^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(comp. ("I", 2, 1), in which N en N' must be put equal to zero). From this it follows that the *simultaneously* acting loads A and $\varkappa B_k$ ($\varkappa$ designing an arbitrary constant) will produce a bending moment $M = -\frac{EI}{r^2}(u'' + u)$, which for every value of $\varkappa$ is *proportional* to $M_k = -\frac{EI}{r^2} U_k$. If the particular value of $\varkappa$ is required for which the combined loads A and $\varkappa B_k$ produce a bending moment not proportional but *equal* to $M_k = -\frac{EI}{r^2} U_k$ (and consequently produce the distortion u_k) it can be remarked, that the single compressive loadsystem $\lambda_k A_k$ is capable of sustaining the prescribed deflection u_k and consequently to rouse the prescribed moment M_k. The loadsystem A therefore stands up for the bending moment $\frac{1}{\lambda_k} M_k$, and the remaining part $\frac{\lambda - 1}{\lambda} M_k$ of the prescribed bending moment M_k must be supplied by the bending loadsystem $\varkappa B_k$ and consequently $\varkappa = \frac{\lambda_k - 1}{\lambda_k}$.

We learn from these remarks that the effect of the "bending" loadsystem B_k is magnified in the proportion $\frac{\lambda_k}{\lambda_k - 1}$ by the presence of the "compressive" loadsystem A.

The "reduced" moment $-\frac{r^2}{EI} M_B \equiv U_B$ belonging to any arbitrary B-system can be expanded into a series

$$-\frac{r^2}{EI} . M_B \equiv U_B = \sum_{k=1}^{\infty} b_k U_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

of the characteristic functions U_k of the A-system under consideration (Comp. "I", 4).

If the ring is subjected to the simultaneous action of the B- and A-system, the magnifying influence of the latter will exert itself to each component B_k of the B-system, and in particular the resultant bending moment M, respectively the resultant function U, due to the joint systems A and B, is represented by

$$-\frac{r^2}{EI} M = U = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - 1} b_k U_k . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The resultant distortion u of the ring is governed by the differential equation

$$u'' + u = U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Summarizing it can be stated, that the stress distribution and the distortion of a circular ring under the combined action of a A- and B-load can be calculated as soon as the characteristic numbers λ_k of the A-system and the corresponding characteristic functions U_k are known. The function $U_B = -\frac{r^2}{EI} M_B$, which has to be expanded into the series (2), must be computed by integrating the third eq. (1), and the coefficients b_k occurring in this series can easily be found by using the orthogonality of the functions U_k.

3. *Application. The ring compressed by two diametral forces P.* With reference to ("I", 2) we replace the two compressing diametral forces P (whose points of attack may coincide with $\varphi = 0$ and $\varphi = \pi$) by their equivalent Fourier-series q and t:

$$q = a_0 + \sum_1^\infty a_k \cos k\varphi + \sum_1^\infty b_k \sin k\varphi\,; \; t = c_0 + \sum_1^\infty c_k \cos k\varphi + \sum_0^\infty d_k \sin k\varphi \quad (5)$$

in which evidently all coefficients a_{2n+1}, b, c and d are zero and in which

$$a_0 = -\frac{P}{\pi r} \qquad a_{2n} = -\frac{2P}{\pi r}$$

so that

$$q = -\frac{P}{\pi r} - \frac{2P}{\pi r} \sum_1^\infty \cos 2k\varphi \qquad t = 0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (6) it follows (comp. "I" 2, 18a, b) that the A and B-loadsystems are represented by

$$(A) \begin{cases} q^* = -\frac{P}{\pi r} + \frac{2P}{\pi r} \sum_1^\infty \frac{\cos 2k\varphi}{4k^2-1} \\ t^* = \frac{2P}{\pi r} \sum_1^\infty \frac{2k \sin 2k\varphi}{4k^2-1} \end{cases} \qquad (B) \begin{cases} q^{**} = -\frac{2P}{\pi r} \sum_1^\infty \frac{4k^2}{4k^2-1} \cos 2k\varphi \\ t^{**} = -\frac{2P}{\pi r} \sum_1^\infty \frac{2k}{4k^2-1} \sin 2k\varphi \end{cases} \quad . \quad (7)$$

The normal force N_0, acting upon the cross-section (φ) of the ring and caused by the A-system is given by

$$N_0 = -\frac{P}{\pi} + \frac{2P}{\pi} \sum_1^\infty \frac{\cos 2k\varphi}{4k^2-1} = -\frac{P}{\pi}\left[1 + \sum_1^\infty \varepsilon_{2k} \cos 2k\varphi\right]; \; \varepsilon_{2k} = -\frac{2}{4k^2-1} \quad (8)$$

and the bending moment M_B belonging to the B-system by

$$M_B = -\frac{2Pr}{\pi} \sum_2^\infty \frac{\cos 2k\varphi}{4k^2-1} = \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$-\frac{2Pr}{\pi}[0{,}3333 \cos 2\varphi + 0{,}06667 \cos 4\varphi + 0{,}02857 \cos 6\varphi + 0{,}01587 \cos 8\varphi + \\ + 0{,}0101 \cos 10\varphi + 0{,}00699 \cos 12\varphi]$$

(comp. the third eq. (1) and the second eq. (7).

With reference to ("I" 8), where an example has been treated analogous with the "A"-problem under consideration (viz. $N = \lambda N_0 = \lambda(1 + \varepsilon_2 \cos 2\varphi + \varepsilon_4 \cos 4\varphi)$), we restrict ourselves to the summing up of the following numerical results with respect to the first two characteristic numbers λ_1 and λ_2 and the two corresponding characteristic functions:

$$\lambda_1 = 1.0336 \frac{3EI}{r^2} \frac{\pi}{P}; \qquad \lambda_2 = 5{,}3628 \frac{3EI}{r^2} \frac{\pi}{P}$$

$$\left.\begin{aligned} U_1/r \backsim V_5^*/r = 0{,}8396 \cos 2\varphi - 0{,}0784 \cos 4\varphi - 0{,}0041 \cos 6\varphi - \\ - 0{,}0014 \cos 8\varphi - 0{,}0005 \cos 10\varphi - 0{,}0003 \cos 12\varphi \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} -\frac{\pi}{Pr} N_0 V_5^* = -\frac{\pi}{Pr} W_5^* = +0{,}8123 \cos 2\varphi - 0{,}3796 \cos 4\varphi - \\ - 0{,}0460 \cos 6\varphi - 0{,}0266 \cos 8\varphi - 0{,}0162 \cos 10\varphi - 0{,}0112 \cos 12\varphi \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} U_2/r \backsim \overline{V}_{11}^*/r = 6{,}9010 \cos 2\varphi + 15{,}3749 \cos 4\varphi - 4{,}9193 \cos 6\varphi + \\ + 0{,}0784 \cos 8\varphi - 0{,}0651 \cos 10\varphi - 0{,}0308 \cos 12\varphi \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} -\frac{\pi}{Pr} N_0 \overline{V}_{11}^* = -\frac{\pi}{Pr} \overline{W}_{11}^* = 1{,}2819 \cos 2\varphi + 14{,}320 \cos 4\varphi - \\ - 10{,}7560 \cos 6\varphi + 0{,}3678 \cos 8\varphi - 0{,}4107 \cos 10\varphi - 0{,}2678 \cos 12\varphi \end{aligned}\right\} \quad (13)$$

(in these latter series all terms following that with $\cos 12\varphi$ are neglected).

In the expansion

$$-\frac{M_B r^2}{EI} = b_1 U_1 + b_2 U_2 + \ldots \ldots \ldots \ldots \quad (14)$$

the coefficients b_1 and b_2 are defined by

$$\left.\begin{aligned} -\int_0^{2\pi} N_0 U_1 \cdot \frac{M_B r^2}{EI} d\varphi = b_1 \int_0^{2\pi} N_0 U_1^2 d\varphi \text{ resp.} \\ -\int_0^{2\pi} N_0 U_2 \frac{M_B r^2}{EI} = b_2 \int_0^{2\pi} N_0 U_2^2 d\varphi \end{aligned}\right\} \ldots \quad (15)$$

If we replace U_1 and U_2 by their approximate values V_5^* and $\overline{V}_{11}^*$ we find:

$$\left.\begin{aligned} b_1 = -\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} V_5^* W_5^* d\varphi; \\ b_2 = -\int_0^{2\pi} \overline{W}_{11}^* \frac{M_B r^2}{EI} d\varphi : \int_0^{2\pi} \overline{V}_{11}^* \overline{W}_{11}^* d\varphi \end{aligned}\right\} \ldots \ldots \quad (16)$$

The computation of these coefficients only requires the evaluation of the integrals

$$\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI}\, d\varphi, \int_0^{2\pi} V_5^*\, W_5^*\, d\varphi \text{ and the analogous ones for } b_2, \quad (17)$$

If we write:

$$M_B = -\frac{2Pr}{\pi} \sum_{k=1}^{6} m_{2k} \cos 2k\varphi, \quad V_5^* = r \sum_{k=1}^{6} v_{2k} \cos 2k\varphi,$$

$$W_5^* = -\frac{Pr}{\pi} \sum_{1}^{6} w_{2k} \cos 2k\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(comp. 9 and 11) we find:

$$\int_0^{2\pi} W_5^* \frac{M_B r^2}{EI}\, d\varphi = \frac{2P^2 r^4}{\pi^2 EI} \sum_{k=1}^{6} \int_0^{2\pi} w_{2k}\, m_{2k} \cos^2 2k\varphi\, d\varphi =$$

$$= \frac{2P^2 r^4}{\pi EI} \sum_{k=1}^{6} w_{2k}\, m_{2k} = 0{,}24348 \frac{2P^2 r^4}{\pi EI}$$

$$\int_0^{2\pi} V_5^*\, W_5^*\, d\varphi = -\frac{Pr^2}{\pi} \sum_{k=1}^{6} \int_0^{2\pi} v_{2k}\, w_{2k} \cos^2 2k\varphi\, d\varphi = -Pr^2 \sum_{k=1}^{6} v_{2k}\, w_{2k} =$$

$$= -0{,}71021\, Pr^2$$

whence

$$b_1 = 0{,}34196 \frac{2Pr^2}{\pi EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Analogously it is found that

$$b_2 = 0.003822 \frac{2Pr^2}{\pi EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The deflections u_1 and u_2 of the ring corresponding with the components $-b_1 \frac{EI}{r^2} U_1, -b_2 \frac{EI}{r^2} U_2$ ... of the bending moment M_B can be calculated from the equations

$$u_1'' + u_1 = b_1 U_1 \qquad u_2'' + u_2 = b_2 U_2 \ldots . \quad . \quad . \quad (21)$$

resp.:

$$u_1'' + u_1 = 0{,}28715 \cos 2\varphi - 0{,}02681 \cos 4\varphi - 0{,}00140 \cos 6\varphi -$$
$$- 0{,}00048 \cos 8\varphi - 0{,}00017 \cos 10\varphi - 0{,}00010 \cos 12\varphi$$

$$u_2'' + u_2 = 0{,}02638 \cos 2\varphi + 0{,}05876 \cos 4\varphi - 0{,}01880 \cos 6\varphi +$$
$$+ 0{,}00030 \cos 8\varphi - 0{,}00025 \cos 10\varphi - 0{,}00012 \cos 12\varphi$$

(comp. (11) and (19), resp. (12) and (20)), from which it follows:

$$\left.\begin{aligned} u_1 &= [-0{,}09572\cos 2\varphi + 0{,}00179\cos 4\varphi + 0{,}00004\cos 6\varphi + \\ &\quad + 0{,}00001\cos 8\varphi + \ldots]\frac{2Pr^3}{EI} \\ u_2 &= [-0{,}00879\cos 2\varphi - 0{,}00392\cos 4\varphi + 0{,}00054\cos 6\varphi + \ldots]\frac{2Pr^3}{\pi EI} \end{aligned}\right\}. \quad (22)$$

The resultant distortion u^* of the ring, caused by the simultaneous action of the A and B loadsystems is given by

$$u^* = \frac{\lambda_1}{\lambda_1 - 1}u_1 + \frac{\lambda_2}{\lambda_2 - 1}u_2 + \ldots = u_1 + u_2 + \ldots + \frac{1}{\lambda_1 - 1}u_1 + \frac{1}{\lambda_2 - 1}u_2 + \ldots \quad (23)$$

The series $u_1 + u_2 +$ ad inf. represents the deflection u of the ring which occurs if no secondary effect is present. This deflection can be calculated by one of the well-known elementary methods (f.i. with the aid of CASTIGLIANO's theorem). The series $\frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 2} + \ldots$ can readily be broken off with the third term in consequence of the fact, that u_k decreases and λ_k increases rather rapidly with increasing k. We therefore write:

$$u^* = u + \frac{u_1}{\lambda_1 - 1} + \frac{u_2}{\lambda_2 - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

At $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ the following results are obtained

$$u_0^* = -\frac{2Pr^3}{\pi EI}\left[0{,}11685 + \frac{0{,}09388}{1{,}0336\frac{3\pi EI}{Pr^2} - 1} + \frac{0{,}01217}{5{,}3628\frac{3\pi EI}{Pr^2} - 1}\right]. \quad (25)$$

$$u_{\pi/2}^* = +\frac{2Pr^3}{\pi EI}\left[0{,}10730 + \frac{0{,}09748}{1{,}0336\frac{3\pi EI}{Pr^2} - 1} + \frac{0{,}00433}{5{,}3628\frac{3\pi EI}{Pr^2} - 1}\right]. \quad (26)$$

The secondary effect to which the "compressive" A-system gives rise in influencing the deflections caused by the "bending" B-system as represented by the second and third terms in these expressions has been expressed in terms of the quotient $\frac{3\pi EI}{Pr^2}$, viz. the quotient of the first critical all-sided pressure q_{cr} of the ring ($q_{cr} = \frac{3EI}{r^2}$) and the "mean normal pressure P/π beloging to the two forces P.

Medicine. — *Some remarks on audiometry.* I. *Symmetric remnants of hearing.* By A. DE KLEYN.

(Communicated at the meeting of November 24, 1945.)

In a preceding communication [1]) we examined in how far the so-called "Symmetriegesetz der erblichen Taubheit" of LANGENBECK was in accordance with our clinical experience. LANGENBECK [2]) thought that in this case he might speak of a so-called law, in connection with the fact that when examining audiometrically people who were deaf-mute by inheritance, he had in most cases found symmetric remnants of hearing. He also spoke of symmetric remnants of hearing when the patients did not show any remnants of hearing on either side.

Our own observations in the Amsterdam clinic and our private practice were not in accordance with the conceptions of LANGENBECK. Nor does this appear to be the case with the observations made by Mrs. VELLEMAN, not published up to now, who in our clinic made an audiometric examination in view of remnants of hearing, of the deaf-mute children of the school for deaf-mute children in Amsterdam. It is true that both of us found symmetric remnants of hearing in several people who were deaf-mute by inheritance, but also in several others, undoubtedly also deaf-mute by inheritance, the remnants of hearing were quite asymmetrical (cf. preceding communcation). This fact in itself would not be a great objection in the heredity-investigation. Such cases only prove that when symmetric remnants of hearing are absent, the conclusion that heredity plays no part, should not be drawn.

Much more important is the question whether one is justified to assume at once a hereditary genesis, when at the examination symmetric remnants of hearing are found. It appeared that this, too, had to be answered in the negative as regards our observations. Nor does the assertion made by LANGENBECK: „Somit kann man die These voll und ganz aufrechterhalten dass die Symmetrie der Hörreste bei erblicher Taubheit bezw. Schwerhörigkeit ein brauchbares Symptom für die Beurteilung der Erblichkeit ist" seem to me to be tenable. Already in the preceding communication some cases were brought forward for this as a proof. Meanwhile the investigation has been continued and more cases could be added.

I. In the first place it has appeared that the remnants of hearing in patients who suffered from an *infectious disease* and after that became dull of hearing, are not seldom symmetric.

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[1]) DE KLEYN, A., Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1190 (1941).

[2]) LANGENBECK, B., Ztschr. f. Hals Nasen u.s.w. Heilk., **39**, 223, 286 (1936).

This concerned patients from the Amsterdam clinic, in whom the anamnesis did not give a single indication for an "Erbkrankheit" and who yet, after having gone through an infectious disease, appeared to be dull of hearing and having symmetric remnants of hearing. A typical example, which was already mentioned in the preceding communication, concerns a deaf-mute boy of 14. (fig. 1).

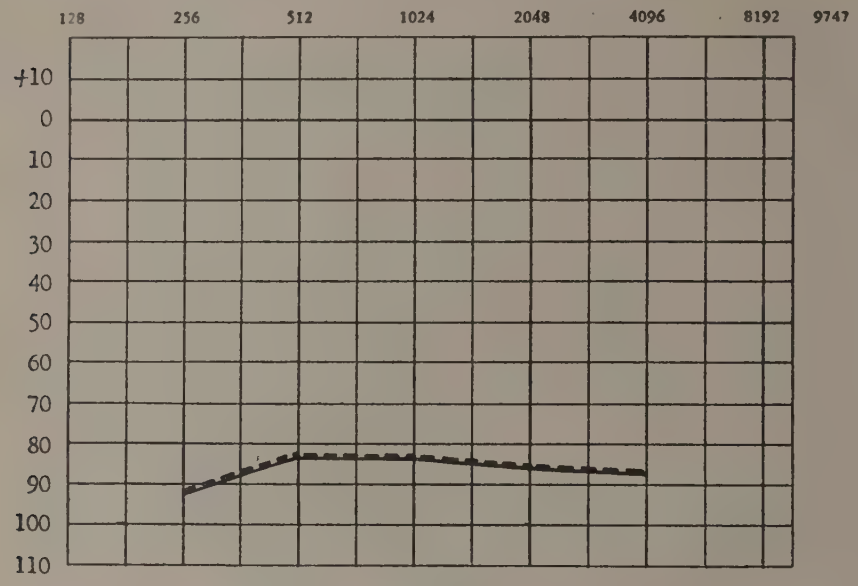

Fig. 1.

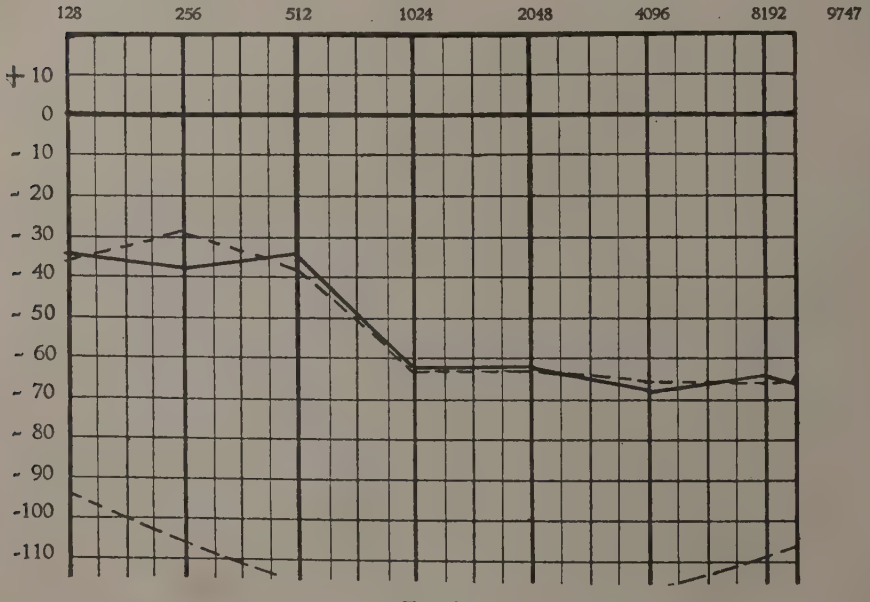

Fig. 2.

At the age of 1½, when he spoke already remarkably well for his age, he got a *parotitis*, after which he became almost completely deaf. In this case the symmetry of the remnants of hearing, as appears from the figure, is very striking. Both parents of the child heard well, while, as far as could be ascertained, no dullness of hearing was found among the various members of the family. Therefore, according to me, there is no single reason to take into consideration, in this case, the possibility of an hereditary genesis.

Also quite symmetric remnants of hearing in a boy of 14, suffering from *congenital* lues, have already been represented in the preceding communication.

In three patients who became dull of hearing after a *typhoid fever* quite symmetric remnants of hearing could also be shown. Fig. 2 shows the audiogram of a lady of 62, who about 25 years ago suddenly became dull of hearing after a typhoid fever, while before that her hearing was excellent. This dullnes of hearing had continued to exist after that and had hardly changed. In the family-anamnesis there occured not a single fact which might point to a hereditary taint.

The further acoustic status was the following:

	Conv. sp.	Lower tone limit	Upper tone limit	C 5 (30″ normal)	RINNE	SCHWABACH	WEBER
AD	1 M	16	f 6	9	+	shortened	medial
AS	1 M	16	f 6	10	+	suortened	

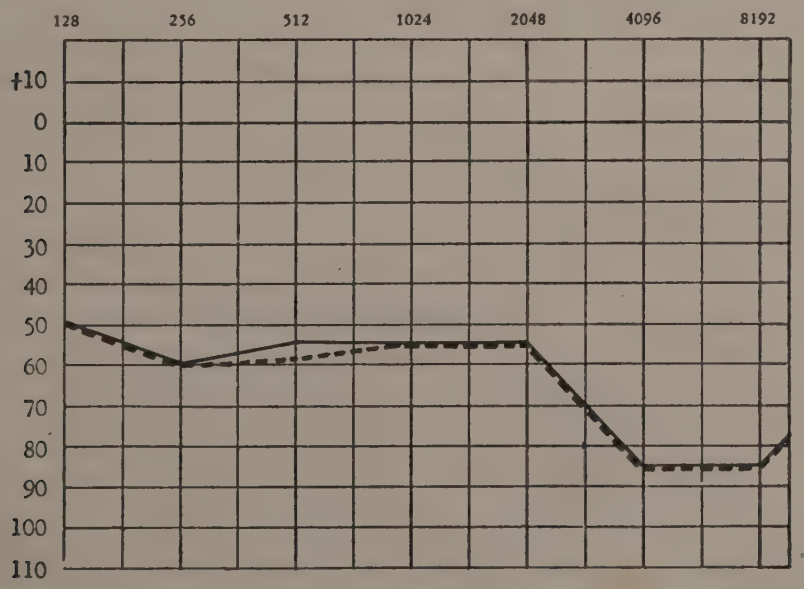

Fig. 3.

II. Also after *otitis media on both sides* one can sometimes observe symmetric remnants of hearing in patients who after these otitides have become dull of hearing on both sides. This, among other things. was the case in a young women, also mentioned in the first communication, who from her 3d year suffered from recidivating otitis media, which led to a dullness of hearing on both sides. A radical operation on the right had to be carried out, on the left a conservative treatment appeared to be sufficient. In the course of years both ears became dry, but the sharpness of hearing had suffered much owing to the many otitides. Accordingly, at the age of 33 a serious dullness of hearing could be shown. The audiograms of the right and of the left ear were completely symmetric, however (fig. 3). Nor were any cases of dullness of hearing found in the extensive family-relations of this lady.

For the rest the acoustic status was:

	Lower tone limit	Upper tone limit	C 5 (30″)	RINNE	SCHWABACH	WEBER
AD	128	n	10	—	± normal	medial
AS	64	n	10	—	± normal	

In order to save his symmetric-law from the possibility of symmetric remnants of hearing being found in patients who have become dull of hearing after an infectious disease, LANGENBERG gave an explanation, mentioned in the preceding publication and according to me carrying little conviction. He also discussed the question whether it is possible that after an otitis on both ears symmetric remnants of hearing are left and thought he must deny this. "Denn dass eine auf entzündlichem Wege entstandene Schwerhörigkeit zufällig gerade auf beide Ohren genau den gleichen Hörausfall hervorrufen sollte, wird man nicht annehmen können." Yet this was the case in the patient just mentioned. Though the one ear had been treated conservatively, the other by operation.

In another patient with otitis media on both sides, who already from a child suffered from this affection, the acoustic status was as follows:

	Whis-pered sp.	Lower tone limit	Upper tone limit	C 5 (30″)	RINNE	SCHWABACH	WEBER
AD	0.3	64	n	18	—	± n	medial
AS	0.2	64	n	18	—	± n	

Fig. 4 shows the audiograms of both ears, also in this case nearly symmetric.

III. Also in patients who have become dull of hearing as a *consequence of noise* in their profession (boiler-makers, gunners, pilots etc.) one may sometimes find a less or greater symmetry of the audiograms. [3]) A typical example of nearly complete symmetry is the following. A man of 39

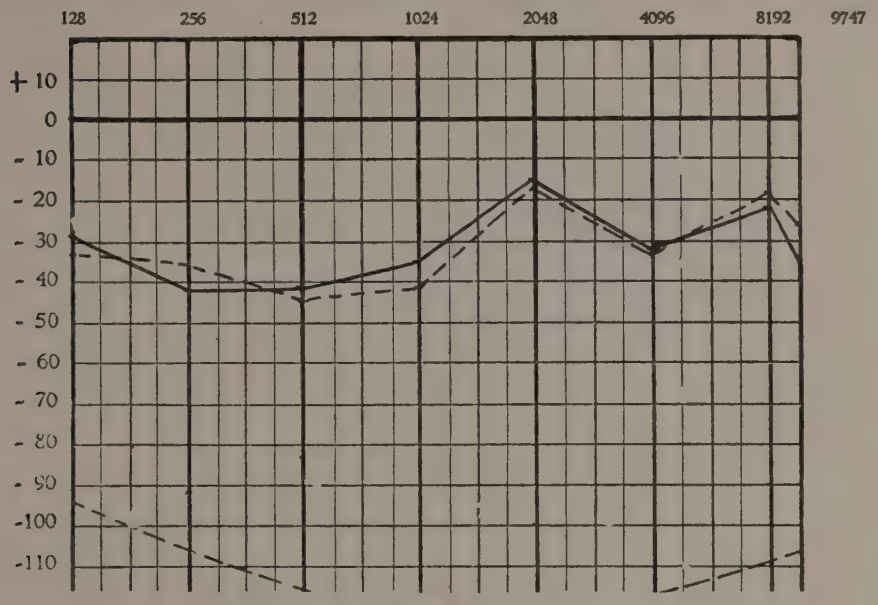

Fig. 4.

had for five years slowly become dull of hearing for high tones, after much artillery-practice. At the acoustic examination the following was found:

	Whispered sp.	Lower tone limit	Upper tone limit	RINNE	SCHWABACH	WEBER
AD	5 M	n	n	+	Lightly shortened	medial
AS	5 M	n	n	+	Lightly shortened	

The audiograms are represented in fig. 5.

Sometimes, however, even one strong detonation suffices to cause dullness of hearing with symmetric remnants of hearing, though in such detonations the remnants of hearing are generally clearly symmetric. Thus a soldier of 37 became completely deaf immediately after a detonation, while before that he had a good hearing. When I examined him about a fortnight after the accident, the quite symmetric remnants of hearing represented in fig. 6 were found. The patient heard no bone-conduction,

[3]) Cf. e.g. in DE WIT, G.: Gehoorbeschadiging door lawaai. Dissertation, A'dam 1940.

conversational speech was heard on both sides ad concham, the lower tone limit was at 128 d.v.

With the slow recovery, which I could not quite follow because of the departure of the patient, the symmetry was lost again and the left ear recovered quicker than the right.

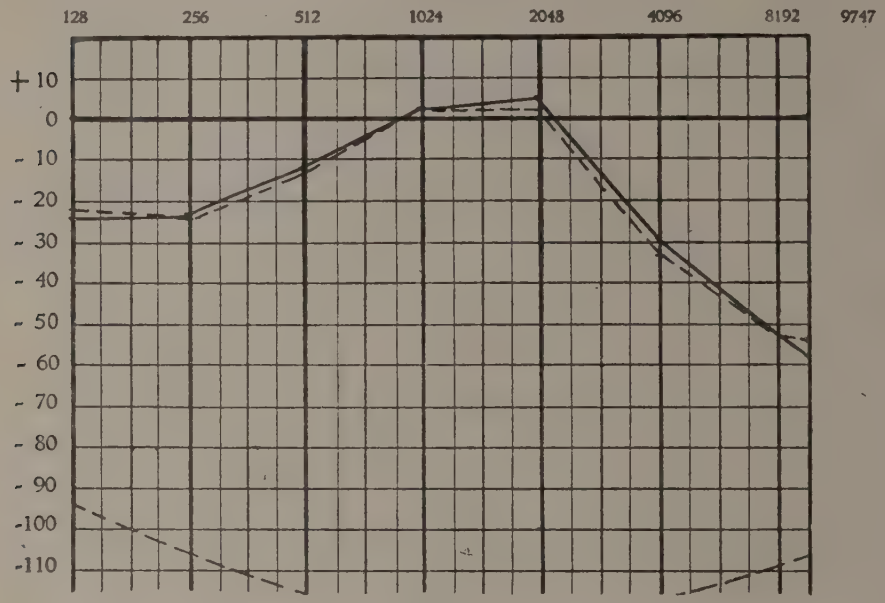

Fig. 5.

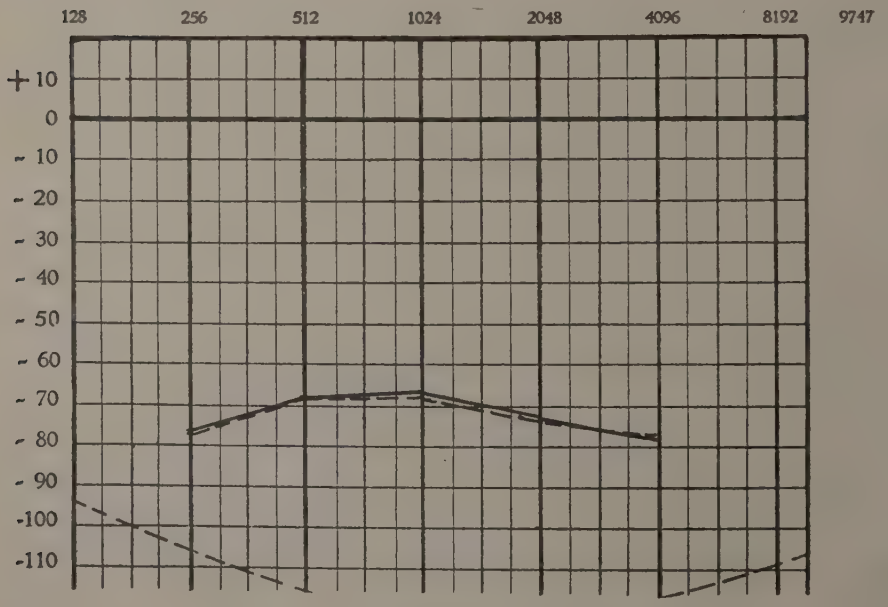

Fig. 6.

IV. A professional illness, with an origin quite different from those just mentioned, is to be found in *boxers*. In all match-boxers examined by me up to now the perception for high tones was disturbed. This was very striking in one of our boxer-champions, a man of 39.

The audiogram of both ears showed a striking symmetry. Som ten audiograms, taken in the course of half a year, all showed, with small variations, the picture of fig. 7.

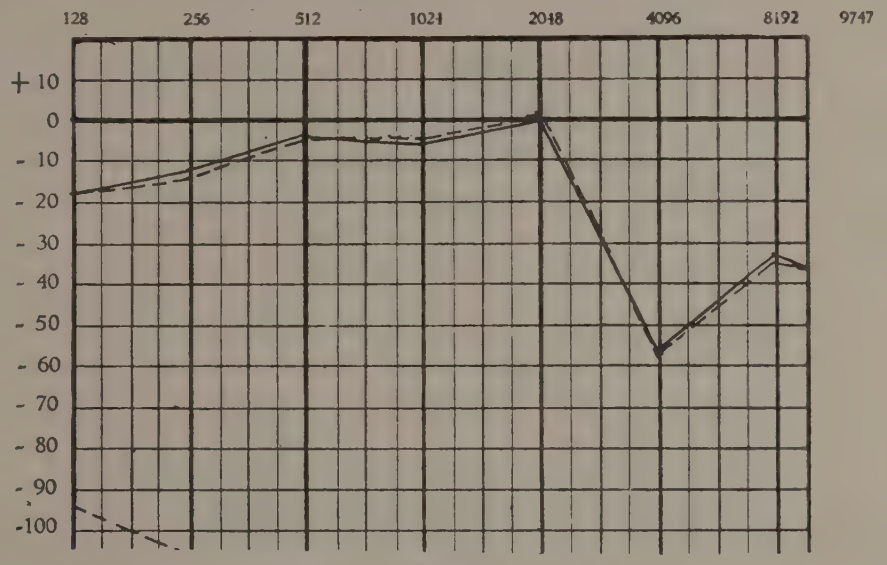

Fig. 7.

V. In all examples mentioned up to now we were dealing with peripheral affections of the octavus system, either of the labyrinth itself or of the N. octavus. Fig. 8 clearly shows, however, that also in *purely central acoustic affections* nearly symmetric remnants of hearing can occur.

The patient concerned, a man of 25, was suffering from the highly localised form of the disease of DUCHENNE-ARAN and showed, beside a central dullness of hearing and central vestibular disturbances in the form of position-nystagmus, a readiness to nystagmus to one tide (Nystagmusbereitschaft) etc. disturbances of speech of a bulbar character and a central paresis of the NN III, VII, IX and XII.

At further acoustic examination the following was found:

	Convers. sp.	Lower tone limit	RINNE	SCHWABACH	WEBER
AD	0.8	64	+	shortened	→ R
AS	0.8	64	+	shortened	

From the above-mentioned examples one can at once draw the conclusion that if injurious influences act on both organs of hearing, the

right and left octavus-system are not seldom affected in exactly the same degree, however varying may be the effect of those injurious influences in each special case. The symmetric remnants of hearing are a clear proof of this.

Even more curious, however, is the fact that several times we found these symmetric remnants of hearing in cases where the loss of function of the two ears had not taken place at the same time by the same cause, but with intervals of some years by different causes.

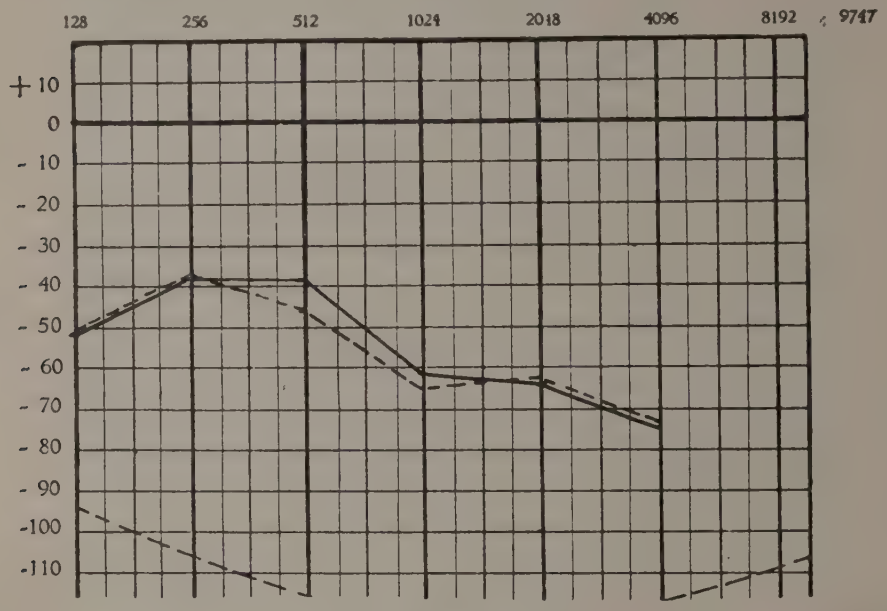

Fig. 8.

In order to explain such observations one might suppose that a certain and bilaterally equal part of the two organs of hearing is more vulnerable than the remaining part and that if both ears are seriously damaged by whatever cause, it is only this more vulnerable part that loses its function. As in different patients not always the same symmetric remnants of hearing are found, the more vulnerable part would have to vary with the individual, as regards size.

As examples we would mention the following cases:

a. The clinical history of the patient already described (fig. 3) points already in this direction. This lady, who was suffering fron an otitis media on both sides, was for many years very dull of hearing in her right, radically operated ear, while the sharpness of hearing of her left ear was still sufficient. Gradually this, too, became less and less and at last reached a level that was completely equal to that of the other side.

b. Another typical example [4]) concerns a lady who became very dull

[4]) Already mentioned in the preceding communication.

of hearing in one ear following upon a gravidity. Many years afterwards, after a malaria, there suddenly arose dullness of hearing in the other ear as well and that to such an extent that the remnants of hearing were quite symmetric (fig. 9). At the further acoustic and otoscopic examination the picture of a dullness of hearing of the middle ear-type on both sides was found (otosclerosis?). In the family no further cases of dullness of hearing were found.

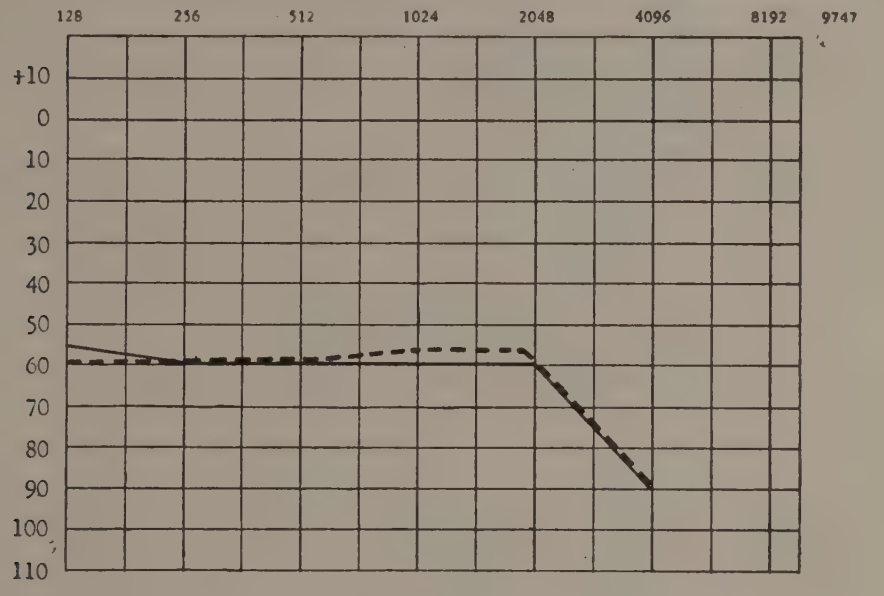

Fig. 9.

The further acoustic findings were:

	Conv. sp.	Lower tone limit	Upper tone limit	C 5 (30″)	RINNE	SCHWABACH	WEBER
AD	0.3	128	h 5	10	—	shortened	medial
AS	0.5	128	a 5	10	—	shortened	

c. Fig. 10 shows the audiogram of a man of 28 [5]) who had been deaf in the left ear as long as he remembered. In his family deafness did not occur. He was a hairdresser and as such he had to talk much with the public. He could do this without difficulty, because his right ear was quite normal. His family doctor also informed us that no dullness of hearing of the right ear had ever been observed in the patient. Some days before his admission into the clinic it appeared, when he awoke,

[5]) See preceding communication.

that he was practically completely deaf. At an extensive internal, neurological and serological examination no cause of this sudden deafness was found. The remnants of hearing were also in this case quite symmetric (fig. 10), whispered and conversational speech were not heard.

d. Finally, clearly symmetric remnants of hearing were also found in a man of 41, who in February 1944 got a heavy object against his right temple and after that became dull of hearing on that side. He had not lost consciousness when this accident happened nor had he vomited. On May 12th, without being able to indicate the cause of this, he had quite suddenly become dull of hearing in the left ear too, which before that time had been quite normal.

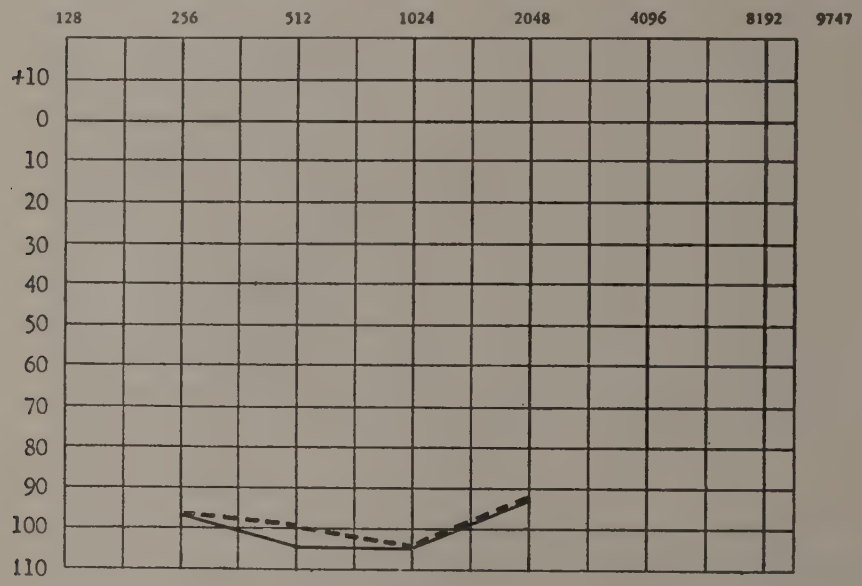

Fig. 10.

When he was admitted into the clinic, röntgenologically there appeared to be no starting-point for a labyrinth-fracture on the right. Internally, neurologically and serologically no anomalies were found, so that the occurring of the sudden dullness of hearing on the left remained completely unexplained.

Vestibularly there was a good excitability on both sides. Acoustically the following was found:

	Conv. sp.	Lower tone limit	Upper tone limit	C 5 (30")	RINNE	SCHWABACH	WEBER
AD	ad conch.	32	12000	8	+	Strongly shortened	→ L
AS	ad conch.	32	12000	7	+	Strongly shortened	

Fig. 11 shows the symmetric remnants of hearing.

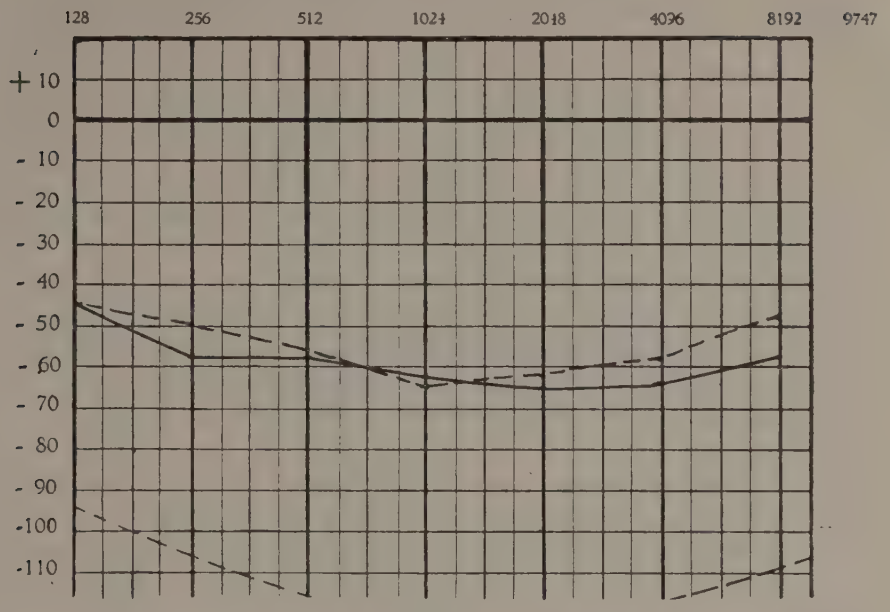

Fig. 11.

Summary.

1. The assertion made by LANGENBECK that symmetric remnants of hearing are so important for the judging whether a dullness of hearing is hereditary or not that one might speak of a "Symmetriegesetz der erblichen Taubheit" is wrong, according to us.

2. It is a fact that if injurious influences act on the organs of hearing this often results in the lesions on the right and on the left side being equally large or in other words, that here symmetric remnants of hearing are left.

3. Sometimes one finds, both in patients with a hereditary taint and in those without one, symmetric remnants of hearing, when the two ears do not lose a great part of their function at the same time, but with intervals of many years.

4. Perhaps the last-mentioned fact can be explained by the supposition that a certain part of the organs of hearing, varying with the individual, but bilaterally equal, is more vulnerable than the remaining part and that if both ears are seriously injured, though this happens with great intervals and by different causes, it is exclusively this more vulnerable part that loses its function. The remaining part will then manifest itself in symmetric remnants of hearing on the left and on the right side.

Medicine. — *Some remarks on audiometry.* II. *The influence of the duration of the sound on the form of the audiogram.* By A. DE KLEYN.

(Communicated at the meeting of November 24, 1945.)

At the meeting of the Dutch Oto-rhino-laryngological Society of Nov. 24th 1940 [1]) VAN DAMME communicated the clinical history of a patient from our Clinic, who at the audiometric examination showed very curious phenomena.

The girl, aged 18, had been admitted into the clinic for tonsillectomy and was a little deaf in both ears, on the right more than on the left. In her anamnesis nothing particular was to be found; deafness did not occur in her family. At the internal and serological examination no anomalies were found.

The tympanic membranes were a little retracted. The sharpness of hearing for whispered speech amounted to 3 M on the right, to 4 M on the left. The lower tone limit was heightened on both sides (256 d.v.) the upper tone limit lowered (6000 d.v. examined by means of the monochord of STRUYCKEN). RINNE was indicated positively on the left, SCHWABACH strongly shortened. Bone-conduction was not perceived well on the right. At the test of WEBER the tuning-fork was lateralised to the left. At the audiometric examination the right ear showed a very curious phenomenon, which was absent on the left.

1. Both for bone- and for air-conduction sounds, when acting a short time, were not heard.

2. These sounds, when acting a long time, were perceived clearly, but only after a certain number of seconds.

3. When the source of sound was interrupted regularly, the sound was only perceived after a certain number of seconds and this number of seconds varied in proportion to the intensity of the source of sound (summation).

4. The longer a tone-sound acted continuously, the lower the tone-intensity, necessary to be perceived.

V. DAMME put the question what was the significance of this perhaps pathological perception. Owing to a coincidence it had not been possible to make a more extensive neurological examination of the patient, so that this latter question had to remain unanswered. During the discussion V. DAMME also communicated that the patient had never suffered from tinnitus.

In connection with a remark made by WAAR that the observation of V. DAMME should be considered as a stimulus-summation in general, I

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1) Ned. Tijdschr. v. Geneesk., 85, 2576 (1941).

thought I had to point out, during the discussion, that the explanation could not be so simple, as the above-mentioned phenomena are not found in normal persons and were only present on one side in the patient of v. DAMME. According to me something analogous is rather found in the abnormal pain-reactions which one can sometimes observe in tabetici. In these cases the patients indicate, when pricked in an ordinary way by the point of a pin, that they feel no pain, if one continues the pressure, however, a sudden pain is felt.

It was the original intention, therefore, to continue the examination of v. DAMME in deaf tabetici because, after what has just been communicated, the possibility did not seem excluded that in them corresponding disturbances of the senses of hearing would occur. This investigation did not take place, owing to war-circumstances. It has appeared, however, that a remarkably high percentage of another group of patients shows the phenomenon of v. DAMME, namely patients who became dull of hearing following upon a detonation.

The audiometric examination was always made by us in the following way: At first the threshold-value for a certain tone was determined, this being *slowly* strengthened [2]) until the tone was perceived by the tested person. Here, for purposes of control, the determinations were repeated some times, so that patients with whom the indications differed more than a few decibles, could be eliminated as tested persons who were not reliable.

After such a determination the same tone was given again, but now lessened 5 decibles in intensity. Then normal persons do not perceive the tone, not even if it acts a few seconds (an action of up to 5 seconds' duration was investigated). In patients who show the phenomenon of v. DAMME this is the case, however. After the tone has acted a few seconds with them, they indicate that they hear it. Now a further investigation was made, each time with an intensity 5 decibles lower, until an intensity was reached where the tone was not even perceived when it acted longer. In this way one obtains two audiograms of each ear: one in which the threshold-value for the different tones has been determined in the ordinary way and one in which the tones were only perceived after a longer action.

A typical example, where the phenomenon of v. DAMME was observed after a detonation, is the following:

St., a man of 40, heard very badly with both ears after a bombing-smash in his direct surroundings. For that matter, there had been very little sharpness of hearing of the right ear as a consequence of an otitis

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[2]) This strengthening slowly is absolutely necessary in order to obtain constant results. Strengthening too quickly causes the threshold to be found varying and too high. The threshold-values for the following tones were always investigated: 128, 256, 512, 1024, 2048, 4096, 8192 and 9747 d.v.

media; accordingly, the tympanic membrane on this side was strongly and irregularly retracted.

The left tympanic membrane showed various haemorrhages in the form of stripes. The bombing-smash took place on February 9th, 1945. Several audiometric examinations always showed the phenomenon of v DAMME, but especially on the left.

Fig. 1 shows the audiogram, taken at the last examination, of the left ear (March 30th, 1945). On that date the whispered speech, which was

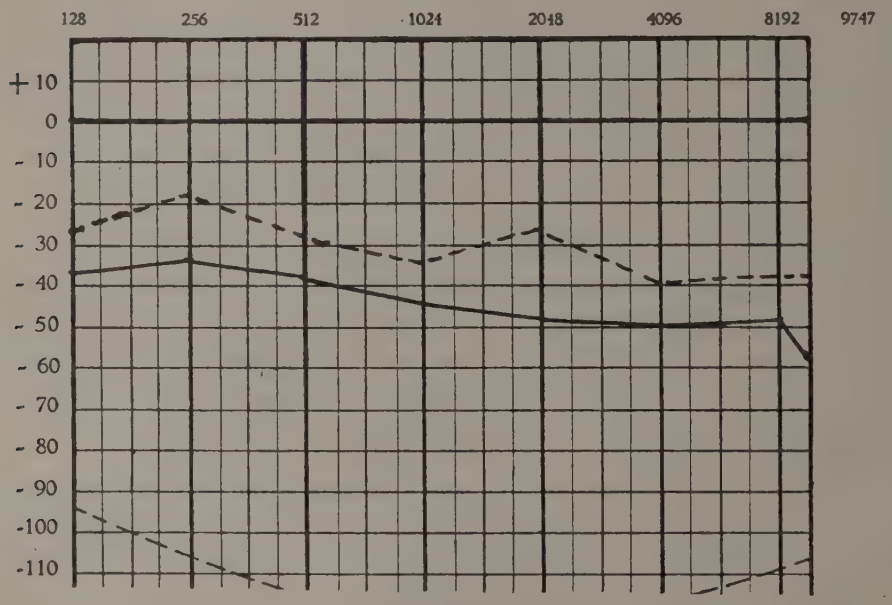

Fig. 1.

not heard at the first examination a month ago, was perceived again at a little more than 5 metres, the lower tone limit was found at 16 d.v., the upper tone limit at g^6. The tuning-fork c^5, heard normally for 30 seconds, was only perceived for 9 seconds.

RINNE was indicated positively, SCHWABACH shortened. WEBER was localised medially. In fig. 1 the extended line means the audiogram obtained by examination in the ordinary way, the interrupted line the audiogram when the different tones act longer.

It is seen that the two audiograms show a difference of not less than 10—21 decibles.

One find another example represented on fig. 2.

C., man of 31, when there was a bombing-smash a year ago, got a rupture in both tympanic membranes, with two permanent big perforations down in front as a final result. There was no secretion. About 4 weeks ago another bombing-smash took place in his surroundings and now his left leg was also seriously injured.

The examination showed that the left ear was practically eliminated. On the right the acoustic status was:

	Whispered sp.	Lower tone limit	Upper tone limit	C 5 (normal 30″)	RINNE	SCHWABACH	WEBER
AD	ad concham	128	a 6	only haerd at smash	—	shortened	→ R

Fig. 2 refers to the two audiograms of the right ear. It is curious that at repeated examinations the phenomenon of v. DAMME was absent at

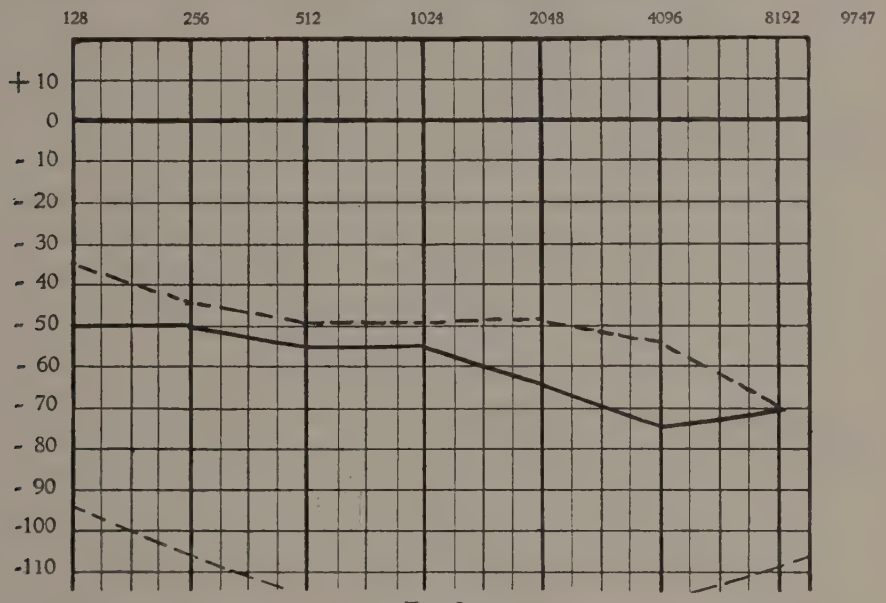

Fig. 2.

the tone 8192 d.v., but was present at all other tones. Tone 9747 was not perceivd when the examination was made in the ordinary way. Only if this tone acted for some seconds at the maximum strength of our audiometer did the patient indicate that he heard the tone.

Not only this patient, but also several others showed the symptom that with them the phenomenon of v. DAMME could be shown for certain tones, but not for others. Thus, to mention only a few examples, it was found in

Pat. W. 31 y

	Detonation	Tymp. membrane	Whispered sp.	Conv. sp.	Phen v. DAMME absent for tones
for a week	AD	Tymp. membrane rupture down in front	0	ad conch.	Only for tones of 1024 d.v.
	AS	idem	0	0.2	of 512, 1024 and 2048 d.v.

Pat. N., 38 y.

	Detonation	Tymp. membrane	Whisp. sp. 3 M	Conv. sp.	Phen. v. DAMME absent for tones
for about 1 year	AD		3 M		For all tones, except 52 d.v.
	AS		ad conch.		of 128, 256 and 4096 d.v.

Pat. St., 35 y.

for 3 year	AD	no anomalies	4 M		of 128 d.v.
	AS	idem	0.75 M		of 1024 and 2048 d.v.

It appears already from these few examples that when there is a detonation the two ears can be struck in a very different degree of strength. This difference was very pronounced in the following patients:

Pat. S., 23, was examined two weeks after a bombing-smash which took place in a room in which there were 8 people. Four persons were killed and four seriously injured. A splinter of a shell seriously injured patient S. in the shoulder and less seriously in the head. After the accident he lost consciousness for two hours. The tympanic membranes showed no anomalies. The acoustic status was:

	Whisp. sp.	Lower tone limit	Upper tone limit	C 5 (norm 30'')	RINNE	SCHWABACH	WEBER
AD	> 5 M	16	n	13	+	shortened	medial
AS	1.5 M	16	n	9	+	shortened	

AD Phen. of V. DAME is absent for all tones except 256 d.v.
AS only for tone 572 d.v.

Such a result was also obtained with patient K., 39, who more than three weeks before our investigation was wounded in the lung and became dull of hearing, when a bombing-smash took place. Only the right tympanic membrane showed a haemorrhage down in front. The acoustic status was as follows:

	Wbisp. sp.	Lower tone	Upper tone	RINNE	SCHWABACH	WEBER
AD	2 M	64	g^6	+	shortened	medial
AS	4 M	16	g^6	+	shortened	

At the audiometric examination the phenomenon of V. DAMME was clearly

present on the left ear (fig. 3 – and – – –); on the right it was entirely absent (fig. 3 –.–.–).

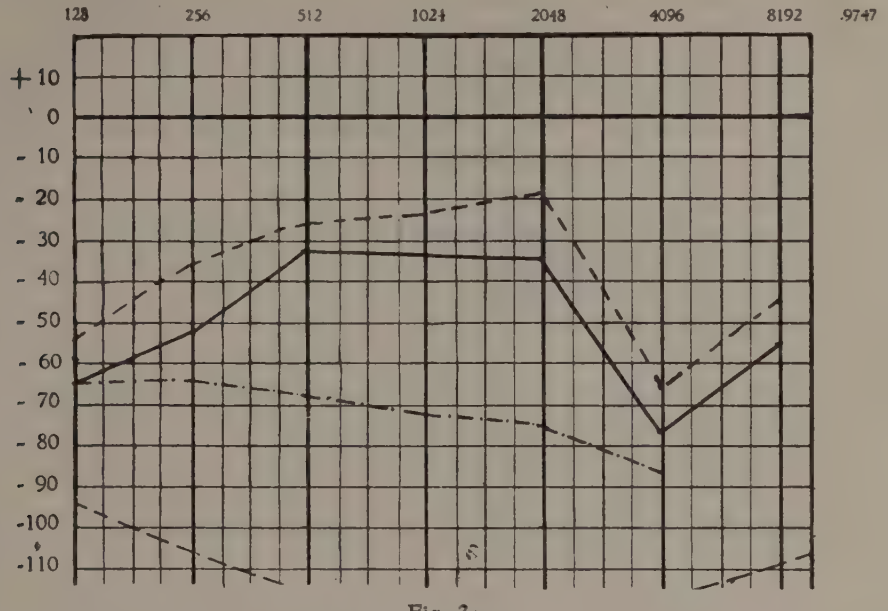

Fig. 3.

Yet more symmetric lesions of hearing can also occur owing to detonation. In reality, as was described in part I of this communication, one finds all sorts of transitions, even completely symmetric remnants of hearing (p. 14).

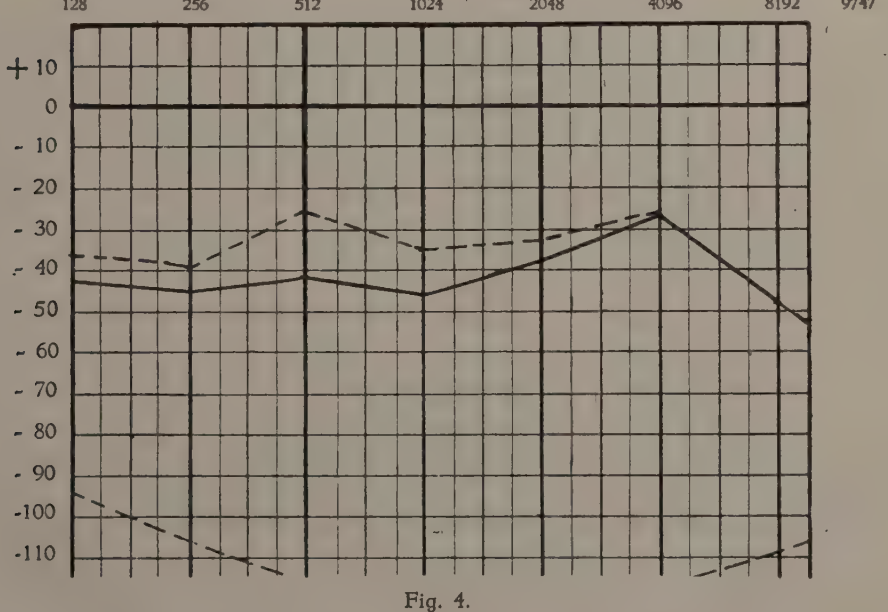

Fig. 4.

In general I got the impression that the phenomenon is most strongly pronounced a short time after the detonation and that if it decreases or disappears, this happens first for certain tones and after that entirely.

The phenomenon can also be observed when an examination is made of the bone-conduction. Generally speaking, however, the finds when there was air- and when there was bone-conduction were not in accordance with another. Thus patient S. (fig. 4), when there was air-conduction, showed the absence of the phenomenon at the tones 4096,

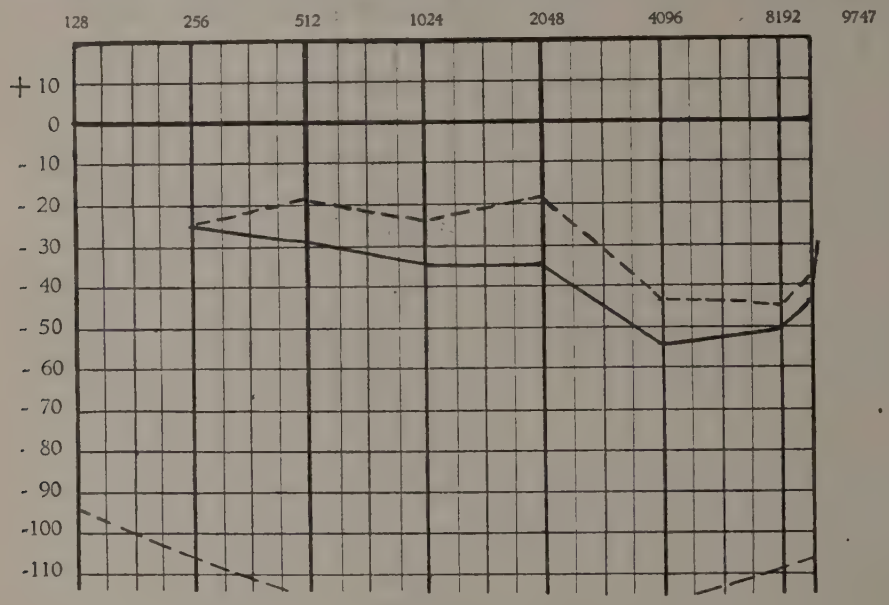

Fig. 5.

8192 and 9747 d.v., when there was bone-conduction only for the tone 256 d.v. (fig. 4 air-conduction, fig. 5 bone-conduction). Inversely, a patient W., a man of 25, who at a bombing-smash was injured in the face and in the head and who after having lost consciousness for about 12 hours appeared to be very dull of hearing, showed the phenomenon of v. DAMME for all tones when there was air-conduction, but for not a single tone when there was bone-conduction.

Summary.

1. In some patients the duration of the sound appears to have a clear influence on the form of the audiogram.

2. As early as 1940 v. DAMME found in a patient from the Amsterdam Otological Clinic that she did not perceive sounds when they acted a short time, but that she did perceive them when they acted longer. In this communication it could be shown that this phenomenon is observed rather frequently in patients who have become dull of hearing by a detonation.

3. In this latter group of patients the phenomenon of v. DAMME is found both when there is air- and when there is bone-conduction, though the anomalies occurring with these two conductions do not run parallel.

4. The phenomenon is often only to be found with certain tones. Generally speaking it seems to be most strongly pronounced a short time after the detonation and if improvement occurs it seems to disappear first for certain and later for all tones.

5. When the detonation occurs the two ears, in most cases, are not struck in an equal degree of strength. Yet, beside very strong asymmetric lesions all sorts of transitions occur to completely symmetric affections, expressing themselves in symmetric remnants of hearing .

Medicine. — *Some remarks about gargoylism* (*Typus E*). By P. G. GERLINGS and A. DE KLEYN. (From the Oto-rhino-laryngological Department, University of Amsterdam.)

(Communicated at the meeting of November 24, 1945.)

In 1936 Mrs. C. DE LANGE and L. WOLTRING [1]) described two boys with a clinical picture resembling gargoylism, but with certain deviations from the typical picture. For this they used the name Typus E, this letter being the initial of their family name. One of the boys was later (1941) admitted into our clinic and died there after a bronchoscopy. A detailed publication by CORNELIA DE LANGE, P. G. GERLINGS, A. DE KLEYN and T. W. LETTINGA about the clinical symptoms and the general post mortem findings appeared a short time ago in the same Acta. [2]) In this communication the otological anomalies were not described in detail. As, however, as far as we know, this is the first time that it was possible to examine the ossa petrosa microscopically in a case of gargoylism and as in this case the central acoustic paths were examined in detail, it also seemed desirable to us to give a more detailed communication of tthe otological findings.

In 1936 the past thistory ran as follows: (C. DE LANGE)

Boy, aged 12, birth normal, walked at the age of a year, but after a short time left off walking and standing. Near the end of the second year the abdomen rapidly became protuberant. He resumed walking when 2½ years old. Two years later attacks of somnolence occurred of 3 to 5 days duration, followed by headache and vomiting. Enuresis nocturna appeared at the age of six. The boy was not able to follow the lower school and was transferred to a school for mental defectives. The attacks of somnolence ceased at the age of 7½ and did not return. In 1932 he was operated for an umbilical hernia. It cannot be stated with certainty when the retardation of growth set in, probably in the patient's second year. As far as the mother can recollect, the boy showed a certain stiffness in both arms. There was a chronis otitis media for several years.

On examination (1936): Length 1.19 m (— 31 cm); weight 40 kg (— 1,5 kg). Dysproportionate dwarf of unwieldy build. Slight dolichocephaly: greatest circumference of the head 54 cm, no ridges, no bossing. Sutures and fontanelles closed. No hypertelorism. The features are without relief. Lips thick, nose broad, eyebrows thick and bushy, hair of the scalp thick and dry. Corneae clear, fundus oculi normal. The boy wears

[1]) DE LANGE, CORN. and WOLTRING, L. Acta Paediatrica, **19**, fasc. 1 (1936).
[2]) Acta Paediatrica ,**31**, 398 (1944).

glasses for myopia. Limbs of coarse and broad build, hands short and thick, somewhat stiff and with limitation of movements. The patient cannot clench his fist. On both hands the little finger is curved in toward the radius. The fingers are in flexion-contracture, the interphalangeal articulations slightly swollen. The nails have watchglassform. There is a marked limitation of movements in the shoulder joints, movements being only possible with co-operation of the shoulderblades. If a hand is laid on the boy's shoulder cracking of the joint can be felt. In both elbowjoints there is a flexion-contracture and the rotation is slightly limited. Thorax symmetric with wide lower aperture.

No findings in the lungs. The heart dullness is normal on percussion, but over the ictus cordis a systolic murmur with accentuation of the second pulmonary sound is heard.

Abdomen is enlarged, circumference 65 cm. The liver is very large, reaches to 1 à 2 cm below the umbilicus, surface smooth, non-tender. The spleen can just be felt when there is respiration, but when there is inspiration it reaches two fingerbreadths below the costal margin. The boy is mentally defective, he is shy and has an introverted nature. At the age of fourteen he was only in the third form of the school for backward children. On the other hand he is fond of simple carpenting and drawing (fig. 1).

Neurological symptoms are lacking.

Bloodcount. Hgb. 90 %, R.B.C. 4.750000, W.B.C. 7300. Platelets 290000, Sed. rate 1st hour 2 mm, 2nd hour 4 mm. Eosinophils 1, polymorphs 49, 5 lymphocytes 45,5, mononuclears 4 %. No hypersegmentation of the polymorphs, no toxic granules. Urine nothing abnormal ;in particular no acetone in the morning portion. TAKATA-ARA's reaction negative; HIJMANS V. D. BERGH direct negative, indirect trace of bilirubin. Galactose tolerance test (40 g. per os): the blood galactose exceeds the normal limit of 40 mg% and the excretion of the galactose is lengthened.

Radiological examination: Marked dolichocephaly, sella turcica normal. No impressiones digitatae. Occipital bone somewhat thin. Sutures without anomalies. [3]) In the elbowjoints the proximal parts of radius and ulna are broad and coarse, especially the capitulum radii. Metacarpals and phalanges abnormally broad and stunted. The capitula of the first phalanges on both sides much deeper indentated than normal. Slight retardation of the ossification of the carpals not as to number but as to size. The distal parts of radius and ulna also broad and coarse.

At the age of 19 the patient was admitted into the Otolaryngological clinic on April 4th 1941, as he did not get sufficient air and became

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[3]) As in this patient afterwards a diagnosis of craniostenosis and closed sutures was made we have again examined the radiagram of his skull. There proved to be only a lateral view, the sella was certainly not enlarged at that time, the coronar suture is distinct, the lambdoid is less so.

short of breath especially when riding on a bicycle. His voice should also have changed, while at a distance an inspiratory stridor was perceptible.

The patient had already been under policlinical treatment from 1931. In 1933 an adenotomy was performed. The clinical history of 2.9.1931 said: on the right side the whispered voice is not heard, at the left at a distance of 0,5 m. Lower tone limit bilaterally at 24 d.v., upper tone limit 18000 respectively 19000 d.v. (Monochord). RINNE positive on both sides. SCHWABACH shortened, WEBER no lateralization.

From the family history it appeared that both parents were in good health and heared well, 2 children had also a good hearing, but one brother (acoustic examination see history), the great-grand-mother on the father's side and three aunts of this latter were dull of hearing. Acoustic examination: both tympanic membranes were normal. In the beginning no exact examination with tuning forks was possible owing to the peculiar negativistic psychological condition af the patient. When he felt more at home in the clinic this examination could be performed much better. Then we found:

		Conversational voice	Lower tone limit	Upper tone limit (monochord)	C 5 (normal 30")	RINNE	SCHWABACH	WEBER
16—12—'41	AD	ad conch.	32	17000	7	+	shortened	→ L
	AS	20 cM	32	20000	12	+	shortened	

Audiogram: see fig. 2.

The examination of the 11-years old brother gave the following result:

		Whispered voice	Lower tone limit	Upper tone limit	C 5 (normal 30")	RINNE	SCHWABACH	WEBER
12—12—'41	AD	ad conch.	16	13000	10	+	shortened	no lateralization
	AS	ad conch.	16	12000	13	+	shortened	

Audiogram: see fig. 3.

Therefore, both in 1931 (see previous history) and in 1941 the acoustic condition of our patient showed the picture of a labyrinthine diminished hearing.

A vestibular examination of the patient could not be performed as he died suddenly after a bronchoscopy.

His brother, however, showed quite normal responses resulting from stimulation of both vestibular apparatuses.

Rhinoscopia anterior: no disturbances; rhinoscopia posterior: did not succeed in consequence of the patient's thick tongue.

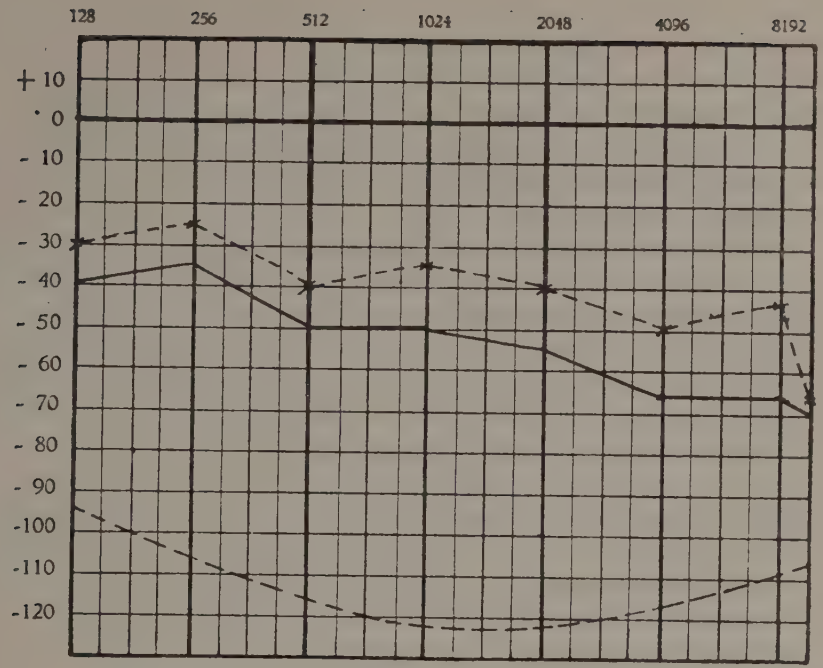

Fig. 2. AD Air Conduction ————————
AS Air Conduction ——×——×——×

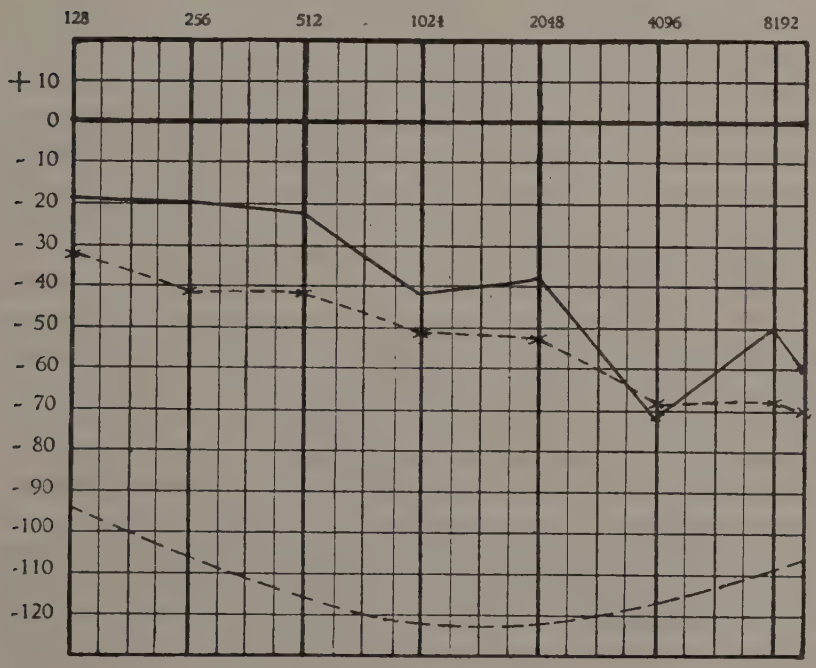

Fig. 3. AD: Air Conduction ————————
AS: Air Conduction ——×——×——×

Where, owing to the same reason, the indirect laryngoscopy had not succeeded, the direct method was used which gave the following result: on the left vocal cord an irregular papillomatic swelling was found which was partly removed. The pathologist (dr. SCHOLTE) reported that the specimen removed for examination consisted of a piece of the vocal cord, covered with pavement epithelium under which cartilage was found. Slight edema was present also.

On 6.12 a bronchocospy was performed under evipan narcosis. The left vocal cord was still thickened. For the rest the glottis was passed easily by the tube. The distance to the bifurcation was very short. The carina was abnormaly thick which seemed to be caused by a swelling of the mucous membrane at the entrance of the left main bronchus. This was the reason why this bronchus was only partly visible. The right main bronchus showed a normal picture.

Another specimen was removed from the left vocal cord. This time microscopical examination proved that the vocal cord was covered with cylindric epithelium; only at one place pavement epithelium was present; in the depth sometimes cartilage, sometimes chondroid or mucoid alteration of the connective tissue was found. It was the same picture as known in cases of tracheopathia chondro-osteoplastica.

On 18.12 the patient refused to stay in the clinic any longer and went home.

In February he was readmitted as the stridor had become worse. Bronchoscopy under evipan narcosis was carried out and showed a thickening of the posterior wall of the thrachea just beneath the glottis. The trachea itself seemed to be narrowed. This time the right main bronchus was easily exposed; again a lumen image of the left main bronchus was not obtainable in consequence of the thickened carina. A portion of the carina was removed for microscopical examination. The pathologist (dr. SCHOLTE) reported: The removed tissue consists of connective tissue with nerve fibres and fat. A diagnosis cannot be made.

The bronchoscopy was repeated on the 3rd of March under evipan narcosis. The trachea was still narrowed by the swelling of its mucous membrane so that even the bronchus-tube for women could pass only with difficulty. A portion both of the carina and the posterior tracheal wall was removed for microscopical examination. Directly after the bronchoscopy the breathing stopped suddenly and although a tracheotomy was immediately performed and the tube was again inserted, the heart soon stopped beating. Subcutaneous injection of lobeline and adrenalin into the heart had no result. A part of the mucous membrane of the larynx and of a part of the trachea were removed directly after death for microscopical examination.

The general pathological findings (dr. LETTINGA) can be summarised as follows [4]):

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[4]) See for details Acta Paediatrica **31**.

In the liver there is slight cirrhosis and an accumulation of glycogen to such an amount that we may say the enlargement of the liver is mainly due to this last fact. In the heartmuscle and the kidneys, which, like the liver, were also fixed for over twentyfour hours in 10 % formalin, no glycogen could be demonstrated. The method of LORRAIN-SMITH-DIETRICH showed only traces of phosphatids in the liver and spleen. Beside these organs, the heart muscle, heart valves, aorta, kidneys, thymus, prostate, larynx, trachea, lungs and adrenals were also examined in this way. The result was negative except for the adrenals where, at several places, numerous black granules were found in the cells.

In the spleen, beside faint signs of a chronic perisplenitis, there is a slight fibro-adenia as is often seen with a concurrent cirrhosis of the liver. The enlargment might be due to the action of the same noxae, which are also responsible for the cirrhosis of the liver.

In the heart muscle we find signs of an acute myocarditis. The sclerotic valves showed chondroid changes. It was difficult to give a good explanation of these changes. As the result of an endocarditis the aspect is more or less uncommon. Very striking, however, is the resemblance to the changes in the perichondrium and contiguous connective tissue of larynx, trachea and vertebral column (neck).

In the wall of the left main bronchus no explanation was found for the swelling observed there at bronchoscopy.

As to the right elbow joint, the presence of nearly all the epiphyseal lines of humerus, radius and ulna at an age of over 19½ years, indicate that there is a considerable retardation in growth. (There is nothing anomalous as regards the ossification.) The cartilage of the joint surface is somewhat thick, which corresponds to the relatively small bones in the epiphysis.

Finally there was strong mucoid degeneration of the connective tissue in various parts of the body.

The laryngological pathological findings were the following: Trachea and main bronchi: Trachea in antero-posterior direction somewhat flattened. Musosa smooth, somewhat red. No narrowing of the main left bronchus at the bifurcation. At a distance of about 1½ cm from the bifurcation there is a little defect caused by exploratory excision. Trachea and main bronchi contain some sanguineous mucus.

The larynx is narrow. Some edema of the plica glossoepiglottica and the mucous membrane of the arytenoid cartilages. On the left, somewhere abount the vocal cord, there is a soft, irregular swelling of a blood-red colour with a diameter of about 1 cm. Joined with and distally from this swelling a defect in the mucous membrane of about 4 cm caused by the removal of tissue during life and after death. The left cord cannot be recognized here. Further below the right vocal cord there are some artificial defects in the mucous membrane. The corpus ossis thyreoidei and the cricoid cartilage are slit in the middle.

Microscopical examination:

Hematoxilin-eosin stains unless otherwise mentioned.

Larynx at approximately the level of the left vocal cord: the boundary of the arytenoid and cricoid cartilage is irregular and not sharply defined in respect of the surroundings. The perichondrium is enlarged and shows several cartilage cells and chondroid and mucoid changes in the connective tissue, which stain red and partly violet with cresylviolet; the staining with mucicarmine is negative. Farther, at greater distance from the arytenoid, on the outer side as well as in the mucous membrane of the larynx there are areas, not clearly circumscribed, where the cells begin somewhat to resemble cartilage cells with mucoid degeneration of the hyalin connective tissue. There are no elastic fibres in this region. In the mucous membrane also rests of the epithelium, sometimes distinctly stratified squamous epithelium, mucous glands, nerves and some blood. At some places proliferation of elastic fibres. Nor is the boundary of the thyroid cartilage sharp. At a short distance there are also chondroid changes in the connective tissue. The cartilage cells of the arytenoid and thyroid cartilage are less flattened at the periphery than usual. The interstitial substance exhibits the so-called asbestos transformation. There are relatively many cartilage cells. These contain fat droplets staining orange-yellow with Sudan III, which are not birefractory. The cells of the perichondrium of the thyroid and arytenoid cartilage have a finely granulated protoplasma often staining a pale brownish-yellow with Sudan III and containing no birefractory substances, even in a region with clear chondroid changes. On the same level in the right half of the larynx we find the same changes in cartilage and perichondrium, but no islands of cartilage or chondroid changes in the mucous membrane. Of the epiglottis the boundary between cartilage and perichondrium at several places is not sharp and the perichondrium shows chondroid changes and mucoid degeneration in the partially hyalin connective tissue. The capsules of the cartilage cells are often broad and changed into very small violet-blue granules, which are also found in the enlarged cartilage cells, where the nuclei are partially lost. Neither is the cartilage of the trachea and *left main bronchus* sharply circumscribed, mainly on the outside and the perichondrium exhibits the same changes as in the larynx. At the place of the biopsy some granulation tissue, fibrin and slight bleeding. The mucous membrane exhibits no further anomalies.

The brain was examined by Mrs. DE LANGE. The outcome of this investigation was entirely negative. That part of the medulla which contains the auditory area, was divided into halves, one being stained according to WEIGERT-PAL, the other with thionine-blue. No abnormal findings were present.

Prof. DE LANGE thus summarised the clinical and the post mortem examination: "Nowhere do the nerve-cells show the faintest resemblance

to those in the classic gargoylism, nor are there traces of a past meningo-encephalitis as one might suppose in view of the fact that attacks of somnolence occurred at the age of 4½."

In our case, therefore, the clinical picture resembled in a large measure that of gargoylism or dyostosis multiplex Hurler, but without the histological substratum of a lipoidosis or expressed differently: one might say that the macroscopical phaenotypus is present; but the microscopical one is absent.

The microscopical examination of both ossa petrosa revealed: *L. temporal bone*. The petrous apex contains mixed lymphoid fat marrow, here and there perivascular bone-resorption in the corticalis. Clear impression of the ganglion GASSERI. The carotis canal shows some dehiscences, there is an extensive plexus venosus caroticus, no particulars on the A. carotis interna and rami sympathici. The sinus petrosus sup. and inf. contain little blood, the veins of the dura of the central cranial fossa are widened and filled with blood.

There is nothing particular about the tube, one peritubar cell. The septum intermusculo-tubaris is largely wanting, the musc. tensor tympani is a little atrophic.

In the middle ear there is a fibrous hyperplastic mucous membrane, the niche of the round window is partly filled with connective tissue, the tympanic membrane is thickened. A few connective tissue traits run through the tympanic recess. The malleus contains cartilage remnants, both malleus and incus have great spaces of marrow. There is a bony growth between malleus and tegmen tympani. The bottom of the tympanic cavity shows a great dehiscence, this is attended with a high position of the bulbus v. jugularis. The tegmen tympani also shows a great dehiscence and several impressions, with PACCHIONIC granulations. Through the dehiscence this PACCHIONIC granulation reaches as far as the mucous membrane of the tymptanic cavity. (Fig. 4.)

In the labyrinthine capsule one sees below the cochlea-turn a small focus of otosclerosis (fig. 5) with strongly dilated vessels, round which there is resorption, while against the basal cochlea-turn a focus with new formation of bone can be seen.

Before the oval window there is a rather large cartilage remnant ("Knorpelfuge").

In the facialis-canal there is a dilated vein, there is a small dehiscence in the bony canal.

In the cochlea (fig. 6) the scalae vestibuli and tympani are normal. In the middle turn a little connective tissue trait runs in the scala tympani. In the ductus cochlearis, which has partly collapsed, the organ of CORTI has been changed into a little hill of cells, the membrane tectoria lies pressed against it. The ganglion spirale shows few ganglion cells in the basal turn, there is secondary glia-proliferation. (Fig. 7.)

The sacculus has been partly torn, the macula sacculi also, the otolithic-

membrane loose above it, in the lumen of the sacculus. The utriculus is not dilated, the macula is intact. The ductus and saccus endolymphaticus are a little too wide. The aquaductus cochlea is normal. The semicircular canals have small cristae ampullares, the cupula of the anterior and posterior vertical canal is wanting.

In the inner auditory canal, which is not too narrow, the crista transversa is somewhat blunt, some irregular contour near the fundus. In the ganglion vestibularis the number of ganglion cells seems small, as in the ganglion spiral secondary glia-proliferation. (Fig. 8.)

There is a strong inhibition of pneumatisation.

The crista pyramidalis sup. has developed strongly, with large spaces of marrow, in which fibrous marrow with wide vessels. The posterior vertical canal lies very superficially, the ampulla reaches as far as the bulbus-wall.

R. temporal bone. Shows the same picture as left. The middle ear shows a strongly hyperplastic fibrous mucous membrane with exsudation containing few cells in the lumen. In the mucous membrane subepithelially, infiltration and small cysts. In the lower cochleaturn some rose-coloured precipitation.

The otosclerosis-focus is exactly symmetrical, still goes on in the promontorium.

In the ductus cochlearis it is difficult to judge the organ of CORTI.

Summary.

L. temporal bone.

1. Strongly inhibited pneumatisation with fibrous hyperplastic mucous membrane.
2. Otosclerosis-focus.
3. Too small number of ganglion cells in the basal cochlea-turn with secondary glia-proliferation.
4. Atrophy of the organ of CORTI.
5. Secondary glia-proliferation in the ganglion vestibulare, where the ganglion-cells have partly disappeared.
6. "Knorpelfuge".
7. Dehiscences bottom of tympanic cavity and tegmen tympani with PACCHIONIC granulations.
8. Congestion of the vessels of the dura of the central cranial fossa.

R. temporal bone.

Same picture as left, except 7.

There is a catarrhal otitis media.

It has often been observed that patients suffering from gargoylism or related illnesses [5]) are deaf or dull of hearing. Thus the first described

[5]) See a.o. HOWARD, E. and RUGGLES, E. Am. Jnl. Roentg. 25, 91 (1931).

Fig. 1.

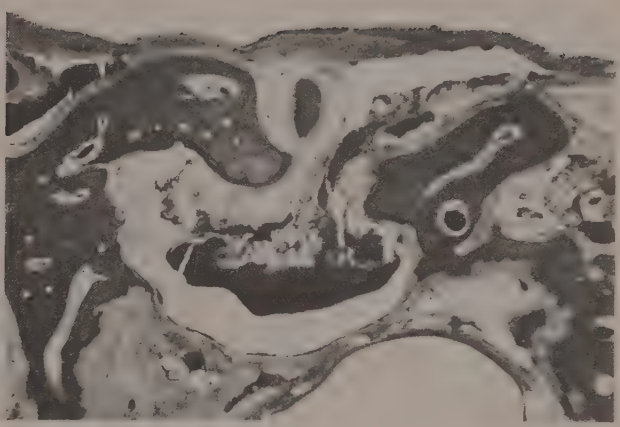

Fig. 4.
Dehiscence of tegmen tympani with Pacchionic granulation.

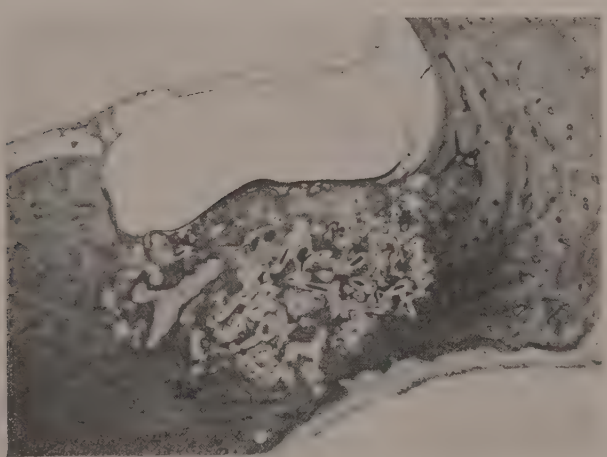

Fig. 5. Focus of otosclerosis.

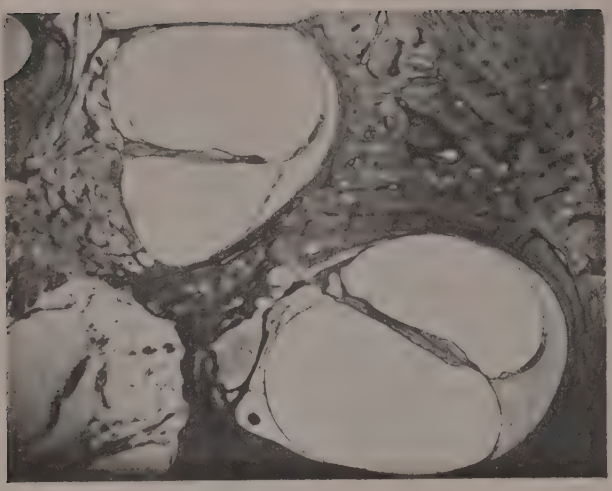

Fig. 6.
Connective tissue tract in scala tympani of the middle turn. Atrophy of the organ of CORTI.

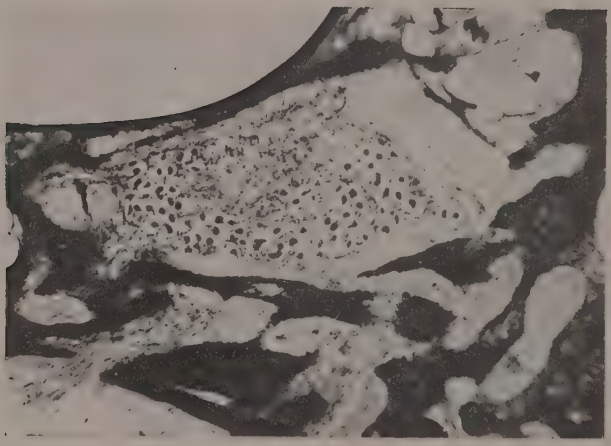

Fig. 7.
Too small number of ganglion cells in the basal cochlea-turn with secondary glio-proliferation.

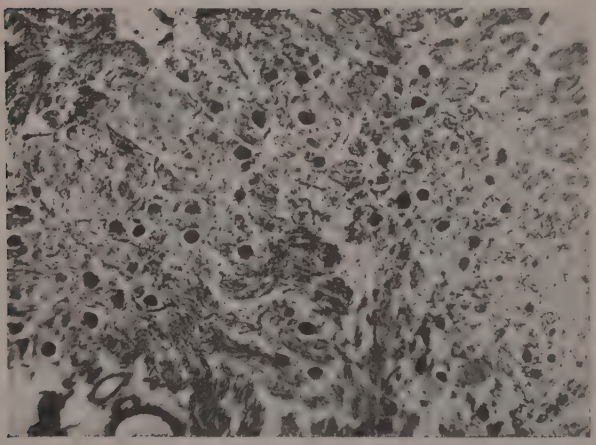

Fig. 8.
Ganglion vestibulare, the ganglion-cells have partly disappeared, secondary glia-proliferation.

little patient of Miss HURLER [6]) was deaf and this was also the case with one of the patients from the clinic of PFAUNDLER, described later by BINSWANGER and ULLRICH [7]).

The first little patient of Miss HURLER died in a psychatric clinic when it was 7 years old. Obduction was possible. Here a hydrocephalus internus and externus were found. The Nn. octavi were macroscopically normal, microscopically the picture of the brain corresponded to that which is found when there is juvenile amaurotic idiocy (examination by SPIELMEYER).

It is curious that just as was observed in our case the PACCHIONIC granulations were very deep: "Das Dach des stark erweiterten, dolichocephalen Schädels is dünn mit tiefen durchtretenden PACCHIONISCHEN Granulationen, so dass sich an Stelle des Knochens, häufig nur häutige Verschlüsse finden."

An exact examination of the ossa petrosa did not take place. Also, they are careful in giving their opinion about the cause of the deafness which the little patient showed. "Ob die Taubheit die wir auch beim Falle P. M. (BINSWANGER) wieder begegnen, auf Konto der Gehirnerkrankung (Schläfenlappenatrophie?) zu setzen, oder in einer Anomalie des Innenohres begründet war, muss offen bleiben."

ZIERL [8]) who also described in detail the first little patient of Miss HURLER, seemed to incline strongest to the opinion that there was a central genesis of the deafness. For he wrote: "Auch bei den Hördefekten des kranken K., die allmählich zur Taubheit auswuchsen, war die Feststellung ihres fraglichen zentralen Ursprung durch eine periphere Erkrankung, die doppelseitige Mittelohrenzündung, behindert. Die Sektion ergab hier eine auffallende Atrophie beider Temporallappen die man für eine zentrale Schwerhörigkeit verantwortlich machen könnte."

As far as our case is concerned, the serious auditory disturbances, which were found during life, were due to an atrophy of the organ of CORTI and not to central lesions. This is in any case contrary to the opinion of DE RUDDER [9]) who thought that the deafness frequently observed as occurring with gargoylism was of a central origin and was due to a phosphatiddiathesis. In our case neither the one nor the other was in accordance with this opinion. One might ask if the scaphocephalus found in the patient might be connected with the deafness. With this form of the skull deafness has been observed more often, though it is more known and examined with the turricephalus [10]). If this connection should be assumed, in our case the deafness could not be explained simply as a

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[6]) HURLER, GERTRUD. Ztschr. Kinderheilk., **24**, 220 (1919).

[7]) BINSWANGER, E. und ULLRICH, O. Ztschr. f. Kinderheilk., **54**, 699 (1933).

[8]) ZIERL, F. Ztschr. f. ges. Neur. und Psych., **131**, 400 (1931).

[9]) DE RUDDER, Ztschr. f. Kinderheilk., **63**, 407 (1942).

[10]) A communication about the deafness occurring with turricephalus will appear shortly, by one of us (GERLINGS).

consequence of the obliteration of the skull-sutures, as, it is true, LETTINGA mentions in his obduction-report "the suture lambdoidea, sagittalis and sutura coronaris are completely obliterated. Of a part of the suture coronaris there is only a shallow groove left", but Mrs. DE LANGE wrote as late as 1936 "sutures without anomalies". At that time, even in 1931, when our patient visited the policlinic for the first time, a serious labyrinthine deafness, however, was already present. It is therefore more probable that both anomalies, i.e. the obliteration of the skull-sutures and the dullness of hearing through atrophy of the organ of CORTI are not directly dependent on each other, but both on the same clinical picture.

Mathematics. — *A theorem about ideals in commutative rings.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 24, 1945.)

Let A and B be two ideals of a commutative ring O. The greatest common divisor (abbreviated g.c.d.) "in idealsense" of A and B is the ideal $(A; B)$ generated by the set of all elements $p + q$, if $p \subset A$ and $q \subset B$. The g.c.d. of two principal ideals (a) and (b) is to be written $(a; b)$. When the ring O is an integral domain with a unity element, of which the elements have the property of the Factor Theorem of Algebra, each pair of elements a and b of O has a g.c.d.; that means an element c, such that all common divisors of a and b are also divisors of c, and conversely all divisors of c are divisors of a and b. We write the g.c.d. in elementsense in this way: $\{a; b\} = c$.

Let O be an integral domain with unity element, whose elements satisfy the Factor Theorem of Algebra; if we suppose a and b to be two elements of O with $\{a; b\} = c$, then generally the g.c.d. in idealsense of (a) and (b) is not (c); in other words, the two notions of g.c.d. generally do not correspond with each other. (B. L. VAN DER WAERDEN, Moderne Algebra I, 2nd. ed. p. 66.)

Theorem. In a principal idealring both notions of g.c.d. correspond with each other; conversely, if O is an integral domain with unity element, for which the basistheorem of HILBERT is satisfied and for which both notions of g.c.d. correspond with each other, O is a principal idealring.

Proof. The first part is simple enough: let O be a principal idealring, then the g.c.d. of (a) and (b) is an ideal $(c) = (a; b)$, generated by an element c and so, that $c = r \,.\, a + s \,.\, b$, while $a = g \,.\, c$ and $b = h \,.\, c$; this means, that $\{a; b\} = c$ and both notions of g.c.d. correspond with each other. Conversely, if in a ring O both notions of g.c.d. correspond with each other, then for each pair of elements a and b of O: $(a; b) = (c)$. Thus for every three elements a_1, a_2 and a_3: $(a_1; a_2; a_3) = ((a_1; a_2); a_3) = (c)$, and as this may obviously be extended to a finite set $a_1, a_2, \ldots, a_r$ of O: $(a_1; a_2; \ldots; a_r) = (c)$.

On account of the supposition that O satisfies the basistheorem of HILBERT, each ideal of O has a finite basis $(a_1; \ldots; a_r)$ and if moreover O is an integral domain with unity element, then O is a principal idealring.

Observation. It is possible to demonstrate by means of an example, if each pair of elements a and b of a ring O defines an ideal $(a; b) = (c)$, generated by an element c, that this information does not include the fact that each ideal of O must be a principal ideal.

Therefore we consider a field K and a function w on K to an ordered valuationgroup Γ. Then we define $w(a)$ for every element a of K; the function w is said to define a valuation of K if $w(a \,.\, b) = w(a) + w(b)$, $w(a + b) \geqq \min(w(a); w(b))$; to each $\alpha \subset \Gamma$ corresponds at least one element $a \subset K$ with $w(a) = \alpha$. The set of elements a with $w(a) \geqq 0$ we call the valuationring O of K. To each ideal A of O we attach a valuation $w(A)$, being the lowest bound of the valuations of its elements. Furthermore we say, an ideal A of O has the symbol f or i according to whether A has elements with the value $w(A)$ or not. An ideal of the symbol f is always a principal ideal; on the other hand an ideal of the symbol i never has a finite basis. When we study a valuationring O of a valued field K with dense valuationgroup Γ, then it is evident that the ideal B of all elements b with $w(b) > 0$ is an ideal of the symbol i and B does not possess a finite basis. Of each pair of ideals in O at least one is a divisor of the other. Thus,when a and b are two elements of O with $w(a) > 0$ and $w(b) > 0$, then (a) must be divisor of (b) — or conversely.

But this means that $(a; b) = (a)$ or (b) for each pair a and b of O, while the prime-ideal B of all elements b with $w(b) > 0$ is not a principal ideal because B has no element with a lowest positive value. The valuationring O, notwithstanding the fact that O is an integral domain with unity element and satisfying for each pair of elements a and b of O the condition $(a; b) = (c)$, cannot be a principal idealring.

Mathematics. — *Ordered groups.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of December 29, 1945.)

§ 1. *Definition.* A set G of elements $a, b, \ldots$ is said to form an ordered group if the conditions expressed in postulates I, II are satisfied:

I. G is a group;

II. G is a simply ordered set, that is, for each pair of distinct elements either $a < b$ or $a > b$, while α) $a < b$, $b < c$ includes $a < c$ and β) moreover $a < b$ includes: $ac < bc$ and $ca < cb$. So the ordering of a group implies the ordering of all its undergroups.

From a topological point of view the ordered groups are to be considered as Hausdorff-spaces; their topological structure is simple on the ground of the property β): the ordered groups are homogeneous in this sense that topological properties of one element are adherent to all elements. If e.g. an element of G is a point of accumulation, then all elements of G satisfy this condition and the group is called dense in itself.

If an ordered set O is dense in itself, then generally O need not be dense-ordered. (*A* set O is be called dense-ordered if for each pair of elements a and b of O there is at least one element c "between" a and b). For ordered groups however we prove:

1. 1. If G is an ordered group dense in itself, then G is densely ordered. If there was namely between two elements a and b no further element, then every element had an immediate successor and predecessor and thus G was not dense in itself. Conclusion: An ordered group G is dense in itself if and only if G is densely ordered. If on the contrary one element is isolated, then every element is necessarily isolated; consequently:

1. 2. An ordered group G is either a discretely ordered group, which means that every element has an immediate successor and predecessor, or a densely ordered group. The property of an ordered group as being discrete resp. dense is as a topological property invariant for a mapping preserving the order.

If two ordered groups of the same ordering-structure are simply-isomorphic, we call them ordered-isomorphic.

1. 3. An ordered group is not finite; neither a first nor a last element exists. For natural m: $a^m > a^{m-1}$ for $a \neq e$ and $a > e$, in other words no element except e is of finite order. It follows from $a > b$ that $a \,.\, b^{-1} \,.\, a > a$, thus G cannot contain a last element a, neither a first element.

§ 2. If for two elements a and b $(a \geqq b > e)$ of an ordered group there exists a natural number n with $b^n > a$, then b is said to be archimedian with regard to a. For $b > e$ and $a \geqq b^{-1} > e$, then b will be called archimedian with regard to a, if b^{-1} is archimedian with regard to a; if $a < e$ as well

as $b < e$ and $a^{-1} \geqq b^{-1}\ e$, then b is said to be archimedian with regard to a, if b^{-1} is archimedian with regard to a^{-1}. If for each pair of elements a and b of an ordered group G a is archimedian with regard to b, then we say that G has an archimedian order and otherwise a non-archimedian order.

If for two elements a and b $(e < a < b)$ a is archimedian with regard to b, then a and b are said to have the same "rank". If $e < a < b$ and for all n: $a^n < b$ then a is said to be of lower rank then b, $a \prec b$. For $a < e$, $b > e$, we define $a \prec b$, if $a^{-1} \prec b$ and if $a < e$, $b < e$, we define $a \prec b$, if $a^{-1} \prec b^{-1}$.

Thus the elements of an archimedian ordered group — exept e — are of the same rank. The property of two elements as being of the same rank, is obviously reflexive, symmetrical and transitive. This enables us to divide the ordered group G into classes of elements: one class contains those and only those elements that are in respect to each other of the same rank. Thus every element of G is contained in only one class. Except for the unity-class E, which contains only the unity-element e, we can state:

2. 1. A class C is not finite, because C contains with an element a also the whole cyclic group $\{a\}$ except the element e.

If A and B are two classes, $a \subset A$, $b \subset B$ and $a \prec b$ then it is obvious that this relation exists also between $a' \subset A$ and $b' \subset B$. For this reason we define the class A of lower rank then the class B, $A \prec B$, if the elements of A are of lower rank than those of B. The relation $A \prec B$ is transitive, in other words:

2. 2. The classes of an ordered group define an ordered set, called the class-set Λ of the ordered group G. The type or order of Λ we call the class-type of G. Λ has the property, that the unity-class E consisting of e precedes all other classes. The class-set of a discretely ordered group moreover has the property that the successor of e belongs to the lowest class after E. If we speak in the following about Λ, then we mean the class-set without the unity-class E, thus:

2. 3. The class-set of a discretely ordered group has a first element. Therefore Λ itself need not be a discretely ordered set. Archimedian ordered groups are of the class-type 1.

We add to every element a of an ordered group a rank $\mathfrak{r}_a$ with the following conditions:

1. $\mathfrak{r}_a = \mathfrak{r}_b$ if and only if a and b belong to one class;
2. $\mathfrak{r}_a < \mathfrak{r}_b$ if and only if a belongs to a lower class than b.

Thus the set of all ranks is a set with the same order-structure as Λ. It is easy to see that

2. 4. $\alpha)$ $\mathfrak{r}_a = \mathfrak{r}_{a^{-1}}$; $\beta)$ from $a < c < b$ follows $\mathfrak{r}_a \leqq \mathfrak{r}_c = \mathfrak{r}_b$; $\gamma)$ $\mathfrak{r}_{ab} = \max(\mathfrak{r}_a; \mathfrak{r}_b)$ and only if $\mathfrak{r}_a < \mathfrak{r}_b$, then $\mathfrak{r}_{ab} = \mathfrak{r}_b$; $\delta)$ we prove: $\mathfrak{r}_{pq} = \mathfrak{r}_{p^{-1}q}$ if $\mathfrak{r}_p \neq \mathfrak{r}_q$; if namely $\mathfrak{r}_p < \mathfrak{r}_q$, then $\mathfrak{r}_{pq} = \mathfrak{r}_q$; as $\mathfrak{r}_p = \mathfrak{r}_{p^{-1}}$, so also $\mathfrak{r}_{p^{-1}} < \mathfrak{r}_q$, therefore $\mathfrak{r}_{p^{-1}q} = \mathfrak{r}_q$ and $\mathfrak{r}_{p^{-1}q} = \mathfrak{r}_{pq}$. For $\mathfrak{r}_p > \mathfrak{r}_q$ there is a similar reasoning. For $\mathfrak{r}_p = \mathfrak{r}_q$ the statement is generally not valid.

2. 5. The classes of densely ordered groups are at once open and closed sets. If a and b are elements of a class A, then $\mathfrak{r}_a = \mathfrak{r}_b$. If c belongs to the interval $(a\,;\,b)$, then we conclude (on the ground of 2. 4. β) $\mathfrak{r}_a = \mathfrak{r}_b = \mathfrak{r}_c$, in other words the interval $(a\,;\,b)$ together with c also contains an interval, that c contains and that is contained in $(a\,;\,b)$, on the ground of the dense ordering. Thus every class A is open and therefore A, being the complement of the sum of all the other classes, is closed.

2. 6. An ordered group is a topological group.

a. The inverse element x^{-1} of an element x is a continuous function of x while the mapping $\alpha(x) = x^{-1}$ presents an inverse ordering mapping, in other words, α is a topological mapping;

b. The product xy of two elements x and y is a continuous function of x and y. Therefore it is sufficient to prove, that if $x \to x_0$ and $y \to y_0$ that $xy \to x_0 y_0$. We have

$$(xy)\,(x_0 y_0)^{-1} = (x x_0)^{-1} \cdot [x_0\,(y y_0^{-1})\,x_0^{-1}].$$

The second term is the product of two elements approaching to e; and so the first term has the limit e and xy has the limit $x_0 y_0$.

2. 7. If an ordered group G has a class A of lowest rank then A-with the exception of e-forms a group; if moreover G is discretely ordered then this group of lowest rank is a cyclic group.

If a and b are elements of A, so $\mathfrak{r}_a = \mathfrak{r}_b$; it follows that $\mathfrak{r}_{ab} \leqq \mathfrak{r}_a$. If $\mathfrak{r}_{ab} < \mathfrak{r}_a$ then necessarily $ab = e$ and a and b are inverse elements. If a is not the inverse element of b, then $\mathfrak{r}_{ab} = \mathfrak{r}_a$ and ab also belongs to the class of lowest rank. If G is discrete then this group of lowest rank is a cyclic one: if namely a is the immediate successor of e then the lowest class A contains — with the exception of e — the cyclical group $\{a\}$.

If moreover A contained an element b not belonging to $\{a\}$, then b would, on the ground of the archimedian ordering of a class, be an element of an interval $(a^m\,;\,a^{m+1})$ with the consequence that $e < b\,.\,a^{-m} < a$, contrary to the definition of a. According to this theorem the discrete and archimedian ordered groups are cyclic groups. We shall return to this subject later on.

2. 8. A discretely ordered group G is archimedian ordered if and only if every Dedekind-cut defines a sault [1]).

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[1]) If we divide an ordered set N into two subsets N_1 and N_2 with the conditions:

I. Every element of N belongs to one, but only one of the sets N_1 and N_2;

II. Neither N_1 nor N_2 is empty;

III. All elements of N_1 precede those of N_2,

then we call this division of N a Dedekind-cut $N_1 \mid N_2$ in N. There are only the following cases that exclude each other:

1. N_1 has a last, N_2 a first element; then we call the cut $N_1 \mid N_2$ a sault in N.

2^a. N_1 has a last, N_2 no first element;

2^b. N_1 has no last, N_2 has a first element; we call the cut in 2^a and 2^b a continuous cut.

3. N_1 has no last, N_2 has no first element; the cut is called a lacuna in the set N. The ordered set N is called continuously ordered, if no cut in N defines a sault or a lacuna.

Necessary. Suppose there be a cut that did not define a sault, then it followed that there existed two elements p and q $(e \leqq p < q)$, so that the interval $(p\,;q)$ contained an infinite number of elements. Let a be the immediate successor of e then the series of elements that followed p would be: $p,\ p\,.\,a,\ p\,.\,a^2,\ \ldots,\ p\,.\,a^n,\ \ldots$ and for no finite value of n we should have $p\,.\,a^n \geqq q$, in other words the group would be non-archimedian ordered.

Sufficient. Suppose that G was non-archimedian ordered then there would exist at least one pair of elements a and b $(e < a < b)$, so that for no finite n: $a^n > b$. If we now divide G into two subsets A and B (with $A + B = G$): A contains all elements α with $\alpha \leqq a^n$ for all n; B contains the remaining elements. B is not empty while $b \subset B$. This cut, however, defines no sault, as A e.g. certainly has no last element. A discretely and archimedean ordered group has the properties: a) no first element, no last element; b) every cut defines a sault. Therefore

2. 9. A discretely and archimedean ordered group G is of the type $\omega^* + \omega$ of the set of all ordinary integers.

Suppose e to be the unity element of G, a the immediate successor of e then G is the cyclic group $\{a\}$ (see above) with the groupoperation $a^m\,.\,a^n = a^{m+n}$, $a^0 = e$, $a^m > a^n$ if $m > n$ with m and n integer. If we add to e the number o, to a^m the number m, then we have:

2. 10. Discretely and archimedian ordered groups G are free cyclic groups; they are ordered-isomorphic with the additive group of all ordinary integers and therefore commutative.

2. 11. A non-archimedean ordered group is not continuously ordered.

If the ordering was continuous then of every cut $A|B$ either A had a last, B no first element, or just the reverse. We shall try to prove, however, that in every non-archimedean ordered group there is to define a non-continuous cut. Suppose $e < a < b$ and a non-archimedean with regard to b. Let the class A contain all elements $\leqq a^n$ for a positive integer n, B the remaining elements; B is not empty, as $b \subset B$. A has no last element, nor has B a first element, for B also contains the elements $ba^{-1} > ba^{-2} > \ldots$, elements that precede b. Therefore a non-archimedean ordered group is never a continuous ordered set or directly: continuously ordered groups are necessarily archimedean ordered groups. Non-archimedean ordered groups therefore are in a topological sense unconnected. If we complete a dense-ordered group then the ordering becomes continuous. By completing such a group the original ordering is maintained. Therefore, non-archimedean ordered groups are not be completed in the usual way, but archimedean densely ordered groups can be completed, as van Dantzig proves us in this way that the transformed series of series converging to e are also e-convergent.

2. 12. A densely ordered group G has the property that to every element $a > e$ there exists an element $b > e$, so that $b^2 < a$.

We divide G into two classes N_1 and N_2 ; N_1 contains the elements b_1

with $b_1^2 \leqq a$ and N_2 the elements b_2 with $b_2^2 > a$. This division defines a Dedekind-cut, for $e \subset N_1$, $a \subset N_2$. Is the cut continuous then there exists an element $b > e$ so that $b^2 = a$ and for the elements $e < b_1 < b : b_1^2 < a$. If the cut defines a lacuna then N_1 has no last element, N_2 no first element, in other words e is not the last element with $e^2 < a$; there are other elements $b_1 > e$ with $b_1^2 < a$. Without a proof we pronounce the following theorem:

2. 13. In a continuous ordered group there exists to every element $a > e$ one element $b > e$ with $b^n = a$.

2. 14. In an archimedean densely ordered group G we can construct a series of elements converging towards e.

Suppose $a_0 > e$ to be a given element of G. According to 2. 12 there exists an element $a_1 > e$ with $a_1^2 < a_0$; further an element $a_2 > e$ with $a_2^2 < a_1$, ..., generally an element $a_n > e$ with $a_n^2 < a_{n-1}$. We prove that the series (a_i) $(i = 1, 2, 3, \ldots)$ penetrates into every interval $(e ; b)$ with $b > e$.

Suppose therefore $e < b_0 < a_0$ and $b^n > a_0$; from the construction of a_i follows: $a_1^2 < a_0$, $a_2^4 < a_0$, $a_3^8 < a_0$, ..., and thus $a_n^n < a_0$ and $a_n^n < b_n$ or $a_n < b$, with which we proved the existence of a series of elements converging towards e.

2. 15. In an archimedean densely ordered group G there exists a denumerable densely ordered subgroup R dense in G.

We choose in G a series (a_i) converging towards e, constructed as in 2. 14. We prove that the intervals formed by the powers of a_i form a denumerable system of neighbourhoods. We simply prove that in every interval $(b ; c)$ is found an interval $(a_n^p ; a_n^{p+1})$. We choose namely an element a_n out of (a_i) so that $a_n^2 < cb^{-1}$. While G is archimedean ordered, there exists an integer number p, so that $a_n^{p-1} \leqq b$ and $a_n^p > b$. But then $a_n^{p+1} = a_n^{p-1} \cdot a_n^2 \leqq a_n^2 \cdot b < c \cdot b^{-1} \cdot b = c$, in other words that the interval $(a_n^p ; a_n^{p+1})$ is contained in the interval $(b ; c)$. So we have proved: All archimedean densely ordered groups G have a denumerable dense subset. Then the continuously archimedean ordered groups G satisfy the conditions: 1) continuous ordering; 2) no first, no last element; 3) G contains a denumerable, in G dense subset R. This means:

2. 16. A continuously archimedean ordered group G is of the same order-structure as the set of all real numbers, while G contains a denumerable subset R dense in G, so that R has the orderstructure of the rational numbers. We prove finally that every continuously archimedean ordered group is ordered-isomorphic with the additive group of all real numbers and to prove this we show first:

2. 17. Every continuously archimedean ordered group G contains a denumerable subgroup R, dense in G and ordered-isomorphic with an additive group R' of rational numbers. We construct the subgroup R with

the aid of a series (a_i) converging in G towards e and so that

$$a_1^{10} = a_0,\ a_2^{10} = a_1, \ldots, a_{n+1}^{10} = a_n, \ldots$$

which is possible according to (2. 13). The group R now exists of all products of a finite number of a_i; R is denumerable and is dense in G. We add

$$a_0 \leftrightarrow 0,\ a_1 \leftrightarrow 0,\ 1, \ldots, a_p \leftrightarrow 10^{-p},\ a_p \,.\, a_q \leftrightarrow 10^{-p} + 10^{-q}.$$

So for R we get a group ordered-isomorphic with an additive group of rational numbers, namely the additive group of all finite decimal fractions.

With this result we prove finally:

2. 18. A continuously archimedean ordered group G is ordered-isomorphic with the additive group G' of all real numbers.

We use the theorem that G contains a denumerable subgroup R, dense in G and ordered isomorphic with an additive group R' of rational numbers. Further we use 2. 16, namely that G and G' have the same orderstructure; if a and b are elements of G $(a > e, b > e)$, a' and b' their mapping-elements in G' then it is sufficient to prove: $ab \leftrightarrow a'b'$. Therefore we choose an element $x > e$ of R, so that $e < x < a$ and $e < x < b$. Then we have in G': $e' < x' < a'$, $e' < x' < b'$. According to the archimedean ordering of G there exist integer numbers n_1 and n_2 with $x^{n_1} \leqq a < x^{n_1+1}$ and $x^{n_2} \leqq b < x^{n_2+1}$. R and R' are ordered isomorphic, thus $x^k \leftrightarrow x'^k$. While G and G' are of the same order-structure, $x'^{n_1} \leqq a'$ and $x'^{n_1+1} > a$. For x and x' the corresponding n_1 and n_2 are equal. From $x^{n_1} \leqq a < x^{n_1+1}$ it follows $e \leqq ax^{-n} < x$ and $e \leqq bx^{-n_2} < x$. If now $x \to e$, which is possible in R, then $ax^{-n_1} \to e$, therefore

$$a = \lim_{\substack{x \to e \\ x > e}} x^{n_1} \text{ and } b = \lim_{\substack{x \to e \\ x > e}} x^{n_2}, \text{ but also } a' = \lim_{\substack{x' \to e' \\ x' > e'}} x'^{n_1} \text{ and } b' = \lim_{\substack{x' \to e' \\ x' > e'}} x'^{n_2}.$$

According to the continuity of the group

$$ab = \lim_{\substack{x \to e \\ x > e}} x^{n_1+n_2} \text{ and } a'\,b' = \lim_{\substack{x' \to e' \\ x' > e'}} x'^{n_1+n_2}$$

and therefore $ab \leftrightarrow a'b'$.

Moreover we see $ab = ba$. Every continuously archimedean ordered group therefore is ordered-isomorphic with the additive group of all real numbers.

2. 19. Every archimedean ordered group is ordered isomorphic with an additive group of real numbers.

Mathematics. — *Some topological problems.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 24, 1945.)

In this paper I shall give a survey about some topological problems; for shortness I am not giving any proofs; these are to be published later on.

When mentioning a set or a space I always mean a separable set or space (regular space with countable base), i.e., topologically speaking, a subset of the HILBERT space R_ω or — when the space has a finite dimension — even a subset of the n-dimensional EUCLIDIAN space R_n.

1. *Topological classification of all microcompact countable spaces.* The topological classification of a certain family of sets — often called the chief problem of topology — has been achieved for but a small number of special families, for instance the well-known family of all closed surfaces.

It is, however, not very difficult to give all topological invariants of the family of all compact countable spaces A. By constructing the transfinite, but always countable sequence of the derived sets of A:

$$A = A_0 \supset A_1 \supset \ldots A_\omega \supset \ldots,$$

there is formed a last not-vacuous derived set A_λ, where λ is an ordinal number of the first or second class. A_λ consists of a finite number — say l — of points. Thus one may attach to each compact countable space A the pair of numbers (λ, l) which we call the *degree* of A; conversely with every degree (λ, l) there exists a compact countable space A (the degree of the vacuous set being $(0, 0)$). It may be proved that the degree is a topological invariant. Hence:

Theorem I. *Two compact countable spaces are different in a topological sense if, and only if, their degrees differ. By considering all possible degrees (λ, l) one has a topological classification of the compact countable spaces.*

Secondly, let us consider a microcompact, not necessarily compact, countable space B. Denote by μ the first ordinal number (of the first or second class) for which the derived set B_μ is compact. B_μ is permitted to be vacuous. Let (λ, l) be the degree of $A = B_\mu$; then we define (μ, λ, l) as the *degree* of B. Thus we may prove:

Theorem II. *Two microcompact countable spaces are different in a topological sense if, and only if, their degrees differ. By considering all possible degrees (μ, λ, l) one has a topological classification of the microcompact countable spaces.*

In particular *we have thus given a topological classification of all* (*e.g. in an* R_n) *closed countable sets.*

I have not been able, however, to give a topological classification of all countable sets, not even of all countable sets with a not-vacuous nucleus [1]).

2. *Continuous classification of all countable and all compact 0-dimensional sets.*

Let us consider the following property of a countable set A: "A is compact and the degree of A is $< (\lambda, l)$ [2]". Denote this property by $(< \lambda, l)$. We may prove:

Theorem III. *All continuous invariants of the family of all countable spaces are given by* $(< \lambda, l)$.

Thus we have shown in particular: 1^0. given an arbitrary non-compact countable space A and an arbitrary countable space B, there exists a continuous mapping of A on B; 2^0. there exists a continuous mapping of a countable compact space A on a countable compact space B if, and only if, the degree of B is the same as or less than that of A.

Considering compact 0-dimensional spaces one has

Theorem IV. *All continuous invariants of the family of all compact 0-dimensional sets are the properties "countability" and "countability and* $(< \lambda, l)$".

Thus we have shown in particular that every compact uncountable 0-dimensional set may be mapped continuously on every compact 0-dimensional set.

Meanwhile I cannot give a topological classification of all compact 0-dimensional sets.

3. *Topological characterisation of all subsets of the* R_1. An M.K.-set (MOORE–KLINE) is defined as a set which 1^0. is compact, 2^0. whose components are points or (closed) simple arcs, 3^0. where no interior point of a simple arc-component S is limit element of the complement $K—S$.

It is known (KLINE) that the M.K.-sets are identical with the *compact* subsets of the R_1. Thus all compact subsets of the R_1 are characterised topologically.

We now more generally define an M.K.-set as a set which satisfies the conditions $2'$ and $3' : 2'$. the components are points or open, half-open or closed simple arcs (i.e. the topological image of an interval $0 < x < 1$, or $0 \leqq x < 1$, or $0 \leqq x \leqq 1$); $3'$. no interior point of a component S which is not a single point is limit element of the complement $K—S$.

Using a compactification-theorem of FREUDENTHAL and myself [3]), saying

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[1]) The nucleus being the largest subset of the given space, which is dense in itself.

[2]) $(\lambda, l) < (\mu, m)$, if $\lambda < \mu$ or $\lambda = \mu$, $l < m$.

[3]) See my thesis "Topologische Studiën", Groningen (1942), hereafter denoted by [1].

that the semicompact [4]) spaces M are identical with the spaces M which may be compactified by an 0-dimensional set N (i.e. $\overline{M} = M + N$), one may prove:

Theorem V. *The (generalised) semicompact M.K.-spaces are identical — in a topological sense — with the subsets of the set of real numbers.*

4. *Topologically ordered sets.*

Sometimes a (topological) space S may be ordered (in the ordinary way), i.e. one may define a relation of order between the points of S "without destroying the topological structure" of S. An example gives an arbitrary 0-dimensional space S: S is homoeomorphic with a subset of R_1 and therefore ordered. More in general Theorem V gives the possibility of ordering the generalised semicompact M.K.-spaces.

On the other hand consider a given ordered set M. By defining the neighbourhoods of a point $m \subset M$ as the open "intervals" (ab), containing m $(a < m < b)$, this relation of order induces a topology in M. This topology of the ordered set M we call the *induced topology*.

There now arises the following question: is it possible to order a certain (topological) space S in such a way that the topology, induced (by the ordering) in S, concurs with the given topology of S? If such an ordering is possible we call S a *perfectly ordered space*. It is easy to see that every compact M.K.-space may be perfectly ordered. If a space can be ordered it is not yet always possible to determine a *perfect ordering:* e.g. the ordered space consisting of the points x with $0 < x < 1$, $x = 2$. The ordered space $0 \leqq x < 1$, $x = 2$ also is not perfectly ordered, but one may determine a perfect ordering by taking the point $x = -1$ instead of the point $x = 2$.

We may now also give an exact definition of the term used in the first sentence of this number: "without destroying the topological structure". We define therefore: a (topological) space S may be *ordered* if S is homoeomorphic with a subspace of a perfectly ordered space.

One may prove

Theorem VI. *Any 0-dimensional set may be perfectly ordered.*

On the other hand a set of dimension > 0 may not always be perfectly ordered, as was already shown above.

Considering well-ordered sets instead of ordered sets one may define *perfectly well-ordered* and *well-ordered* sets in just the same way. The theory of these sets is mainly exhausted by the following theorems.

Theorem VII. *The microcompact countable spaces are identical with those sets which may be perfectly well-ordered.*

Theorem VIII. *The countable spaces with a vacuous nucleus* [1]) *are identical with those sets which may be well-ordered.*

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[4]) A space is semicompact if every point has arbitrarily small neighbourhoods with compact boundaries (ZIPPIN).

5. *The problem of acontinuity.*

Two sets A and B we call *continuously different* if there does not exist a continuous mapping of A on B, and conversely. The family $\{A, B\}$ is called an acontinuous family. An arbitrary family of sets $\{A, B, \ldots, X, Y, \ldots\}$ is called an *acontinuous family* if any two sets X and Y of the family are continuously different. Then all sets of the family are called continuously different. A simple arc and the set consisting of two points give us a simple example of two continuously different sets.

It is easy to show that for every natural number n there exists an acontinuous family of n sets. To construct, however, infinite acontinuous families offers a far more difficult problem. Yet they exist, and by constructing well chosen sets one may prove

Theorem IX. *There exist acontinuous families of the potency* $\aleph$ [5], *the sets of which are all compact, connected and of dimension* n (n *being* $\geqq 1$).

Theorem X. *There exist acontinuous families of the potency* $2^{\aleph}$, *the sets of which are all connected and of dimension* n ($n \geqq 1$).

Because the potency of all compact sets is $\aleph$, and the potency of all sets is $2^{\aleph}$, it is impossible in this respect to give further-reaching results. Theorems IX and X in particular show the existence of $\aleph$ topologically different compact and $2^{\aleph}$ topologically different sets respectively.

Take a given arbitrary family of sets F. We form all possible acontinuous families F_i, each F_i consisting of sets belonging to F. Let m_i be the potency of F_i. If there exists an m_j with $m_j \geqq m_i$ for every index i, then m_j is called the *degree of acontinuity* of the family F. If the potency of F_i is finite for every i, but there does not exist an m_j with $m_j \geqq m_i$ for every i, then the degree of acontinuity of F is ω by definition. We may construct families with degree of acontinuity ω. Therefore the degree of acontinuity of a family may be

$$1, 2, \ldots, n, \ldots, \omega, \aleph_0, \aleph, 2^{\aleph}.$$

We denote the degree of acontinuity

of the family of all	by
countable sets	δ_1
compact 0-dimensional sets	δ_2
countable or microcompact 0-dimensional sets	δ_3
compact connected sets	δ_4
compact sets	δ_5
connected sets	δ_6
sets	δ_7
compact sets with an uncountable number of components	δ_8

[5] $\aleph_0$, $\aleph$ are the potencies of the sets of all natural and all real numbers respectively.

Theorem XI. $\delta_1 = \delta_2 = \delta_8 = 1$
$\delta_3 = 2$
$\delta_4 = \delta_5 = \aleph$
$\delta_6 = \delta_7 = 2^{\aleph}$

I cannot, however, determine the degree of acontinuity δ_0 of the family of all 0-dimensional sets. I only know $\delta_0 \geqq 2$.

Consider an arbitrary family F and all acontinuous families F_i, consisting of sets belonging to F, each of them containing the same set $A \subset F$. One may define like we did above, the *degree of acontinuity of A in respect to F: the local degree of acontinuity.*

It is not difficult to prove:

Theorem XII. *The local degree of acontinuity of a simple arc in respect to the family of all compact sets is 2.*

6. *The extension of mappings.*

The problem of the extension of continuous and topological mappings has been studied extensively in topology. We shall add some new results to this extension-theory.

Considering not only separable, but also more in general completely regular spaces we may prove:

Theorem XIII. *Any continuous mapping f of a completely regular (resp. separable) space R on a completely regular (resp. separable) space R' may be extended to a continuous mapping $\overline{f}$ of a bicompact normal (resp. (bi)compact separable) space $\overline{R}$ on a bicompact normal (resp. (bi)compact separable) space $\overline{R}'$; R and R' are dense in $\overline{R}$ and $\overline{R}'$.*

Moreover it is possible to construct for any bicompact normal (resp. (bi)compact separable) space $\overline{R}'$ (R' lying everywhere dense in $\overline{R}'$) a required $\overline{R}$ and $\overline{f}$.

The analogon of this theorem for RIESZ-spaces holds also true, but is less important.

By using the method of topological products (LEFSCHETZ), the proof of Theorem XIII is fairly simple.

This theorem does not apply for every bicompactification $\overline{R}$ of R; on the contrary: the bicompactification $\overline{R}$ of R depends on $\overline{R}'$ and f.

Therefore, if R is a subset of a given bicompact space $\overline{R}$ there arises the problem under what circumstances a continuous extension remains possible.

With regard to *metrical* compact spaces $\overline{R}$ and $\overline{R}'$ it is known that a continuous mapping f of R on R' may be extended to a continuous mapping $\overline{f}$ of $\overline{R}$ on $\overline{R}'$ if, and only if, f is a *uniformly continuous mapping*. But the

property of uniform continuity is not a topological invariant and depends on the given metric; therefore it is of no use in not-metrical spaces and of little use in respect to topology at all.

We shall now give (topologically invariant) conditions for f, R, resp. R', with regard to the inbedding in $\overline{R}$, resp. $\overline{R}'$, on which continuous extension is possible. These conditions are:

1^0. f is a *closed continuous mapping*, i.e. a continuous transformation, mapping any closed subset of R on a closed (sub)set of R';

2^0. R is *quasiconnected in respect to* $\overline{R}-R$.

This implies: taking an arbitrary point $p \subset \overline{R}-R$, there exist arbitrarily small neighbourhoods $U(p,R)$ of p in R ($U(p,R)$ apparently does not contain the point p), in which any two subsets having p for limit element are quasiconnected. *Two subsets* S_1 *and* S_2 *of a space* S *are quasiconnected* (in this space), if any open and closed subset of S containing S_1 also contains at least one point of S_2.

3^0. $\overline{R}'-R'$ is *discontinuous*, i.e. $\overline{R}'-R'$ does not contain a subset being a continuum.

One may prove

Theorem XIV. *In a (separable) space are given two compact subsets* $\overline{R} = R + A$ *and* $\overline{R}' = R' + A'$. *If* R *is quasiconnected in respect to* $A = \overline{R} - R$ *and if* $A' = \overline{R}' - R'$ *is discontinuous, any closed continuous mapping* f *of* R *on* R' *may be extended to a (closed) continuous mapping* $\overline{f}$ *of* $\overline{R}$ *on* $\overline{R}'$.

Remarks. The analogon of this theorem for bicompact normal spaces $\overline{R}$ and $\overline{R}'$ holds true as well; in this case however condition 3^0. has to undergo a small change. — Because every topological mapping is a closed continuous mapping, the theorem holds true in particular for topological mappings f. — A special case of this theorem has already been proved in [1]. — In this theorem are given sufficient conditions for continuous extension. I have examined too how far these conditions are necessary. They appear to be fairly general, but not absolutely necessary. It appears in particular that it is necessary — topologically speaking — to involve special conditions 1^0. for the continuous mapping f, 2^0. for the inbedding of R in $\overline{R}$, 3^0. for the inbedding of R' in $\overline{R}'$, if for *every* f of the (by the given conditions determined) class of special continuous mappings there may exist a continuous extension $\overline{f}$.

Let D be the discontinuum of CANTOR, in other words a perfect (i.e. compact, dense in itself) 0-dimensional set. D is sometimes called *homogeneous*, because for any two points d_1 and d_2 of D there exists a topological mapping of D on itself, which maps d_1 on d_2 (or conversely).

How far is it possible to determine a topological mapping of D on itself which maps a given subset D_1 of D on a given subset D_2 of D? Of course D_1 and D_2 must be chosen as homoeomorphic sets. The answer gives

Theorem XV. *D_1 and D_2 are homoeomorphic closed proper subsets of the discontinuum D. A topological mapping t of D_1 on D_2 may be extended to a topological mapping of D on itself, if and only if, the boundary $R[D_1]$ (of D_1 in respect to D) is mapped by t on the boundary $R[D_2]$.*

This theorem is particularly of importance if the boundaries are identical with D_1 and D_2 themselves. This is the case when D_1 and D_2 are countable or when D_1 and D_2 are nowhere dense in D. In these cases any topological mapping of D_1 on D_2 may be extended to D.

Finally we are giving a generalisation of an extension-theorem of STOÏLOW.

Theorem XVI. *$\bar{I} = I + A$ is a compact set. I is a set consisting of isolated points, the limit elements of which form the compact set A. The same conditions hold true for $\bar{K} = K + B$. Any topological mapping of A on B may now be extended to a topological mapping of $\bar{I}$ on $\bar{K}$.*

Mathematics. — *Sur un principe de variation de* GAUSS *dans la théorie du potentiel.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 24, 1945.)

§ 1. Dans un article précédent [1]) j'ai énoncé une propriété de maximum portant sur les distributions de masse positive sur la frontière Σ d'un ensemble ouvert Ω à frontière bornée entièrement extérieure. Dans ce qui suit nous donnerons une démonstration plus simple, d'où résultera aussi le rapport avec un problème déjà traité.

Théorème. *Soit θ une distribution de masse positive dans Ω avec potentiel V. Soit $\varrho(e)$ une distribution de masse positive sur Σ, dont le potentiel U satisfait à*

$$\left.\begin{array}{l} U \leqq V \text{ dans } \Omega + \Sigma \\ U = V \text{ sur le complément de } \Omega + \Sigma \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Alors l'intégrale

$$F[\lambda] \equiv \int_{\Sigma} (U - 2V)\, d\varrho\,(e_Q). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

atteint un minimum pour la distribution $\varrho = \underline{\mu}$ et un maximum pour $\varrho = \bar{\mu}$. Ici $\underline{\mu}$ signifie la distribution sur Σ, résultant du balayage de θ sur Σ, $\bar{\mu}$ celle résultant de l'extrémisation de θ. On a

$$\varrho\,(e) = \lambda_e\, \bar{\mu}\,(e) + (1 - \lambda_e)\, \underline{\mu}\,(e). \qquad (0 \leqq \lambda \leqq 1) \quad . \quad . \quad . \quad (3)$$

Démonstration. Quant á la propriété de minimum et la relation (3) nous renvoyons à l'article mentionné et la littérature là donnée. Il s'agit donc de la propriété de maximum. Pour là, nous démontrons que, étant donnée une distribution ϱ non identique à $\bar{\mu}$, il existe une variation finie $\delta\varrho$, dont la variation correspondante δF est positive. La variation $\delta\varrho$ doit etre permise, c.à.d. le potentiel de la distribution $\varrho + \delta\varrho$ doit satisfaire aux conditions (1). La variation $\delta\lambda$, correspondant avec $\delta\varrho$, suit de (3): [2])

$$\delta\lambda = \frac{\delta\varrho}{\bar{\mu} - \underline{\mu}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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[1]) Sur un principe de variation de GAUSS dans la théorie du potentiel. Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 50—61 (1941).

[2]) Il faut remarquer, que $\varrho = 0$ sur tous les ensembles où $\bar{\mu} = \underline{\mu} = 0$; sur ces ensembles on a $\delta\varrho = 0$.

On trouve, aux termes du 2e ordre en $\delta\lambda$ près, si l'on pose

$$\iint \frac{1}{r}\, d\varrho\, d\sigma = I(\varrho, \sigma),$$

$$\delta F = \{2I(\lambda\bar{\mu}, \delta\lambda\bar{\mu}) - 2I(\delta\lambda\mu, \lambda\underline{\mu}) - 2I(\lambda\bar{\mu}, \delta\lambda\underline{\mu}) + 2I(\lambda\underline{\mu}, \delta\lambda\underline{\mu})\} +$$

$$+ \{2I(\underline{\mu}, \delta\lambda\bar{\mu}) - 2I(\underline{\mu}, \delta\lambda\underline{\mu}) + 2\int_{\Sigma} V\, d(\delta\lambda\,.\,\underline{\mu}) - 2\int V\, d(\delta\lambda\,.\,\bar{\mu})\}$$

$$= \delta_1 F + \delta_2 F.$$

Il suit de cette formule, que $\delta_2 F$ est égale à la variation de F pour $\lambda = 0$, si la distribution $\underline{\mu}$ correspondant à $\lambda = 0$ est variée par $\delta\varrho$ (la question si $\delta\varrho$, variation permise pour ϱ, est une variation permise pour $\underline{\mu}$, reste ouverte). Or, F atteint son minimum pour $\lambda = 0$ et nous savons que ce minimum n'est atteint que pour la distribution $\underline{\mu}$ [3]). De plus, pour cela il n'est pas nécessaire d'imposer les conditions (1). Il s'ensuit

$$\delta_2 F > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Il ne reste donc que de déterminer une variation $\delta\varrho$ telle que $\delta_1 F \geqq 0$. Pour cela nous réduirons $\delta_1 F$. Posons

$$\overline{U}_\lambda = \int_{\Sigma} \frac{d\lambda\bar{\mu}}{r}, \qquad \underline{U}_\lambda = \int_{\Sigma} \frac{d\lambda\underline{\mu}}{r}.$$

Il suit

$$\tfrac{1}{2}\delta_1 F = \int_{\Sigma} (\overline{U}_\lambda - \underline{U}_\lambda)\, d(\delta\lambda\,.\,\bar{\mu}) - \int_{\Sigma} (\overline{U}_\lambda - \underline{U}_\lambda)\, d(\delta\lambda\,.\,\underline{\mu}). \quad . \quad . \quad (6)$$

A cause de (3) on a

$$\overline{U}_\lambda - \underline{U}_\lambda = \int_{\Sigma} \frac{d\varrho}{r} - \int \frac{d\underline{\mu}}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

De plus (voir [1]))

$$\int \frac{d\bar{\mu}}{r} \leqq \int \frac{d\varrho}{r} \leqq \int \frac{d\underline{\mu}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Aux points stables de Σ le membre à droite de (8) est égal aux membre à gauche, de sorte que dans ces points

$$\overline{U}_\lambda = \underline{U}_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Le balayage n'apporte pas de masse sur l'ensemble des points irréguliers, c.à.d. là on a $\underline{\mu} = 0$. L'extrémisation n'apporte pas de masse sur les points

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[3]) C. DE LA VALLÉE POUSSIN. Les méthodes nouvelles de la théorie du potentiel et le problème généralisé de DIRICHLET. Act. Sci. et industr. 578 (Paris 1937).

instables et là est $\bar{\mu} = 0$. Puisque chaque point irrégulier est instable, on a $\underline{\mu} = \bar{\mu} = 0$ sur l'ensemble des points irréguliers. Il s'ensuit que dans (6) les intégrales ne doivent être étendues que sur les points de Σ qui sont instable mais régulier. Pour cet ensemble E on a (sauf dans un cas qui sera exclu) $\bar{\mu} = 0$ et $\underline{\mu} > 0$. En effet si l'on avait $\underline{\mu} \equiv 0$ sur E, alors $\underline{\mu}$ était $\not\equiv 0$ seulement sur l'ensemble des points stables et il résultera du théorème d'unicité de l'extrémisation [4]) que $\bar{\mu} = \underline{\mu}$ sur tout Σ; la propiété de maximum n'existe alors pas. Ce cas excluant on a donc $\bar{\mu} = 0$ et $\underline{\mu} > 0$ sur E; en particulier E n'est pas vide (même a une capacité > 0, puisque $\underline{\mu} = 0$ sur les ensembles de capacité zéro). On a donc

$$\tfrac{1}{2}\,\delta_1 F = \int_E (\underline{U}_\lambda - \bar{U}_\lambda)\, d\,(\delta\lambda \,.\, \underline{\mu}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Puisque ϱ n'est pas identiquement égal á $\bar{\mu}$ sur Σ, sur E, ϱ n'est certainement pas égal à $\bar{\mu}$. Autrement la distribution ϱ ne porterait que de la masse sur l'ensemble des points stables et alors on aurait $\varrho = \bar{\mu}$. Il s'ensuit que λ n'est pas $= 1$ sur tous les sous-ensembles de E. On voit aussi qu'il y a des sous-ensembles où $\lambda \not\equiv 1$ et $\underline{\mu} > 0$. Autrement on avait en particulier, puisque $\underline{\mu}(E) > 0$, $\varrho(E) = \bar{\mu}(E) = 0$, où suivrait $\varrho \equiv \bar{\mu}$ et donc $\lambda \equiv 1$.

Considérons, pour $\varepsilon > 0$ donné, la variation

$$\delta\varrho = \varepsilon\,(\bar{\mu} - \varrho)$$

et

$$\delta\lambda = \varepsilon\,\frac{\bar{\mu} - \varrho}{\bar{\mu} - \underline{\mu}}.$$

Pour tous les sous-ensembles de E on a $\delta\varrho \leqq 0$ et $\delta\lambda \geqq 0$. A cause de (7) et (8) il suit que pour ce $\delta\varrho$ et $\delta\lambda$ on a

$$\delta_1 F \geqq 0.$$

$\varepsilon(\bar{\mu} - \varrho)$ est une variation permise. En effet, c'est le cas pour $\bar{\mu} - \varrho$, qui donne l'accroissement de ϱ à $\bar{\mu}$. Le potentiel de la distribution $\varepsilon(\bar{\mu} - \varrho)$ est donc zéro sur le complément de $\Omega + \Sigma$, tandis que à fortiori $U \leqq V$ reste vrai sur $\Omega + \Sigma$.

En prenant $\varepsilon = 1$, on voit que F pour $\bar{\mu}$ est plus grand que F pour ϱ. Il suit de ce qui précède que F ne peut atteindre un maximum pour aucun $\varrho \not\equiv \bar{\mu}$. En variant ainsi, ϱ tend vers $\bar{\mu}$ sur E et donc (voir ce qui précède) sur tout Σ. Le théorème est donc montré.

Nous montrons encore que, si $\lambda \not\equiv 1$, l'ensemble des points stables de Σ contient des sous-ensembles où $\bar{\mu} > 0$ et $\lambda \not\equiv 1$. Autrement on avait $\varrho = \bar{\mu}$ sur tous les sous-ensembles de l'ensemble des points stables, sauf éventuellement sur ceux où $\bar{\mu} = 0$. Alors l'ensemble des points instables ne

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[4]) M. BRELOT. Critères de régularité et de stabilité. Bull. Ac. Royale de Belgique **25**, 136 (1939).

portait pas de masse, puisque autrement le potentiel ne serait pas conservé hors $\Omega + \Sigma$. Il suivrait du théorème d'unicité déjà mentionné qu'alors $\varrho = \overline{\mu}$ sur Σ et donc $\lambda \equiv 1$.

La masse totale de la distribution-ϱ est égale à celle de la distribution θ si Ω est borné (intégration sur un surface contenant $\Omega + \Sigma$). Si Ω n'est pas borné on a une déperdition de masse [5]).

2. Dans ce qui suit sera supposé que Ω est borné. Si l'on prend pour θ la masse 1, placée au point P de Ω, on trouve sur Σ une distribution $\varrho^P(e)$, fonction de P. Déjà l.c. [1]) était considéré l'intégrale

$$\varphi(P) = \int_{\Sigma} f(Q)\, d\varrho^P(e_Q) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

où $f(Q)$ est une fonction continue sur Σ. Etant donné $f(Q)$, à l'ensemble des ϱ correspond un ensemble de fonctions $\varphi(P)$, tous définies dans Ω. Pour $\varrho^P = \underline{\mu}^P$ cette fonction est égale à la solution du problème de DIRICHLET généralisé $\underline{\varphi}(P)$ pour valeurs-frontières $f(Q)$ (WIENER); pour $\varrho^P = \overline{\mu}^P$, φ égale la fonction $\overline{\varphi}(P)$, que l'on construit à l'aide d'une approximation de $\Omega + \Sigma$ par une suite décroissante d'ensembles ouverts (KELDYCH–LAVRENTIEFF–BRELOT) [6]). Dans ces cas spéciaux $\varphi(P)$ est une fonction harmonique de P; la question si c'est le cas en général sera traité plus tard. L.c. [1]) est montré que dans Ω on a

$$\underline{\varphi}(P) \leqq \varphi(P) \leqq \overline{\varphi}(P) \text{ ou } \overline{\varphi}(P) \leqq \varphi(P) \leqq \underline{\varphi}(P) \quad . \quad (12a, 12b)$$

Si $f(Q)$ est tel, que $f(Q)$ peut être étendu à une fonction qui est sous-harmonique (super-harmonique) dans un ensemble ouvert contenant Ω, alors on a la première (seconde) inégalité; dans le cas général on ne peut pas discerner entre ces deux relations.

On peut étudier $\varphi(P)$ au voisinage de la frontière, donc les propriétés de φ si P tend vers le point-frontière Q. On obtient une théorie analogue à celle, déjà connue, de $\underline{\varphi}$ et $\overline{\varphi}$.

Un point Q de Σ sera appelé ϱ-régulier si on a pour chaque fonction continue $f(Q)$ sur Σ que $\varphi(P) \to f(Q)$ si $P \to Q$.

On a le critère suivant:

Pour que le point Q_0 soit ϱ-régulier, il faut et il suffit que pour une seule fonction $f(Q)$ avec $f(Q_0) = 0$ et $f(Q) > 0 (Q \neq Q_0)$ on a pour la fonction correspondante $\varphi(P) \to 0$ si $P \to Q_0$.

La démonstration est analogue à celle dans le cas de la fonction de WIENER et repose seulement sur le fait que φ dépend linéairement de f et que de $|f(Q)| \leqq \varepsilon$ il suit que $|\varphi| \leqq \varepsilon$.

Nécessaire et suffisant pour que Q_0 soit ϱ-régulier est que $\varrho^P(e)$ tend vers la masse 1 dans Q_0 si $P \to Q_0$.

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[5]) voir l.c. [3]) p. 34.

[6]) voir l.c. [4]) p. 125—137.

Dans un point-frontière stable on a $\overline{\varphi}(P) \to f(Q)$ si $P \to Q$, dans un point régulier $\underline{\varphi}(P) \to f(Q)$ et puisque chaque point-frontière stable est régulier, il suit de (12*a*, 12*b*).

Les points-frontière stables de Σ sont ϱ-réguliers. Les points ϱ-réguliers sont partout dense sur Σ.

Par contre on a

Un point irrégulier est ϱ-irrégulier.

En effet, soit R un point fixe dans Ω; on a

$$\int \frac{d\,\overline{\mu}^P}{RQ} \leqq \int \frac{d\,\varrho^P}{RQ} \leqq \int \frac{d\,\underline{\mu}^P}{RQ} \leqq \frac{1}{PR}.$$

Si on avait

$$\int \frac{d\,\varrho^P}{r} \to \frac{1}{RQ} \text{ si } P \to Q, \text{ il suivrait } \int \frac{d\,\underline{\mu}^P}{RQ} \to \frac{1}{RQ},$$

de sorte que la fonction de GREEN de Ω tendrait vers 0 dans Q; alors Q était régulier.

Cependant on n'a pas que chaque point régulier est ϱ-régulier. On le voit déjà dans le cas où $\varrho^P = \overline{\mu}^P$; la ϱ-régularité est alors identique avec la stabilité. Sauf quand $\overline{\mu} \equiv \underline{\mu}^P$ il y a alors des points réguliers qui sont instables. En effet si $\overline{\mu}^P \equiv \underline{\mu}^P$, alors $\overline{\varphi}(P) = \underline{\varphi}(P)$ pour tout $f(Q)$ et l'ensemble des points stables est identique à celui des points réguliers. Si inversement ces ensembles sont identique, les intégrale

$$\int_\Sigma \frac{d\,\overline{\mu}^P}{r} \text{ et } \int_\Sigma \frac{d\,\underline{\mu}^P}{r}$$

sont égales sur le complément de $\Omega + \Sigma$ et aux points réguliers (égal au potentiel de la masse 1 dans P). En vertu d'un théorème d'unicité [7]) il suit alors $\overline{\mu}^P \equiv \underline{\mu}^P$.

Remarquons que $\varrho^P(e)$ est resté indéterminé comme fonction de P (on peut prendre p. ex. $\varrho^P = \underline{\mu}^P$ dans $\Omega_1 \subset \Omega$ et $\varrho^P = \overline{\mu}^P$ dans $\Omega - \Omega_1$). Il est donc clair qu'il faut imposer des conditions à $\varrho^P(e)$ afin de pouvoir obtenir des propriétés générales. Nous exigeons:

Pour chaque $e \subset \Sigma$, $\varrho^P(e)$ est une fonction harmonique de P.

Alors nous pouvons montrer

Sauf quand $\varrho^P(e)$ est identique à $\underline{\mu}^P(e)$, il y a des points réguliers qui sont ϱ-irréguliers.

Supposons, que chaque point régulier était ϱ-régulier. Les fonctions

$$\int_\Sigma f(Q)\, d\,\underline{\mu}^P \text{ et } \int_\Sigma f(Q)\, d\,\varrho^P$$

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[7]) M. BRELOT. Fonctions sous-harmoniques et balayage. Bull. Ac. Royale de Belgique **24**, 301—312, 421—436 (1938) en part. p. 421.

étaient harmonique dans Ω, continues sur Σ et y prenaient des valeurs-frontières $f(Q)$, sauf dans les points d'un ensemble de capacité zéro. Puisque dans un ensemble ouvert il ne puissent pas exister deux fonctions harmoniques différentes qui prennent sur la frontière, sauf sur un ensemble de capacité zéro, les mêmes valeurs-frontières, il suit que $\varphi(P) = \underline{\varphi}(P)$ $(P \in \Omega)$ pour toute continue $f(Q)$.

En vertu de la définition les potentiels

$$\int \frac{d\,\underline{\mu}^P}{r} \quad \text{et} \quad \int \frac{d\,\underline{\varrho}^P}{r}$$

sont egaux hors Ω. Si nous montrons qu'ils sont égaux dans les points réguliers, il résulte du théorème d'unicité (l.c. 7) que $\underline{\mu}^P \equiv \underline{\varrho}^P$. Afin de montrer ceci, nous suivons une méthode de BRELOT (l.c. 7) p. 424). Soit Q_0 un point régulier de Σ et $\{\Omega_n\}$ une suite croissante d'ensembles ouverts réguliers contenus dans Ω avec limite Ω. Si $\underline{\mu}_n^P$ désigne la distribution sur Σ_n, obtenue par balayage sur Σ_n de la masse 1 dans P (P est fixe) et $U_n(R)$ le potentiel de cette distribution, alors si $n \to \infty$, $U_n(R)$ tend vers une fonction $W(R)$, qui égale $1/PR$ sur le complément de Ω et qui dans Ω égale la solution généralisée pour valeurs-frontières $1/PQ$; — W est presque-sousharmonique et égale à

$$\overline{\left(\left(-\frac{1}{PR}\right), -\frac{1}{PR}\right)}.$$

Soit σ une sphère de centre Q_0 et rayon ε. On a (médiation spatiale)

$$\int_\sigma U_n(R)\,dm = \int_\sigma dm \int_{\Sigma_n} \frac{d\,\underline{\mu}_n^P}{r} = \int_{\Sigma_n} d\,\underline{\mu}_n^P \int_\sigma \frac{dm}{r}.$$

Puisque $\int_\sigma \frac{dm}{r}$ est une fonction continue partout, la limite pour $n \to \infty$ existe et vaut

$$\int_\Sigma d\,\underline{\mu}^P \int_\sigma \frac{dm}{r} = \int_\Sigma d\,\underline{\varrho}^P \int \frac{dm}{r} = \int_\sigma U(R)\,dm$$

où $U(R)$ désigne le potentiel de $\underline{\varrho}^P$. En désignant par M_σ la médiation spatiale sur σ on a donc

$$\lim_{n \to \infty} M_\sigma U_n = M_\sigma U.$$

On en tire

$$M_\sigma(-U) = M_\sigma(-W)$$

et si $\varepsilon \to 0$,

$$-U(Q_0) = -\widehat{W}(Q_0) = -\frac{1}{PQ_0}$$

et ceci est, Q_0 étant régulier, la valeur de $\int \frac{d\mu}{r}$ dans Q_0, ce qu'il faut montrer [8]).

L'inverse, c.à.d. si $\varrho^P \equiv \underline{\mu}^P$ les points réguliers sont ϱ-réguliers et inversement, est évident.

La même démonstration est valable pour le théorème suivant plus général

Sauf quand $\varrho^P \equiv \underline{\mu}^P$, l'ensemble des points ϱ-irréguliers a une capacité positive.

Il suit de ce qui précède que la différence de $\varphi(P)$ et $\underline{\varphi}(P)$ au voisinage de Σ se trouve sur l'ensemble des points réguliers instables de Σ.

Cependant la distribution générale ϱ^P manque de quelques propriétés fondamentales de $\underline{\mu}^P$ et $\bar{\mu}^P$. Ainsi, ce n'est pas vrai qu'il n'y a pas de masses sur l'ensemble des points ϱ-irréguliers, comme est le cas pour $\bar{\mu}^P$ et $\underline{\mu}^P$. On le voit par l'exemple de $\varrho^P = \frac{1}{2}(\underline{\mu}^P + \bar{\mu}^P)$. Alors les points ϱ-irréguliers sont les points instables donc *a*) les points irréguliers et *b*) les points réguliers instables. Sur ce dernier ensemble $\underline{\mu}^P$ et donc ϱ^P n'est pas identiquement nulle.

De plus, le théorème d'unicité du balayage et de l'extrémisation n'est plus vrai en général. Une distribution de masse sur Σ n'est pas déterminée par la valeur du potentiel sur le complément de $\Omega + \Sigma$ et aux points ϱ-réguliers. P ex. les distributions $\bar{\mu}^P$ et $\frac{1}{2}(\bar{\mu}^P + \underline{\mu}^P)$, ont même potentiel hors $\Omega + \Sigma$ et aux points stables, donc aux points qui sont réguliers pour les deux distributions.

Remarque. On obtient des distributions ϱ^P, en appliquant le procédé de WIENER–BRELOT, que j'ai généralisé pour ensemble quelconques [9]), à un sous-ensemble arbitraire de $\Omega + \Sigma$ avec frontière Σ. La distribution ainsi obtenue ne porte pas de masse sur l'ensemble des points ϱ-irréguliers. L'application de ce procédé n'épuise donc pas l'ensemble des ϱ^P permis.

3. Il y a de la connexion entre l'étude de la distribution générale ϱ^P (e) et le problème suivant, traité antérieurement et ne résolu que dans un cas particulier [10]).

A chaque ensemble ouvert borné Ω avec valeurs-frontières $f(Q)$ on fait correspondre une fonction $\varphi(P)$ harmonique dans Ω tel que

1. *$\varphi(P)$ est égal à la solution du problème de* DIRICHLET *si celle existe;*
2. *$\varphi(P)$ est une fonctionnelle linéaire bornée de $f(Q)$;*
3. *Si $\Omega_1 + \Sigma_1 \subset \Omega$ et $\varphi(P)$ corresponds à Ω et $f(Q)$, alors $\varphi(P)$ corresponds à $\Omega' = \Omega - (\Omega_1 + \Sigma_1)$ pour valeurs-frontières f sur Σ et φ sur Σ_1.*

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[8]) Par $\hat{W}$ on désigne la fonction sous-harmonique, uniquement déterminée, qui est presque partout (c. à. d. sauf sur un ensemble de capacité nulle) égale à W.

[9]) Extension du problème de DIRICHLET pour ensembles quelconques. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 497—511 (1940).

[10]) A. F. MONNA. On the DIRICHLET problem and the method of sweeping out. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 491—498 (1939).

L'ensemble des $\varphi(P)$ n'est pas vide: $\underline{\varphi}(P)$ satisfait aux conditions. On demande à montrer que c'est la seule solution.

Il est déjà montré, que chaque fonction φ est de la forme

$$\varphi(P) = \int_{\Sigma} f(Q)\, d\, \varrho^P(e_Q),$$

et il résulte des démonstrations l.c. 10) que la distribution ϱ^P dans cette intégrale satisfait aux conditions, imposées aux distributions de § 1 [11]). Il suit alors de ce qui précède dans le cas où Ω n'a qu'une frontière entièrement extérieure:

Chaque fonction $\varphi(P)$ qu'on peut ajouter à Ω au sens indiqué satisfait dans chaque point P à une des inégalités

$$\underline{\varphi}(P) \leqq \varphi(P) \leqq \overline{\varphi}(P) \text{ ou } \underline{\varphi}(P) \geqq \varphi(P) \geqq \overline{\varphi}(P).$$

Remarquons que, à cause de la première condition — qui ne fut pas posée aux §§ 1 et 2 — on ne peut pas avoir dans Ω, $\overline{\varphi} = \varphi(P)$, c.à.d. φ n'est pas une fonction permise (sauf quand $\overline{\varphi} \equiv \underline{\varphi}$). En effet, KELDYCH et LAVRENTIEFF ont donné un exemple d'un domaine régulier pour lequel $\overline{\varphi}$ ne coïncide pas avec la solution du problème de DIRICHLET [12]).

Il suit de ces inégalités que $\varphi(P) = \underline{\varphi}(P)$ si pour tout P dans Ω on a $\overline{\varphi}(P) = \underline{\varphi}(P)$. Or, ce dernier est vrai lorsque l'ensemble des points instables de Σ a une capacité nulle. Puisque l'ensemble des points irréguliers a une capacité nulle, il suit:

Le problème posé a une solution unique si Ω n'a qu'une frontière extérieure et si l'ensemble des points réguliers instables a une capacité nulle.

Il suit des propriétés géométriques des points instables que cette condition est moins restrictive que celle donnée l.c. 10).

A ce problème s'attache encore un autre. $\underline{\varphi}(P)$ satisfait aux conditions du commencement de ce paragraphe et on a

$$\underline{\varphi}(P) = \int_{\Sigma} f(Q)\, d\, \underline{\mu}^P(Q).$$

La question se pose si ici $\underline{\mu}^P$ est déterminé uniquement. C.à.d. *de*

$$\underline{\varphi}(P) = \int_{\Sigma} f(Q)\, d\, \varrho^P(Q),$$

suit-il nécessairement $\varrho^P \equiv \underline{\mu}^P$?

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[11]) Il n'est pas indiqué l.c. [10]) que le potentiel de ϱ^P dans cette intégrale est $\leqq 1/PQ$ sur Σ; cependant, cela résulte immédiatement des autres inégalités à l'aide de la propriété que la valeur d'une fonction super-harmonique dans un point est égale à la plus limite de la fonction dans ce point sur tout quasi-voisinage.

[12]) Voir: M. BRELOT. Problème de DIRICHLET et majorantes harmoniques. Bull. des sci. math. t. **53**, 79—96, 115—128 (1939).

La réponse est affirmative. Soit R un point dans le complément de $\Omega + \Sigma$, alors on a $\underline{\varphi} = \frac{1}{PR}$ si $f(Q) = 1/RQ$, de sorte que

$$\int_{\Sigma} \frac{d\varrho^P}{RQ} = \frac{1}{RP}.$$

L'égalité de ϱ^P et $\underline{\mu}^P$ suit alors du théorème d'unicité déjà utilisé, si nous montrons que cette égalité reste vraie si R est un point-frontière régulier. On le démontre comme antérieurement (p. 59).

Apeldoorn, août 1944.

Mathematics. — *Sur le problème de la mesure.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 24, 1945.)

Soit E un ensemble infini. TARSKI a montré [1]), qu'il est possible de faire correspondre à chaque sous ensemble V de E un nombre réel non-négatif tel que

1. $m(V)$ est additif au sens restreint;
2. $m(V) = 0$ si V consiste d'un seul point;
3. $m(V)$ est non identiquement nulle.

En acceptant l'hypothèse du continu, BANACH et KURATOWSKI ont montré [2]) qu'une telle correspondance n'est plus possible si on remplace l'additivité au sens restreint par l'additivité au sens complet et si E a la puissance du continu.

Cependant il est possible de généraliser la deuxième condition.

Soit S un système de sous-ensembles de E avec propriétés

a. La somme et le produit d'un nombre fini d'ensembles de S appartiennent à S;

b. E n'appartient pas à S.

Alors il est possible de déterminer une mesure satisfaisant à

1. *$m(V)$ est additif au sens restreint;*
2. $m(E) = 1$;
3. $m(V) = 0$ *pour tout* $V \in S$;
4. $-1 \leq m(V) \leq +1$.

Soit (M) l'espace le BANACH de toutes les fonctions réelles bornées définies sur E, où la norme est définie par

$$\|f(x)\| = \text{fin sup}_{x \in E} |f(x)|.$$

Les fonctions bornées, qui sont partout nulle dans E sauf éventuellement aux points d'un ensemble appartenant à S, constituent un sous-espace linéaire A de (M). Pour $f(x) \in A$, on a

$$\|1 - f(x)\| = \text{fin sup}_{x \in E} |1 - f(x)| \geqq 1.$$

Par φ_V sera désigné la fonction caractéristique du sous-ensemble V de E; $\varphi_V \in (M)$. Soit d la plus courte distance de φ_E à A. On tire de l'inégalité précédente $d = 1$. En vertu d'un théorème connu il existe dans (M) une

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[1]) Une contribution à la théorie de la mesure. Fund. Math. t. XV, 42—50.

[2]) Sur une généralisation du problème de la mesure. Fund. Math. t. XIV, 127—131.

fonctionnelle linéaire F qui $= 0$ pour tous les éléments de A, qui $= 1$ pour φ_E et dont la norme vaut $\frac{1}{d} = 1$ [3]). Alors

$$m(V) = F(\varphi_V)$$

est la mesure cherchée. En effet, à cause de la linéarité de F, m est additif au sens restreint. Si $V \in S$, on a $\varphi_V \in A$, donc $m(V) = 0$. De plus $m(E) = F(\varphi_E) = 1$ et

$$|m(V)| \leqq \|F\| \,.\, 1 = 1.$$

Soit $\aleph_\alpha$ le nombre cardinal de E. Alors la troisième condition peut être remplacée par la suivante:

3*. $m(V) = 0$ *pour tous les sous-ensembles* V *de* E *dont le nombre cardinal est* $< \aleph_\alpha$.

Pour cela il faut montrer, que les ensembles avec nombre cardinal $< \aleph_\alpha$ constituent un système S comme ci-dessus. Soient $\aleph_{\alpha_1}$ et $\aleph_{\alpha_2}$ resp. les nombres cardinaux de V_1 et V_2, et soit $\aleph_{\alpha_1} \geqq \aleph_{\alpha_2}$, alors on a

$$\aleph_{\alpha_1} + \aleph_{\alpha_2} \leqq 2\,\aleph_{\alpha_1} = \aleph_{\alpha_1} < \aleph_\alpha,$$

de sorte que aussi $V_1 + V_2$ appartient à S; le même est vrai pour le produit. E lui même n'appartient pas à S.

Dans cette dernière forme, où 3 est remplacé par la nouvelle condition 3*, il est encore possible de remplacer 4 par:

$$0 \leqq m(V) \leqq 1,$$

et cela comme il suit.

Posons pour tout $V \subset E$

$$m_1(V) = \operatorname*{fin\,sup}_{T \subset V} m(T).$$

On a si $V_1 \,.\, V_2 = 0$

$$m_1(V_1 + V_2) = m_1(V_1) + m_1(V_2).$$

De plus $0 \leqq m_1(V) \leqq 1$ et $m_1(E) = m(E) = 1$. Si le nombre cardinal de l'ensemble V est $< \aleph_\alpha$, alors chaque sous-ensemble de V a le nombre cardinal $< \aleph_\alpha$, de sorte que $m(T) = 0$ pour tout $T \subset V$; donc $m_1(V) = 0$.

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[3]) S. BANACH. Théorie des opérations linéaires (Warschau 1932) p. 57. Ce théorème n'exige par que l'espace A est fermé, pourvu que l'élément, n'appartenant pas à A, a une distance positive de A.

Mathematics. — *Een betrekking voor de polynomen van* LAGUERRE *en van* HERMITE. By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 24, 1945.)

1. Wij bewijzen voor de polynomen $L_n^{(\alpha)}$ van LAGUERRE [1]) de volgende betrekking

$$\sum_{n=0}^{\infty} \binom{m+n+\alpha}{m} L_n^{(\alpha)}(x) \cdot t^n = \left. \begin{array}{l} \\ = \dfrac{1}{(1-t)^{m+1+\alpha}} \cdot e^{-\frac{xt}{1-t}} L_m^{(\alpha)}\left(\dfrac{xt}{1-t}\right), (|t|<1) \end{array} \right\} \quad . \quad . \quad . \quad (1)$$

Het bewijs geven wij door middel van volledige inductie. Voor $m = 0$ luidt (1):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = \frac{1}{(1-t)^{1+\alpha}} e^{-\frac{xt}{1-t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

en wij krijgen dus de bekende formule, waardoor $L_n^{(\alpha)}$ met behulp van een voortbrengende functie wordt gedefinieerd.

Veronderstellen wij dat (1) juist is voor een zekere waarde van m. Wij vermenigvuldigen beide leden met $t^{m+1+\alpha}$ en differentiëren naar t. Links ontstaat dan

$$(m+1)\, t^{m+\alpha} \sum_{0}^{\infty} \binom{m+n+\alpha+1}{m+1} L_n^{(\alpha)}(x) \cdot t^n \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In het rechterlid krijgen wij

$$\left. \begin{array}{l} \dfrac{d}{dt}\left[\left(\dfrac{t}{1-t}\right)^{m+1+\alpha} e^{-\frac{xt}{1-t}} L_m^{(\alpha)}\left(\dfrac{xt}{1-t}\right)\right] = \\ = \dfrac{t^{m+\alpha}}{(1-t)^{m+\alpha+2}} \cdot e^{-\frac{xt}{1-t}} \left[(m+1+\alpha) L_m^{(\alpha)} - \dfrac{xt}{1-t} L_m^{(\alpha)} + \dfrac{xt}{1-t} (L_m^{(\alpha)})' \right] \end{array} \right\} , \quad (4)$$

wat op grond van de betrekking

$$y \frac{d}{dy} L_m^{(\alpha)}(y) = -(\alpha + m + 1 - y) L_m^{(\alpha)}(y) + (m+1) L_{m+1}^{(\alpha)}(y) \quad (5)$$

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[1]) De notatie voor deze polynomen is niet overal dezelfde; wij nemen

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-x)^r}{r!}.$$

overgaat in

$$\frac{t^{m+\alpha}}{(1-t)^{m+2+\alpha}} \cdot e^{-\frac{xt}{1-t}} (m+1)\, L_{m+1}^{(\alpha)}\left(\frac{xt}{1-t}\right). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Uit (3) en (6) volgt, dat (1) juist is voor $m+1$.

2. Uit (1) kan men een analoge betrekking afleiden voor de polynomen van HERMITE door gebruik te maken van:

$$\left.\begin{aligned} H_{2n}(x) &= (-1)^n\, 2^{2n}\, n!\, L_n^{(-1/2)}(x^2) \\ H_{2n+1}(x) &= (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(\frac{1}{2})}(x^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Wij vinden respectievelijk:

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n-\frac{1}{2}}{m} \cdot \frac{(-1)^n\, t^n}{2^{2n}\, n!}\, H_{2n}(x) = \\ &= \frac{(-1)^m}{2^{2m}\, m!\, (1-t)^{m+\frac{1}{2}}} \cdot e^{-\frac{x^2 t}{1-t}}\, H_{2m}\left(x\sqrt{\frac{t}{1-t}}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (8^a)$$

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n+\frac{1}{2}}{m} \cdot \frac{(-1)^n\, t^{n+\frac{1}{2}}}{2^{2n+1}\, n!}\, H_{2n+1}(x) = \\ &= \frac{(-1)^m}{2^{2m+1}\, m!\, (1-t)^{m+1}} \cdot e^{-\frac{x^2 t}{1-t}}\, H_{2m+1}\left(x\sqrt{\frac{t}{1-t}}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (8^b)$$

3. Wij noemen enige bijzondere gevallen van (1) en (8). Voor $t = \frac{1}{2}$ gaat (1) over in

$$\sum_{n=0}^{\infty} \binom{m+n+\alpha}{m} \cdot \frac{L_n^{(\alpha)}(x)}{2^n} = 2^{m+1+\alpha} \cdot e^{-x} \cdot L_m^{(\alpha)}(x) \quad . \quad . \quad . \quad (9)$$

terwijl men voor (8) vindt:

$$\sum_{n=0}^{\infty} \binom{m+n-\frac{1}{2}}{m} \cdot \frac{(-1)^n}{2^{3n} \cdot n!}\, H_{2n}(x) = \frac{(-1)^m}{2^{m-\frac{1}{2}} \cdot m!} \cdot e^{-x^2}\, H_{2m}(x) \quad (10^a)$$

$$\sum_{n=0}^{\infty} \binom{m+n+\frac{1}{2}}{m} \cdot \frac{(-1)^n}{2^{3(n+\frac{1}{2})} \cdot n!}\, H_{2n+1}(x) = \frac{(-1)^m}{2^m\, m!} \cdot e^{-x^2} \cdot H_{2m+1}(x). \quad (10^b)$$

Voor $m = 0$ volgt uit (8)

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, t^n}{2^{2n}\, n!}\, H_{2n}(x) = \frac{1}{(1-t)^{1/2}} \cdot e^{-\frac{x^2 t}{1-t}} \quad . \quad . \quad . \quad . \quad (11^a)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, t^n}{2^{2n} \cdot n!}\, H_{2n+1}(x) = \frac{x}{(1-t)^{3/2}} \cdot e^{-\frac{x^2 t}{1-t}}. \quad . \quad . \quad . \quad (11^b)$$

4. Hier volgt een toepassing der verkregen formules. Onlangs hebben TRICOMI en DOETSCH [2]) de volgende betrekking afgeleid:

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)}\, t^n = e^t\, (xt)^{-\frac{1}{2}\alpha}\, J_\alpha\left(2\sqrt{xt}\right) \qquad (\alpha > -1). \quad . \quad (12)$$

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[2]) TRICOMI, Rend. Lincei **21**, 332—335 (1935).
DOETSCH, idem **22**, 300—304 (1935).

Een oudere relatie, die verband legt tussen de polynomen van LAGUERRE en de Besselse functies is de formule van LE ROY

$$n!\, e^{-x}\, x^{\frac{1}{2}\alpha}\, L_n^{(\alpha)}(x) = \int_0^\infty e^{-t}\, t^{n+\frac{1}{2}\alpha}\, J_\alpha(2\sqrt{xt})\, dt. \quad . \quad . \quad . \quad (13)$$

Wij leiden (13) uit (12) af op de volgende wijze:

$$\int_0^\infty e^{-t}\, t^{n+\frac{1}{2}\alpha}\, J_\alpha(2\sqrt{xt})\, dt = \int_0^\infty x^{\frac{1}{2}\alpha}\, e^{-2t}\, t^{n+\alpha} \left[\sum_{p=0}^\infty \frac{L_p^{(\alpha)}(x) \,.\, t^p}{\Gamma(\alpha+p+1)} \right] dt =$$

$$= x^{\frac{1}{2}\alpha} \sum_{p=0}^\infty \frac{L_p^{(\alpha)}(x)}{\Gamma(\alpha+p+1)} \int_0^\infty e^{-2t}\, t^{n+p+\alpha}\, dt =$$

$$= x^{\frac{1}{2}\alpha} \sum_{p=0}^\infty \frac{L_p^{(\alpha)}(x)}{\Gamma(\alpha+p+1)} \cdot \frac{1}{2^{n+p+\alpha+1}} \int_0^\infty e^{-\tau}\, \tau^{n+p+\alpha}\, d\tau =$$

$$= \frac{n!\, x^{\frac{1}{2}\alpha}}{2^{n+\alpha+1}} \sum_{p=0}^\infty \binom{n+p+\alpha}{n} \frac{L_p^{(\alpha)}(x)}{2^p}.$$

Op grond van (9) is dit gelijk aan

$$n!\, e^{-x}\, x^{\frac{1}{2}\alpha}\, L_n^{(\alpha)}(x).$$

5. De betrekking (1) kan worden benut om zekere bepaalde integralen te berekenen, waarvan de integrand het product van twee functies van LAGUERRE bevat.

Stelt men $\frac{t}{1-t} = a$, dus $t = \frac{a}{1+a}$, dan wordt (1)

$$e^{-ax}\, L_m^{(\alpha)}(ax) = \frac{1}{(1+a)^{m+1+\alpha}} \sum_{n=0}^\infty \binom{m+n+\alpha}{m} \cdot \frac{a^n}{(1+a)^n}\, L_n^{(\alpha)}(x) \quad (14)$$

welke betrekking opgevat kan worden als de ontwikkeling van de in het linkerlid staande functie naar polynomen van LAGUERRE. Hieruit volgt

$$\left.\begin{aligned} \int_0^\infty x^\alpha \,.\, e^{-(a+1)x}\, L_m^{(\alpha)}(ax)\, dx &= \frac{1}{(1+a)^{m+1+\alpha}} \binom{m+\alpha}{m} \Gamma(\alpha+1) = \\ &= \frac{\Gamma(m+1+\alpha)}{m!} \cdot \frac{1}{(1+a)^{m+1+\alpha}} \end{aligned}\right\} \quad (15)$$

of wel, als men $\frac{a+1}{a} = \lambda$ stelt

$$\int_0^\infty x^\alpha \,.\, e^{-\lambda x}\, L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(m+1+\alpha)}{m!} \cdot \frac{(\lambda-1)^m}{\lambda^{m+1+\alpha}}. \quad . \quad . \quad (16)$$

Algemener krijgt men door beide leden van (14) met $x^\alpha \,.\, e^{-x}\, L_p^{(\alpha)}(x)$ te vermenigvuldigen:

$$\int_0^\infty x^\alpha \,.\, e^{-(a+1)x}\, L_m^{(\alpha)}(a\,x) \,.\, L_p^{(\alpha)}(x)\, dx =$$

$$= \binom{m+p+\alpha}{m} \frac{a^p}{(1+a)^{m+p+1+\alpha}} \int_0^\infty e^{-x}\, x^\alpha \,\{L_p^{(\alpha)}(x)\}^2\, dx$$

of, daar de integraal in het rechterlid gelijk is aan $\frac{\Gamma(p+\alpha+1)}{p!}$:

$$\left.\begin{aligned} &\int_0^\infty x^\alpha \,.\, e^{-(a+1)x}\, L_m^{(\alpha)}(a\,x) \,.\, L_p^{(\alpha)}(x)\, dx = \\ &= \frac{\Gamma(m+p+\alpha+1)}{p!\,m!} \cdot \frac{a^p}{(1+a)^{m+p+1+\alpha}} \end{aligned}\right\} \quad \ldots \quad (17)$$

welk resultaat men nog schrijven kan in de meer symmetrische gedaante

$$\left.\begin{aligned} &\int_0^\infty x^\alpha\, e^{-(a+b)x}\, L_m^{(\alpha)}(a\,x) \,.\, L_p^{(\alpha)}(b\,x)\, d\,x = \\ &= \frac{\Gamma(m+p+\alpha+1)}{m!\,p!} \cdot \frac{a^p\, b^m}{(a+b)^{m+p+\alpha+1}} \end{aligned}\right\} \quad \ldots \quad (18)$$

6. Stelt men in (8^a) $\frac{t}{1-t} = a^2$, dus $t = \frac{a^2}{1+a^2}$, dan krijgt men

$$e^{-a^2x^2}\, H_{2m}(a\,x) = \frac{2^{2m}\, m!\,(-1)^m}{(1+a^2)^{m+\frac{1}{2}}} \sum_{n=0}^{\infty} \binom{m+n-\frac{1}{2}}{m} \frac{(-1)^n a^{2n}}{2^{2n}\, n!\,(1+a^2)^n} H_{2n}(x) \quad (19)$$

waaruit volgt

$$\left.\begin{aligned} \int_{-\infty}^{\infty} e^{-(a^2+1)x^2}\, H_{2m}(a\,x)\, d\,x &= \frac{(-1)^m\, 2^{2m} \,.\, m!}{(1+a^2)^{m+\frac{1}{2}}} \cdot \frac{\Gamma(m+\frac{1}{2})}{m!\,\Gamma(\frac{1}{2})} \sqrt{\pi} = \\ &= \frac{(-1)^m\,(2\,m)!}{m!\,(1+a^2)^{m+\frac{1}{2}}} \sqrt{\pi} \end{aligned}\right\} . \quad (20)$$

Stelt men nog $\frac{a^2+1}{a^2} = s$, dan kan men, daar $H_{2m}(x)$ een even functie is, (20) als volgt schrijven

$$\int_0^\infty e^{-sx^2}\, H_{2m}(x)\, d\,x = \frac{(2\,m)!}{m!} \frac{(1-s)^m}{s^{m+\frac{1}{2}}} \frac{\sqrt{\pi}}{2} . \quad \ldots \quad (21)$$

Een algemenere formule wordt verkregen, door beide leden van (19) te vermenigvuldigen met $e^{x^2} . H_{2p}(x)$. Men verkrijgt dan

$$\int_{-\infty}^{\infty} e^{-(a^2+1)x^2} H_{2m}(ax) . H_{2p}(x)\, dx =$$

$$= \frac{(-1)^m 2^{2m} m!}{(1+a^2)^{m+\frac{1}{2}}} \binom{m+p-\frac{1}{2}}{m} \cdot \frac{(-1)^p a^{2p}}{2^{2p} . p! (1+a^2)^p} \int_{-\infty}^{\infty} e^{-x^2} \{H_{2p}(x)\}^2 dx$$

of, daar de integraal in het rechterlid gelijk is aan $2^{2p} . (2p)! \sqrt{\pi}$:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} e^{-(a^2+1)x^2} H_{2m}(ax) . H_{2p}(x)\, dx = \\ &= (-1)^{m+p} . 2^{2(m+p)} \Gamma(m+p+\tfrac{1}{2}) \frac{a^{2p}}{(1+a^2)^{p+m+\frac{1}{2}}} \end{aligned}\right\} \quad . \quad . \quad (22)$$

welke betrekking men nog schrijven kan in de symmetrische gedaante

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} . H_{2m}(ax) . H_{2p}(bx)\, dx = \\ &= (-1)^{m+p} . 2^{2(m+p)} . \Gamma(m+p+\tfrac{1}{2}) \cdot \frac{a^{2p} b^{2m}}{(a^2+b^2)^{p+m+\frac{1}{2}}} \end{aligned}\right\} \quad . \quad . \quad (23)$$

Een overeenkomstig resultaat voor een integraal waarbij het product van twee polynomen van HERMITE van *oneven* index voorkomt, wordt met dezelfde methode van (8^b) verkregen. Men heeft

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} H_{2m+1}(ax) . H_{2p+1}(bx)\, dx = \\ &= (-1)^{m+p} . 2^{2(m+p+1)} \Gamma(m+p+\tfrac{3}{2}) \cdot \frac{a^{2p+1} b^{2m+1}}{(a^2+b^2)^{p+m+\frac{3}{2}}} \end{aligned}\right\} \quad . \quad . \quad (24)$$

7. De in de beide vorige paragrafen verkregen uitkomsten kunnen op hun beurt gebruikt worden om met behulp der relatie (1) integralen van nog algemeene vorm te berekenen. Wij vermenigvuldigen daartoe beide leden van (14) met $x^\alpha . e^{-(b+1)x} L_p^{(\alpha)}(bx)$. Na toepassing van (17) ontstaat dan

$$\int_0^{\infty} e^{-(a+b+1)x} . x^\alpha . L_m^{(\alpha)}(ax) . L_p^{(\alpha)}(bx)\, dx =$$

$$= \frac{1}{(1+a)^{m+1+\alpha}} \sum_{n=0}^{\infty} \binom{m+n+\alpha}{m} \cdot \frac{a^n}{(1+a)^n} \cdot \frac{\Gamma(p+n+\alpha+1)}{n!\, p!} \cdot \frac{b^n}{(1+b)^{p+n+\alpha+1}} =$$

$$= \frac{1}{m!\, p! (1+a)^{m+1+\alpha} (1+b)^{p+1+\alpha}} \sum_{n=0}^{\infty} \frac{(m+n+\alpha)!\, (p+n+\alpha)!}{n!\, (n+a)!} \frac{(ab)^n}{(1+a)^n (1+b)^n} =$$

$$= \frac{\Gamma(m+\alpha+1)\,.\,\Gamma(p+\alpha+1)}{\Gamma(\alpha+1)\,.\,m!\,p!\,(1+a)^{m+1+\alpha}(1+b)^{p+1+\alpha}} \times$$

$$F\left(m+\alpha+1, p+\alpha+1, \alpha+1, \frac{ab}{(1+a)(1+b)}\right)$$

waaruit met behulp der transformatieformule voor hypergeometrische functies:

$$F(a, b, c, y) = (1-y)^{c-a-b}\, F(c-a, c-b, c, y)$$

volgt:

$$\int_0^\infty e^{-(a+b+1)x}\, x^\alpha\,.\, L_m^{(\alpha)}(ax)\,.\, L_p^{(\alpha)}(bx)\, dx =$$

$$= \frac{\Gamma(m+\alpha+1)\,.\,\Gamma(p+\alpha+1)\,.\,(1+a)^p\,(1+b)^m}{m!\,p!\,\Gamma(\alpha+1)\,.\,(1+a+b)^{m+p+\alpha+1}} \times$$

$$F\left(-m, -p, \alpha+1, \frac{ab}{(1+a)(1+b)}\right).$$

Zonder moeite kan men deze uitkomst schrijven in de meer symmetrische gedaante:

$$\left.\begin{aligned} &\int_0^\infty e^{-\lambda x}\, x^\alpha\, L_m^{(\alpha)}(ax)\,.\, L_p^{(\alpha)}(bx)\, dx = \\ &\frac{\Gamma(m+\alpha+1)\,.\,\Gamma(p+\alpha+1)}{m!\,p!\,\Gamma(\alpha+1)} \cdot \frac{(\lambda-a)^m(\lambda-b)^p}{\lambda^{m+p+\alpha+1}} \cdot \\ &F\left(-m, -p, \alpha+1, \frac{ab}{(\lambda-a)\,(\lambda-b)}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

Voor de polynomen van HERMITE vindt men overeenkomstige resultaten, b.v. door in (25) resp. $\alpha = -\frac{1}{2}$ en $\alpha = \frac{1}{2}$ te substitueren en de betrekkingen (7) toe te passen. Na enige herleiding ontstaat

$$\left.\begin{aligned} &\int_0^\infty e^{-\lambda^2 x^2}\, H_{2m}(a\,x)\,.\, H_{2p}(b\,x)\, d\,x = \\ &= (-1)^{m+p}\,.\, 2^{2m+2p-1}\,.\, \frac{\Gamma(m+\frac{1}{2})\,.\,\Gamma(p+\frac{1}{2})}{\sqrt{\pi}} \cdot \\ &\frac{(\lambda^2-a^2)^m\,(\lambda^2-b^2)^p}{\lambda^{2m+2p+1}}\,.\, F\left(-m, -p, \tfrac{1}{2}, \frac{a^2\,b^2}{(\lambda^2-a^2)(\lambda^2-b^2)}\right) \end{aligned}\right\} \quad . \quad . \quad (26^a)$$

$$\left.\begin{aligned} &\int_0^\infty e^{-\lambda^2 x^2}\, H_{2m+1}(a\,x)\,.\, H_{2p+1}(b\,x)\, d\,x = \\ &= (-1)^{m+p}\,.\, 2^{2m+2p+1}\,.\, \frac{\Gamma(m+\frac{3}{2})\,.\,\Gamma(p+\frac{3}{2})}{\Gamma(\frac{3}{2})} \cdot \\ &\frac{ab\,(\lambda^2-a^2)^m\,(\lambda^2-b^2)^p}{\lambda^{2m+2p+3}}\, F\left(-m, -p, \tfrac{3}{2}, \frac{a^2\,b^2}{(\lambda^2-a^2)(\lambda^2-b^2)}\right) \end{aligned}\right\} \quad . \quad (26^b)$$

De hypergeometrische functies in de rechterleden van (25) en (26) zijn polynomen van het vierde argument en van een graad gelijk aan het kleinste der getallen $m + 1$ en $p + 1$.

8. De betrekkingen (25) en (26) schijnen wel de meest algemene bepaalde integralen te bevatten, die men met behulp van de betrekking (1) kan berekenen. De vorige formules (14, 16, 18, 21, 23, 24) zijn daarvan bijzondere gevallen. Formule (21) werd het eerst afgeleid door DOETSCH [3]) uit een beschouwing over de eigenschappen van de differentiaalvergelijking van de lineaire warmtegeleiding; dezelfde leidde ook nog een bijzonder geval af van (26), nl. dat waarbij $\lambda = a$. De relaties (25) en (26) zijn echter op hun beurt bijzondere gevallen van de nog meer algemene betrekkingen, die door MAYR [4]) zijn afgeleid voor de integralen

$$\int_0^\infty e^{-\lambda x}\, x^{\sigma-1}\, L_m^{(\alpha)}(a x)\, .\, L_p^{(\beta)}(b x)\, dx \quad . \quad . \quad . \quad . \quad . \quad (27^a)$$

$$\int_0^\infty e^{-\lambda^2 x^2}\, .\, x^{\sigma-1}\, H_\nu(a x)\, .\, H_\mu(b x)\, dx \quad . \quad . \quad . \quad . \quad . \quad (27^b)$$

die als oplossingen van bepaalde differentiaalvergelijkingen uitgedrukt kunnen worden door polynomen van APPELL. Onze resultaten (25) en (26) komen, afgezien van een door hem niet nader expliciet bepaalde constante factor, bij MAYR voor (l.c. pg. 601, 603). Dezelfde merkt op dat zekere integralen van de gedaante (27^a) voor de golfmechanica van betekenis zijn. Bepaalde gevallen van deze door SCHRÖDINGER [5]) beschouwde uitdrukkingen kunnen dus ook met (25) worden berekend.

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[3]) DOETSCH, Integraleigenschaften der Hermiteschen Polynome, Math. Zeitschr. **32**, 587—599 (1930).

[4]) MAYR, Integraleigenschaften der Hermiteschen und Laguerreschen Polynome, Math. Zeitschr. **39**, 597—604 (1935).

[5]) SCHRÖDINGER, Ann. der Physik **80**, 847 (1926).

Mathematics. — *Een (2,2) congruentie, die een tweevoudig oneindig stelsel van quadratische regelscharen bevat.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 24, 1945.)

1. Indertijd hebben wij met behulp van de afbeelding van de lijnenruimte op de punten van R_5 en op grond van de stelling van DARBOUX–SEGRE voor het oppervlak van VERONESE, de stralencongruenties onderzocht welke een tweevoudig oneindig stelsel van quadratische regelscharen bevatten [1]). Afgezien van het triviale geval van de lineaire congruentie (die ∞^3 regelscharen bevat) vonden wij in hoofdzaak *twee* typen:

a. de koordencongruentie (1.3) van een cubische ruimtekromme en de duale congruentie (3.1).

b. een speciale (2.2) congruentie, nl. die der rechten, welke een quadratisch oppervlak Q raken en een rechte l snijden, die Q raakt; de congruentie is zelf-duaal.

Daarbij voegen zich dan nog een (1.2) en de duale (2.1) congruentie, die echter beschouwd kunnen worden als bijzondere gevallen zowel van type *a* als van type *b* en waarbij de cubische ruimtekromme ontaard of wel het oppervlak Q singulier is.

Naderhand hebben wij kennis genomen van een publicatie van BALDUS [2]) over hetzelfde onderwerp, waarin met behulp van de methodes der synthetische meetkunde eveneens de congruenties worden bepaald met ∞^2 regelscharen. Aangezien bij dit onderzoek alleen het type *a* (met het bijzondere geval) gevonden wordt, komen wij hier op de opgave terug, wijzen de oorzaak aan van het onvolledig resultaat van BALDUS en maken nog enige opmerkingen over de congruentie *b*), die naar het schijnt minder bekend is.

2. De gedachtengang van BALDUS is in het kort de volgende. Als een congruentie ∞^2 regelscharen bevat, zal een willekeurige rechte g van de congruentie tot ∞^1 dezer regelscharen behoren en deze ∞^1 regelscharen zullen tezamen de gehele congruentie (die irreducibel verondersteld wordt) opbouwen. In een brandpunt van g wordt g door een consecutieve rechte van de congruentie gesneden; deze laatste rechte behoort dus tot een der genoemde ∞^1 regelscharen, zeg R, waar g zelf toe behoort. Een niet ontaarde regelschaar bevat echter slechts rechten, die elkaar onderling kruisen.

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[1]) BOTTEMA, Strahlenkongruenzen mit einem zweigliedrigen System von quadratischen Regelscharen, Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, 1276—1281 (1940).

[2]) BALDUS, Über Strahlensysteme, welche unendlich viele Regelflächen 2. Grades enthalten, diss. Erlangen (1910).

De regelschaar R is dus een ontaarde regelschaar. Er zijn, afgezien van verder gaande degeneratie, twee vormen van ontaarde regelscharen, nl. de quadratische kegels en (duaal) de krommen van de tweede klasse.

BALDUS onderscheidt nu *twee* mogelijkheden: de ontaarde regelscharen R_1 en R_2, die bij de beide brandpunten van g behoren, *zijn allebei kegels* dan wel *allebei krommen van de tweede klasse*. Het verdere onderzoek doet dan zien, dat de congruentie een koordencongruentie (1.3) van een cubische ruimtekromme, resp. de duale (3.1) congruentie is. De gevallen waarbij R_1 of R_2 verdere ontaarding vertonen, leiden dan tot de (1.2) of de (2.1) congruentie, die als bijzonder geval van de algemene oplossing te beschouwen is.

Het is wel duidelijk, dat hier de mogelijkheid over het hoofd gezien is, *dat van de beide regelscharen R_1 en R_2 de ene tot een quadratische kegel, de andere daarentegen tot een kromme van de tweede klasse ontaard is*. Deze omstandigheid doet zich bij de congruentie b) zoals wij dadelijk zullen zien, inderdaad voor.

3. In ons bovengenoemd artikel, waarin het vraagstuk analytisch is behandeld, hebben wij het coördinatenstelsel zo gekozen, dat het quadratisch oppervlak Q de vergelijking

$$x_1^2 - x_3^2 + 2\,x_2\,x_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

verkrijgt, terwijl de singuliere rechte l door

$$x_2 = x_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

wordt voorgesteld. De rechte l raakt Q in het punt $P \equiv (0001)$. De congruentie der rechten, die aan Q raken en l snijden, kan dan worden weergegeven door

$$\left.\begin{array}{lll} p_{12} = \nu^2 & , \quad p_{13} = \nu(\lambda - \mu) & , \quad p_{14} = \mu^2 + \lambda^2 \\ p_{34} = \mu^2 - \lambda^2 & , \quad p_{42} = \nu(\lambda + \mu) & , \quad p_{23} = 0 \end{array}\right\} \quad . \quad . \quad (3)$$

waarin p_{ij} de Plückerse lijncoördinaten en λ, μ, ν drie homogene parameters aanduiden. In de afbeelding van KLEIN correspondeert met de congruentie een op de fundamentale variëteit $\Omega \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$ gelegen oppervlak van de vierde graad V, dat ontstaan kan door een oppervlak F van VERONESE te projecteren op een R_4 uit een punt dat gelegen is in een vlak, dat een op F gelegen kegelsnede bevat, maar niet op F zelf gelegen is. Het oppervlak V bevat, evenals F, ∞^2 kegelsneden en deze kegelsneden zijn de beeldfiguren van de ∞^2 tot de congruentie behorende regelscharen. Men verkrijgt de analytische voorstelling van een dergelijke regelschaar door in (3) de substitutie

$$\nu = a\lambda + b\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

uit te voeren, waarbij de niet-homogene parameters a en b niet beide gelijk

aan nul zijn. Voor de vergelijking van het quadratische oppervlak, dat deze regelschaar draagt, vindt men

$$\left.\begin{array}{r} 2(a+b)x_1x_2-(a^2-b^2)x_1x_3+2x_2^2-2(a-b)x_2x_3-(a+b)^2x_2x_4+ \\ +(a^2+b^2)x_3^2=0 \end{array}\right\} \quad (5)$$

en uit het feit, dat men deze vergelijking schrijven kan in de vorm:

$$\{(a+b)x_1+2x_2-(a-b)x_3\}^2-(a+b)^2(x_1^2-x_3^2+2x_2x_4)=0 \quad (6)$$

volgt, dat dit quadratisch oppervlak de rechte l bevat en aan Q raakt volgens een kegelsnede, die door P gaat. De ∞^2 quadratische oppervlakken zijn omgekeerd door deze voorwaarde bepaald. Van elk der oppervlakken behoort tot de congruentie die regelschaar, welke de rechte l bevat.

4. De beide brandpunten van een rechte g der congruentie zijn haar snijpunt met l en haar raakpunt met Q. Voor de rechte (λ, μ, ν) zijn dit resp. de punten

$$G_1 \equiv \{\nu, 0, 0, \lambda+\mu\} \text{ en } G_2 \equiv \{-\nu(\lambda+\mu), \nu^2, \nu(\lambda-\mu), -2\lambda\mu\}.$$

De beide ontaarde regelscharen R_1 en R_2, die bij deze punten behoren, zijn resp. de *kegel* met G_1 tot top, welke Q omhult en de *kegelsnede* welke de doorsnede is van Q met het vlak door l en g en die dus in G_2 aan g raakt. De aldus beschreven regelscharen vormen tevens de gehele verzameling ontaarde regelscharen, die de congruentie bevat. Immers de discriminant van (5) is

$$D \equiv (a+b)^4(a-b)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Voor $a+b=0$ ontstaat $(x_2-ax_3)^2=0$, dus een kegelsnede waarvan het vlak door l gaat; voor $a-b=0$ de kegel $(ax_1+x_2)^2-a^2(x_1^2-x_3^2+2x_2x_4)=0$, die Q omhult en het op l gelegen punt $(a001)$ tot top heeft.

5. De ∞^2 quadratische oppervlakken (5), die de regelscharen der congruentie dragen, bevatten elk nog een tweede regelschaar. Onderzoekt men de verzameling der rechten van deze toegevoegde regelscharen, dan blijkt deze niet een congruentie maar een complex te zijn en wel het tangentencomplex van Q. Men kan de overeenkomstige vraag stellen voor de onder a) genoemde koordencongruentie; ook daar vormen de toegevoegde regelscharen een complex: het secantencomplex der cubische ruimtekromme.

6. De (2,2) congruentie welke bestaat uit de rechten, die een quadratisch oppervlak Q raken en een *willekeurige* rechte l snijden, is in de lijnenmeetkunde welbekend [1]). Aangezien er steeds een projectieve transformatie bestaat, die Q invariant laat en twee punten van Q toevoegt aan twee willekeurige andere punten van Q, zijn alle congruenties, waarbij l met Q twee *verschillende* punten gemeen heeft, onderling projectief. Het geval, waarbij l aan Q raakt, schijnt niet de aandacht te hebben getrokken.

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[1]) STURM, Liniengeometrie, II (1893), 328—329.

Mathematics. — *Hexagonal three-webs.* By J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of November 24, 1945.)

Summary. In "Geometrie der Gewebe" by BLASCHKE and BOL [1]) the necessary and sufficient condition is given, which has to be satisfied in order that a 3-web be hexagonal. But this condition is ony usable when the three vector fields of the tangent vectors of the curves are known. This paper deals with the case that the 3-web is given by a cubic differential form, which coefficients are functions of the coordinates, and gives the condition which has to be satisfied by these coefficients in order that the 3-web involved be hexagonal.

A 3-web in a two-dimensional space is defined as a triple infinite system of curves with the property that through every point of a definite region passes one curve of each system. Let the curves through an arbitrarily chosen point P of the region belonging to the first, second and third system be denoted by C_1, C_2 and C_3 respectively. Consider a point A of C_1. Let B be the point of intersection of the (3)-curve through A and C_2, C the point of intersection of the (1)-curve through B and C_3, D the point of intersection of the (2)-curve through C and C_1 and so on till we get the points A, B, C, D, E, F and G, where G lies just as A on C_1. If this point G coincides with A for every choice of P and A the 3-web is called a *hexagonal 3-web.*

In the following we consider the 3-web given by the null directions of a cubic differential form

$$p_{hij}\, dy^h\, dy^i\, dy^j, \qquad (h, i, \ldots = 1, 2) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is supposed to have in each point of a certain region three different real null directions. The purpose of this paper is to derive the conditions which has to be satisfied by the p_{hij} (y^1, y^2) in order that this 3-web be hexagonal.

We shall arrive at the following result:

The (symmetrical) tensor p_{hij} determines a tensor g^{hi} up to a multiplicative factor by the equation

$$p_{hij}\, g^{hi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We choose one of the solutions of this equation (different from the zero solution), which has always the rank two. Therefore there exists a covariant tensor g_{hi} defined by

$$g_{ij}\, g^{jh} = A_i^h \begin{cases} = 0, & h \neq i \\ = 1, & h = i \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[1]) Springer, Berlin 1938, par. 17.

The g^{hi} and g_{ih} are used for the raising and lowering of indices. Then a covariant derivative can be defined by the equation

$$\nabla_j g_{hi} = p_{jhi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the curvature affinor of this covariant derivative is denoted by $N_{kji}^{\ \ h}$, we may write

$$N_{ji} = N_{hji}^{\ \ \ h} = n_h p_{ji}^h + n\, g_{ji}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which equation determines the coefficients n_h and n uniquely.

A hexagonal 3-web is characterised by the relation

$$\partial_1 n_2 - \partial_2 n_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which means that n_i has to be a gradient.

In order to prove this theorem we choose the coordinate system so that the curves $y^1 =$ constant and $y^2 =$ constant form two of the three systems. Then the directions (1.0) and (0.1) satisfy the equation

$$p_{hij}\, dy^h\, dy^i\, dy^j = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

from which it follows $p_{111} = p_{222} = 0$. As $p_{112} \neq 0$, because the three null directions are supposed to be different, and the 3-web only depends on the ratio of the components p_{hij}, we may take

$$p_{111} = p_{222} = 0 \qquad p_{112} = 1 \qquad p_{221} = -\beta \neq 0. \quad . \quad . \quad . \quad (8)$$

From this we see that the vector $(\beta, 1)$ is a tangent vector of the curve of the third system. Now a 3-web is hexagonal if the third system has an equation of the form

$$\varphi(y^1) = \psi(y^2) + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where C is a parameter of the system [2]). If $(\beta, 1)$ is the tangent vector of this system we have

$$\beta = \frac{\psi'(y^2)}{\varphi'(y^1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In this case β is the product of a function of y^1 and a function of y^2 and therefore

$$\partial_1 \partial_2 \log \beta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Conversely if β satisfies the equation (11) it can be written in the form (10) from which (9) follows by integration. Thus (11) is a necessary and sufficient condition that the 3-web be hexagonal.

From (2) and (8) we obtain the ratio of the components of the tensor g^{hi}. We choose

$$g^{11} = 2, \qquad g^{12} = \frac{1}{\beta}, \qquad g^{22} = \frac{2}{\beta^2}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

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[2]) See f.i. W. BLASCHKE und G. BOL, l.c. pag. 17.

from which it follows

$$G_{11}=\frac{2}{3},\ g_{12}=-\frac{1}{3}\beta,\ g_{22}=\frac{2}{3}\beta^2. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The parameters Π^h_{ji} of the covariant derivative are on account of (4)

$$\Pi^h_{ji}=\left\{\begin{matrix}h\\ji\end{matrix}\right\}-\frac{1}{2}p^{\ h}_{ji}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $\left\{\begin{matrix}h\\ji\end{matrix}\right\}$ are the CHRISTOFFEL symbols belonging to the tensor g_{hi} and

$$p^h_{ji}=g^{hl}\,p_{jil}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

With respect to the chosen coordinate system p^h_{ji} has the following components

$$p^1_{11}=\frac{1}{\beta},\ p^1_{12}=-1,\ p^1_{22}=-2\,\beta,\ p^2_{11}=\frac{2}{\beta^2},\ p^2_{12}=-\frac{1}{\beta},\ p^2_{22}=-1, \quad (16)$$

whereas the components of p^{hi}_j are

$$p^{11}_1=\frac{3}{\beta},\ p^{12}_1=\frac{3}{\beta^2},\ p^{22}_1=0,\ p^{11}_2=0,\ p^{12}_2=-\frac{3}{\beta},\ p^{22}_2=-\frac{3}{\beta^2} \quad . \quad (17)$$

The Π^h_{ji} computed from (14) are

$$\left.\begin{aligned}\Pi^1_{11}&=-\frac{1}{3}\frac{\beta_1}{\beta}-\frac{1}{2\beta},\ \Pi^1_{12}=\frac{2}{3}\beta_1-\frac{1}{2},\ \Pi^1_{22}=-\frac{4}{3}\beta_1\,\beta+\beta,\ (\beta_i=\partial_i\,\beta)\\ \Pi^2_{11}&=-\frac{2}{3}\frac{\beta_1}{\beta^2}-\frac{1}{\beta^2},\ \Pi^2_{12}=\frac{4}{3}\frac{\beta_1}{\beta}+\frac{1}{2\beta},\ \Pi^2_{22}=\frac{\beta_2}{\beta}-\frac{2}{3}\beta_1+\frac{1}{2}\end{aligned}\right\} \quad (18)$$

Then the N_{ji} can be found from

$$N_{ji}\equiv\partial_h\,\Pi^h_{ji}-\partial_j\,\Pi^h_{hi}+\Pi^h_{hl}\,\Pi^l_{ji}-\Pi^h_{jl}\,\Pi^l_{hi} \quad . \quad . \quad . \quad (19)$$

This leads to

$$\left.\begin{aligned}N_{11}&=-\frac{4}{3}\frac{\beta_{11}}{\beta}-\frac{2}{3}\frac{\beta_{12}}{\beta^2}+\frac{2}{3}\frac{\beta_1\beta_2}{\beta^3}+\frac{\beta_2}{\beta^3}-\frac{\beta_1}{\beta^2}-\frac{3}{2}\frac{1}{\beta^2}\\ N_{12}&=\frac{2}{3}\beta_{11}+\frac{1}{3}\frac{\beta_{12}}{\beta}-\frac{1}{3}\frac{\beta_1\beta_2}{\beta^2}-\frac{1}{2}\frac{\beta_2}{\beta^2}-\frac{\beta_1}{\beta}+\frac{3}{4}\frac{1}{\beta}=N_{21}\\ N_{22}&=-\frac{4}{3}\beta_{11}\,\beta-\frac{2}{3}\beta_{12}+\frac{2}{3}\frac{\beta_1\beta_2}{\beta}-\frac{1}{2}\frac{\beta_2}{\beta}+2\,\beta_1-\frac{3}{2}.\end{aligned}\right\} \quad (20)$$

From

$$p^{ji}_l\,p^h_{ji}=I\,A^h_l \quad ; \quad I=\frac{9}{\beta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and (5) it follows

$$n_i = \frac{1}{I} N_{hj} p_i^{hj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This equation enables us to compute the components of n_i. They are

$$n_1 = -\partial_1 \log \beta \; ; \; n_2 = \frac{1}{2} \partial_2 \log \beta \quad . \quad . \quad . \quad . \quad . \quad (23)$$

from which it follows

$$\theta \equiv \partial_1 n_2 - \partial_2 n_1 = \frac{3}{2} \partial_1 \partial_2 \log \beta. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

And as we have seen is a hexagonal 3-web exactly characterised by the vanishing of this expression. But with these considerations the proof is not yet complete. It is true that the result is independent of the special choice of the coordinate system, the left hand side of the equation (24) being an affinor, but we still have to show that θ remains unaltered when p_{hij} and g_{ij} are multiplied by arbitrary factors, for only the ratio of the components of these tensors are determined by the 3-web.

If g_{ij} is multiplied by σ^2 and p_{ij}^h by λ, the Π_{ji}^h will change into

$$\Pi_{ji}^h \rightarrow \Pi_{ji}^h + T_{ji}^h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

with

$$T_{ji}^h = A_j^h \sigma_i + A_i^h \sigma_j - g_{ij} \sigma^h - \frac{1}{2} (\lambda - 1) p_{ji}^h, (\partial_i = \partial_i \log \sigma) \; . \quad (26)$$

As a consequence we have

$$N_{ij} \rightarrow N_{ij} + \nabla_h T_{ji}^h - \nabla_j T_{hi}^h + T_{hl}^h T_{ji}^l - T_{jl}^h T_{hi}^l \; . \quad . \quad (27)$$

For the calculation of this expression the following relations are useful:

$$\nabla_j g^{hi} = -p_j^{hi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\left.\begin{aligned} \nabla_{[k} p_{j]i}^h = g^{hm} \nabla_{[k} p_{j]im} - p_{[k}^{lm} p_{j]im} = g^{hm} \nabla_{[k} \nabla_{j]} g_{im} - \\ - p_{[k}^{km} p_{j]im} = -\frac{1}{2} g^{hm} (N_{kjim} + N_{kjmi}) - p_{[k}^{km} p_{j]im} \end{aligned}\right\} . \quad (29)$$

from which

$$\nabla_h p_{ji}^h = -2 N_{ji} + c g_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where the coefficient c need not be indicated more precisely. It turns out that

$$N_{ji} \rightarrow \lambda N_{ji} + p_{ji}^h (-\lambda \sigma_h - \frac{1}{2} \partial_h \lambda) + C g_{ji} \quad . \quad . \quad . \quad (31)$$

Consequently

$$n_i \to n_i - \sigma_i - \frac{1}{2}\,\partial_h \log \lambda, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which shows that the new n_i is obtained from the first one by addition of a gradient. So the expression

$$\theta = \partial_1 n_2 - \partial n_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

is invariant and our special choice of g_{hi} and p_{hij} has had no influence upon the result:

$\theta = 0$ *characterises a hexagonal 3-web.*

Botany. — *The importance of the ion activity of the medium and the root potential for the kation-absorption by the plant.* By A. C. SCHUFFELEN and R. LOOSJES. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of November 24, 1945.)

1. For the study of the absorption of kations by the plant, we have previously constructed a working hypothesis (1), starting from the physical chemical diffusion process and discussing the factors which have a direct influence on the absorption of the ions.

In the absorption we distinguish intake and outgo:

$$\text{absorption} = \text{intake} - \text{outgo}.$$

The two parts of the process may now be separated, which, in fact, is experimentally possible, through the application of artificial radio-active elements (2).

Our hypothesis led to the construction of two formulae, which are here given in a corrected form:

$$\text{Intake} = K.0.(a_i)_m.e^{\frac{-nF}{RT}E} \quad \ldots\ldots\ldots \quad \text{(I)}$$

$$\text{Outgo} = K.0.(a_i)w \quad \ldots\ldots\ldots \quad \text{(II)}$$

Here K is a constant, $0 =$ the root area, $(a_i)_m =$ the activity of the ion in the medium, $(a_i)_w =$ the activity of the ion in the periphery of the root, $n =$ the valency of the ion, $E =$ the electric potential difference between root and medium, $R =$ the gas constant, $T =$ the absolute temperature and $F =$ the Faraday (96500 Coulombs).

Qualitatively it has been possible to confirm these relations (2, 3).

2. On further investigation, however, it was found that the formula given for the intake of the kations is too simple. In deducting this formula, we did not reckon with the fact that there is absorption on the surface of root and medium, as on any other surface. Owing to this accumulation of ions on the root wall another correction is necessary, apart from the one given in (I) for the electric potential difference. Only then is it rightly possible quantitatively to apply the law of the diffusion on the absorption of ions by the plant.

From the investigations of CERNESCU (4) of the velocity of the establishment of the absorption equilibrium with porous adsorbentia, and from those of HOAGLAND and BROYER (7) on the influence of the temperature and the oxygen pressure of the medium on the absorption of potassium ions by the plant root we may conclude that the adsorption equilibrium is

established many times more rapidly than the intake of the ions from the surface in the plant. From this it follows, that for the calculation of the concentration in the surface of the root we may apply an adsorption formula.

By transposing the formula of LANGMUIR into a suitable form, the two corrections may be made simultaneously. For the adsorption of gases the original form is:

$$\Gamma = c \cdot \frac{b \cdot a}{1 + b \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

in which Γ = the quantity of gas adsorbed, b = a constant, a = the concentration of gas in the gaseous phase and c = also a constant (the maximal quantity that can be adsorbed).

This formula, which applies to gases, so to uncharged particles, may according to HERMANS (5), be developed for ions, so for charged particles, as follows.

HÜCKEL (6) shows that b is proportionate to an e power of the energy which is liberated on adsorption. If we call this energy Q, then calculating in mols we may write:

$$b = k \cdot e^{\frac{Q}{RT}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}$$

Formula IV applies to gases as well as to ions. For gases Q is a constant with a certain gas and a certain surface. This is not the case with ions, as here the energy which is liberated on adsorption also depends on the electric potential difference between the surface and its medium. As a rule this potential difference is not constant, but varies and depends among other things on the composition of the medium.

But Q may be divided into a part Q', which is independent of the potential difference mentioned, and another part, which does depend on it.

For the adsorption of kations we may then write:

$$Q = Q' - nF \cdot E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V)}$$

If we introduce V into IV we get:

$$b = k \cdot e^{\frac{Q'}{RT}} \cdot e^{\frac{-nF}{RT}E}$$

As Q' is constant we may also write a constant for the e-power of Q' (T is assumed constant). If we call it b_0 it follows that

$$b = b_0 \cdot e^{\frac{-nF}{RT}E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VI)}$$

Introducing VI into III we get a formula suitable to our purpose:

$$\Gamma = c \frac{b_0 \cdot a \cdot e^{\frac{-nF}{RT}}}{1 + b_0 \cdot a \cdot e^{\frac{-nF}{RT}E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VII)}$$

If in VII we replace a by $(a_i)_m$, and introduce the form found for Γ into the one-sided diffusion formula (intake part), we get the formula for the intake of the kations, corrected for the electric potential difference and the adsorption (VIII). It is here assumed that the intake is proportionate to the quantity adsorbed on the surface of root and medium.

$$\text{Intake} = K \,.\, 0 \,.\, c \,.\, \frac{b_0 \,.\, (a_i)_m \,.\, e^{\frac{-nF}{RT}E}}{1 + b_0 \,.\, (a_i)_m \,.\, e^{\frac{-nF}{RT}E}} \quad . \quad . \quad \text{(VIII)}$$

If the second term of the nominator become small with regard to 1, then VIII will be identical with I, and our original formula applies. This will only rarely be the case, as on decreasing $(a_i)_m$ the potential difference between the root and its surroundings will always increase (E becomes more and more negative). The product of $(a_i)_m$ and the e power of E will therefore remain great.

For the experimental testing of the formula it is desirable to modify it a little more. If the results of the experiment are reduced to equal root area, 0 becomes constant. K and c are constants, so that we may introduce a new constant for $K \,.\, c \,.\, 0$.

As a rectilinear relation is more easily checked than a hyperbolical one, we also write VIII in the reciprocal form. Calling the intake i we get:

$$\frac{1}{i} = C \,.\, \frac{1}{(a_i)_m \,.\, e^{\frac{-nF}{RT}E}} + C' \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX)}$$

So if we make a diagram from $\frac{1}{i}$ against $\frac{1}{(a_i)_m \,.\, e^{\frac{-nF}{RT}E}}$ it must be possible to draw a straight line through the measuring points.

3. We have checked the above theory by taking measurements concerning the absorption of potassium ions by oat roots. The oats (Mansholt III) for the experiments were grown according to HOAGLAND and BROYER

TABLE 1.

Data for the absorption experiments.

Series	Date of sowing	Date of experiment	Solution	Period of absorption	Temperature	Average root weight per experiment mg dry matter	Composition of root at commencement mg aeq. K/100 g dry matter
II	1 June 1943	30 June 1943	KNO_3—600 cm³	8 h	18–20 °C	325	15.4
III	1 „ 1943	2 July 1943	K_2SO_4—600 cm³	8 h	19–21 °C	303	22.3
IV	6 July 1943	27 „ 1943	KCl —600 cm³	9 h	23–26 °C	210	22.5
VII	27 „ 1943	17 Aug. 1943	KCl —700 cm³	7¾ h	19–22 °C	143	51.7
VIII	27 „ 1943	18 „ 1943	KOl —700 cm³	8 h	21–22 °C	158	38.2

(7), as described before (3). The absorption experiments were made in the same way as described before for oat plants (3), but now only the cut roots were used (also according to HOAGLAND and BROYER).

Besides the potassium absorption we also measured the root potentials

TABLE 2[1]).

Experimental data of the absorption.

Series	Nr	Medium	$(a_i)_m \times 10^5$	pH	Root potential E mV	Absorption: mg aeq K per 100 g dry matter	$\frac{100}{i}$	$\frac{1}{(a_i)_m e^{\frac{-nF}{RT}E}}$
II	3	KNO_3-sol.	58	4.25	—23	26.1	3.83	692
	4	„	77	4.24	—21	25.6	3.91	562
	5	„	96	4.24	—20	25.1	3.98	467
	6	„	190	4.27	—15	30.0	3.33	280
	7	„	372	4.15	—11	34.8	2.87	173
	8	„	522	4.25	— 9	30.9	3.23	127
	9	„	728	4.15	— 8	37.2	2.69	100
	10	„	900	4.14	— 7	36.9	2.71	84
III	4	K_2SO_4-sol.	15	4.73	—91	11.9	8.40	170
	5	„	31	4.67	—82	14.2	7.04	122
	6	„	61	4.52	—72	17.7	5.65	92
	7	„	122	4.44	—60	19·1	5.23	75
	8	„	239	4.33	—46	19.7	5.07	67
	9	„	469	4.33	—37	23.5	4.25	49
	10	„	906	4.25	—31	25.9	3.86	32
IV	3	KCl-sol.	39	4.72	—44	41.3	2.42	446
	4	„	77	4.58	—35	38.6	2.59	321
	5	„	152	4.46	—25	46.9	2.13	243
	6	„	299	4.41	—16	43.9	2.38	176
	8	„	1138	4.51	— 6	63.3	1.58	69
	9	„	2210	4.53	— 3	76.3	1.31	40
	10	„	4150	4.51	0	85.8	1.17	24
VII	3	KCl-sol.	31	5.36	—56	22.3	4.48	280
	5	„	150	5.46	—37	30.6	3.27	152
	7	„	588	5.24	—20	26.9	3.72	76
	9	„	2200	5.35	—13	26.3	2.75	27
VIII	3	KCl-sol.	38	5.05	—38	31.7	3.16	578
	5	„	150	4.98	—17	36.0	2.78	338
	7	„	588	4.73	— 5	39.3	2.54	140
	9	„	2200	5.02	— 4	43.4	2.31	42

[1]) The numbering of the experiments has been retained. As the absorption from mixtures was also examined, only certain numbers could be used for our purpose in this investigation. The first numbers of a series had to be omitted, as so much K was absorbed from these solutions that $(a_i)_m$ did not remain sufficiently constant during the experiment. $(a_i)_m$ was calculated from the concentration and activity coefficients, e is the average of 10 determinations, i is the average of 2 determinations.

in the various solutions, according to LUNDEGÅRDH (8). For this purpose we used the roots of plants from the same medium as those used for the absorption experiments. The war made it impossible for us to experiment with artificial radio-active potassium. Therefore it was impossible to

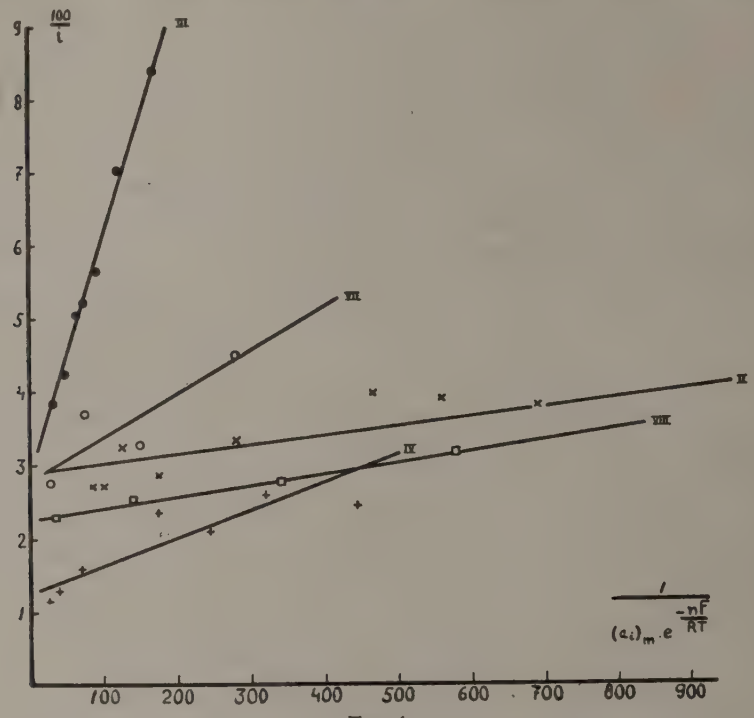

Fig. 1.

determine the actual intake. But owing to our selecting for the experiment roots with a very low activity of the potassium ions in the periphery, the outgo was very slight. Therefore we assume that we may neglect it, and that we may take the intake to be equal to the absorption.

The data and the results of the experiments are collected in Tables 1 and 2.

In Figure 1 the relation of equation IX has been set out. It is seen to agree with the relation quite reasonably. We may conclude that the theory can describe the absorption of a single ion.

4. The application of the adsorption equation to the absorption of substances by living matter is not new. As far back as 1910 GROS (9) applied the absorption formula in accounting for the haemolysis of blood corpuscles through ammonia. In 1934 VAN DEN HONERT (10) obtained good results with the formula of FREUNDLICH in describing the absorption of phosphate ions by sugar cane. In amplification of these investigations we think we have now found a base which supports the application of the adsorption formula for the absorption problem of plants.

We have, however, used an adsorption formula entirely different from the one used by these investigators. Although our formula has a theoretical base, it is yet worth while to see, if our material can also be described with other adsorption formulae. For this purpose we chose LANGMUIR's formula for gases (III), which is occasionally also applied to ions (11), the corrected formula of LANGMUIR (VII) and the empiric formula of FREUNDLICH ($\lg i = \lg (a_i)_m + C$).

In principle it appears that all three can be used, so that we have asked ourselves which gives the best description. To answer this question we first adjusted the lines which can be drawn through our measuring points, according to the method of the smallest squares. Next we determined the deviations of the points measured from the line (as i) and calculated the differences in percents as standard error.

As the value of $(a_i)_m$ is many times more accurate than those of i and E, we only adjusted for these two data. This means that LANGMUIR's formula for gases and that of FREUNDLICH were only adjusted for i, and LANGMUIR's formula for ions for i and E. The results of these calculations are found in Table 3.

TABLE 3.

The standard error $\left(\sigma = \sqrt{\frac{\Sigma \triangle}{n-1}}\right)$ in procents.

Series	Observation numbers	Langmuir for ions	Langmuir for gases	Freundlich
II	8	8.8	8.3	6.6
III	7	2.6	8.5	6.5
IV	7	12.9	24.7	10.0
VII	4	11.3	16.7	17.3
VIII	4	1.7	5.0	1.0
Aver.	30	8.8	14.8	9.2

The standard error of LANGMUIR's formula for gases is evidently the greatest. Only case II is an exception. Here, however, the difference in E is not great (see Table 2), so that the correction to be made is small. FREUNDLICH's formula and the corrected formula of LANGMUIR give about the same result. If, in judging the standard error we also reckon with the fact that the corrected formula of LANGMUIR had to be adjusted not only for i, but also for E, it is evident that the former formula is best. From other experiments we know that the standard deviation for the determination of i is 8.5 %, which is entirely in keeping with the figures found here. Hence we think that we have found in our formula the right base for the description of ion intake. It would also seem to us that the formula given will be very suitable for the adsorption of electrolytes to other adsorbentia. More especially we expect that it will be applicable on a larger sphere than LANGMUIR's formula for gases, because the influence of the

potential difference between dispersion medium and disperged phase have been taken into account.

Summary.

Our working hypothesis, in which the diffusion law was only corrected for the electric potential difference between root and medium, had to be amplified by a correction for the adsorption of the ions to the root wall. The two corrections can be made simultaneously with the formula of LANGMUIR for ions, as deducted by HERMANS. The experimental results were in agreement with the theory within the limits of error of the determinations.

We wish to express our thanks to Dr. J. J. HERMANS for his help in the development of our working hypothesis.

Résumé.

Notre hypothese de l'absorption des ions pour les racines des plantes corrigeant la loi de diffusion seulement pour la potentièle électrique entre le racine et le milieu (formule I), doit être completé avec une correction de l'adsorption des ions sur le surface des racines. Avec la formule de LANGMUIR pour les ions réalisé par HERMANS nous pouvons appliquer les deux corrections à même temps (formule VII et VIII). Les resultates experimentelles sont en rapport avec la théorie entre les limites d'erreurs (fig. 1, table 3).

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Palaeontology. — *Corals from the Upper Kalibeng beds (Upper Pliocene) of Java.* By J. H. F. UMBGROVE. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of December 29, 1945.)

The name Upper Kalibeng beds was introduced by members of the Geological Survey of Java to indicate a series of strata of Upper Pliocene age in the Kendeng region between Trinil and Surabaya. They were mapped as such by J. DUYFJES on the sheets 109 (Lamongan), 110 (Modjokerto), 115 (Surabaya) and 116 (Sidoardjo) of the Geological map of Java 1 : 100000 published in the year 1938. The same author published a sumary of the stratigraphy of the whole region in 1936 [1]).

In the western part of the region the locality Sonde, which is well known for the Pliocene Molluscs described by K. MARTIN, belongs to these Upper Kalibeng beds. Corals from the neighbourhood of Sonde and Trinil were described by J. FELIX. At present Dr. C. O. VAN REGTEREN ALTENA is studying the Mollusca of the Kendeng region. In the publication of the first part of his results a complete list is given of localities from which the collection of fossils were made by the Geological Survey [2]).

I am publishing here the results of my examination of the corals collected from the Upper Kalibeng beds by members of the Geological Survey. Consequently the numbers of the localities mentioned in the list below correspond to those of VAN REGTEREN ALTENA's table I. They are the following localities. (Sheets 93 and 99 have not yet been published! They are situated to the West of the sheets 109, 110 mentioned above.)

Locality M. 6, sheet 99B, W. of Sempol, Limestone
„ M. 250, „ 93B, Soloriver near Gadjah, Limestone
„ M. 251, „ „ „ „ „ N.W. of Padasmalang, sandy marl
„ M. 252, „ „ „ „ „ „ „ „ „ „ „ „ „
„ M. 253, „ „ „ „ „ „ „ „ „ „ „ Limestone
„ M. 254, „ „ „ „ „ „ „ „ „ „ „ „
„ M. 257, „ „ „ „ „ S. of the mouth of R. Alastoewa near Sonde, sandy marl
„ M. 258, „ „ „ „ „ idem
„ M. 260, „ „ „ „ „ near Bangoenredjo Kidoel

The list gives a review of 35 corals from the Upper Kalibeng beds. The species of 29 could be identified. Twenty of these belong to still living species, i.e. 69 percent. This percentage figure agrees with the figure 68,8 resulting from my revised list of the corals described by J. FELIX from the region of Trinil and Sonde.

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1) J. DUYFJES. Zur Geologie und Stratigraphie des Kendenggebietes zwischen Trinil und Soerabaja (Java). De Ingenieur in Nederl. Indië **3** (1936).

2) C. O. VAN REGTEREN ALTENA. The marine Mollusca of the Kendeng beds (East Java). Gastropoda, part I. Leidsche Geologische Mededeelingen. Vol. X (1938).

In the accompanying list of corals nos. 5, 16, 33 and 35 belong to well known fossil species; no. 1, 2, 3 and no. 30 will be amply described in my papers on the corals from the Lower Pliocene reef of Prupuk and the Lower Pleistocene Putjangan beds (Both the manuscrips are ready for the press). Nos. 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 23 and 24 were identified without doubt by comparison with large suites of recent corals. Nos. 20 and 21, though probably still living species, could not be identified as the specimens are too fragmentary. It is not necessary in my opinion to give full descriptions of the twenty-four coral species enumerated above. Therefore we may restrict ourselves in devoting only paleontological descriptions or taxonomical comments on the remaining 11 species.

Number of species	Coral species from the Upper Kalibeng beds	Recent	Numbers of Localities
1	Acanthocyatus spinosa nov. spec.		251
2	„ grayi E. H.	+	251, 253, 257
3	Heterocyathus aequicostatus E. H.	+	252, 257, 260
4	Seriatopora micrommata Felix, var.		252, 258
5	Pocillopora Jenkinsi Reuss		252, 260
6	Stylophora solida nov. spec.		253
7	Antillophyllia constricta (Brügg.)	+	252, 260
8	Cyphastraea seraila (Forsk.)	+	260
9	Leptastraee purpurea (Dana)	+	253
10	„ transversa (Klunz.)	+	260
11	Galaxea clavus (Dana)	+	260
12	Favia speciosa (Dana)	+	253
13	„ favus (Forsk.)		
14	Goniastraea retiformis (Lam.)	+	253, 254
15	„ hombroni (Reuss)	+	253
16	„ simplicitexta Umbgrove		260
17	Coeloria lamellina (Ehrenb.)	+	253, 260
18	„ daedalea (Forsk.)	+	253, 260
19	Platygyra phrygia (Ell. et Sol.)	+	253
20	Merulina spec. (cf. ampliata Ell. et Sol.)		252
21	Lobophyllia spec.		260
22	Symphyllia recta (Dana)	+	253, 260
23	Diploastraea heliopora (Lam)	+	250, 253
24	Fungia cyclolites Lam.	+	257, 260
25	„ sibogae v. d. Horst	+	260
26	„ somervillei Gard.	+	260
27	„ costulata Ortmann	+	260
28	„ inaequicostata Gerth		6
29	„ echinata Pallas	+	260
30	„ cf. repanda Dana		6
31	„ cf. concinna Verrill		6
32	Pachyseris curvata Martin		6
33	Cyathoseris lophiophora Felix		252, 260
34	Pavona microstoma Umbgrove		252
35	Dictyaraea anomala Reuss.		260

Seriatopora micrommata Felix var. javana nov. var.
Pl. I, fig. 7 and 8.

The branchlets resemble Seriatopora micrommata which is known from the Upper Miocene (Tertiary *f* and *g*) of Borneo and Java. They differ, however, by (1) a greater diameter of the branches, up to 5 mm, (2) a greater number of longitudinally arranged calices, (3) the absence of a columellar papilla on the directive septa, (4) the calices being less prominent.

Perhaps some Seriatopora fragments from Gunung Linggapadang, Prupuk may also belong to the present variety.

Stylophora solida nova spec.
Pl. II, fig. 9 and 10.

The single specimen is a fragment of a large massive growth; it is 125 mm high and has a diameter of 100 mm. The calices are circular and have a diameter of 1 mm or slightly more. Corallites crowded, up to 1 mm distant, usually less. Six thin equal septa reach the columella, secondaries and tertiaries extending only as low ridges along the calicular walls. Coenenchyma formed by a loose dissepimental structure. Upper surface of the colony not preserved.

Leptastrea transversa (Klunzinger).
Pl. II, fig. 13.

1913 Leptastrea transversa Klunzinger, Felix, Java, p. 348
1918 „ „ „ , Vaughan, p. 94, plate 31, fig. 1, 1*a* (with synonyms)

A figure is here given of part of one of the specimens, showing the septal arrangement which is characteristic of this species.

Distribution: Recent: Indo-Pacific; Pliocene of Duku Pengkol, Java.

Symphyllia recta (Dana).

1913 Symphyllia cf. sinuosa Quoy et Gaim., Felix, Trinil, Palaeontogr. 60, p. 343.
1913 „ acuta Quelch, Felix, Trinil, Palaeontogr. 60, p. 343.
1915 „ Molengraaffi Felix, Palaeont. v. Timor II, p. 10.
1928 „ recta (Dana), Matthai, Catal. Madrep. corals British Mus. vol. 7, p. 227 Pls (with synonymy).
1940 „ recta (Dana), Umbgrove, Zoolog. Meded. 22, p. 288 (with synonymy).

The specimens are all fragments of large colonies undoubtedly belonging to the still living species S. recta. The Siboga specimen is from a pleistocene reef limestone 80 metres above sea level. I restudied a syntype of S. Molengraaffi Felix in the Institute of Mining at Delft. From a comparison with

a suite of recent specimens of Symphyllia it is obvious that FELIX's species belongs to S. recta (Dana). The differences mentioned by FELIX fall within the limits of variability of that species. From the description of Symphyllia acuta FELIX it may be deduced that FELIX studied not Symphyllia agaricia E. H. (= S. acuta Quelch according to Matthai, 1928, p. 233) but S. recta Dana as defined by MATTHAI (1928, p. 277).

Fungia sibogae Van der Horst.
Pl. I, fig. 1 and fig. 2.

The specimen is damaged; especially the edge of the disc is preserved in only one spot. However, the undamaged part allows an identification. The description of the recent Siboga specimen given by VAN DER HORST (Siboga Exped. Monogr. XVI *b*, 1921) is in every respect applicable to the fossil specimen; only in the Javanese fossil the central part of the lower surface is slightly convex, whereas the recent specimen has a slightly holowed aboral surface.

Moreover the recent specimen shows an individual prominence of the septa round the axial fossa which is absent in the fossil one.

Fungia somervillei Gardiner.
Pl. I, fig. 3 and 4.

1921. Fungia somervillei VAN DER HORST, Siboga Exp. Monogr. XVI 6, (with synonyms).

Although only one fragment of a disc is present, the specimen shows so many highly characteristic features that they must be described at some length. The disc had an elliptical shape. From the fragment it may be inferred that it must have had a length of about 140 mm, a breath of 80 mm; thin along the edge of the disc (10 mm) but much thickened in the centre (40 mm). The lower surface is very irregularly undulating. Wall imperforate. Costae low, subequal; those of the first cycle slightly more prominent. All costae covered with small spinose and sub-equal granules. Costae near the centre of the disc interrupted; in the centre loosing themselves in an irregular pattern. Septa rounded along the edge of the disc; the first cycles thickened towards the central fossa and more prominent than those of the thinner septa; densely and finely granulate. Some of the septa showing an arrangement of grains in lines perpendicular to the edges of the septa. Edges of the septa locally with small teeth. The axial fossa is 40 mm long.

The species has the characteristics of F. somervillei as described by GARDINER but it is more elongate. In this respect the specimen described by VAN DER HORST is an intermediate form, but that specimen is only 20 mm high. Without studying a large suite of this species it is impossible to form an idea of its variability. There is, however, no reason to describe the present specimen under a new name, as it agrees in all principal characteristics with F. somervillei.

Fungia costulata Ortmann.
Pl. I, fig. 5 and 6.

1902. Fungia costulata Döderlein, Abh. Senckenb. Naturforsch. Gesellsch. vol. 27 (with synonymy).

The lower surface of the specimen is undulating, showing a concavity in the centre. Costae subequal, many times interrupted, running nearly straight to the centre of the disc, where they form an irregularly winding pattern. Costae covered by small and some larger irregular spinose granulations. Septa of subequal height, in their peripheral part strongly granulate even on their upper and outer gently rounded and free edges. Only the youngest cycle of septa very thin and low. Septa not dentated. Diameter of the disc 60 mm, height 18 mm.

Distribution: Recent: Ceylon, New Britain, Blanche Bay.

Fungia inaequicostata Gerth.

1925. Fungia inaequicostata Gerth, Java, Leidsche Geolog. Meded. I, p. 41, plate 6, fig. 1, 1a.

The sepeal edges are not well preserved. It may, however, be seen that they are crowded with fine granules, which bear distinct teeth (8 per centimeter). In the specimens studied the theca is imperforate. The species seems to me allied to Fungia decipiens (MARTIN, 1880, Pl. 25, fig. 3) which appears to have less prominent costae.

It is, however, very difficult to compare the two as in MARTIN's specimens the marginal parts are not preserved while in GERTH's and in my specimens of F. inaequicostata the central area is either covered or worn off.

Fungia echinata Pallas.

A worn off specimen of an elongated, rather flat Fungia, length 115 mm, breadth 65 mm, shows on its lower surface spines which have suffered much by diagenetic processes. The habitus of the corallum, the strong granulation of the septa and the shape and arrangement of the spines show this species to belong to Fungia echinata. The specimen was broken in several parts along the longer axis. After reconstruction the length of the axial fossa could be estimated at about 30 mm. The upper surface is worn off. It seems to me, however, that there is no reason to doubt the specific identity with the recent F. echinata.

I mentioned this specimen and Thiel's F. brachystoma when discussing the recent corals from Togian.

Fungia cf. repanda Dana.

A fragment of a large disc (diameter 130 mm) shows the characteristics of a Fungia belonging to the repanda group. I am inclined to consider the specimen Fungia repanda Dana on account of the perforations, which

(apart from openings due to erosion) occur in the peripheral area of the underside.

However, the greatest part of the lower side is covered by strongly attached rock material. It may therefore be doubted whether we are dealing with F. repanda or with the closely allied F. concinna. That is the reason for placing a "cf" before the species name. However, it is a question of no great importance, because transitional forms between F. repanda and F. concinna have often been found among recent corals.

Fungia cf. concinna Verrill.

Aboral surface imperforate. Diameter of disc. 73 mm. The strongly developed costae of the first cycles bear very distinct rough granules; the thinner costae of the higher cycles show smaller but distinct blunt spinuliform granules, which occur also in the central part of the disc. The form must be considered closely allied to Fungia concinna Verrill, a well known recent species, which has been found in Neogene strata as well.

The septal edges are badly damaged, but small parts are preserved and show dentations as in F. concinna.

Pavona microstoma Umbgrove.
Pl. II, fig. 11 and 12.

1925. Pavona microstoma Umbgrove, Gerth, Nias, Leidsche Geol. Meded. I, p. 35.

1926. Pavona microstoma Umbgrove, Sumatra, Wetensch. Meded. Nr. 4, p. 43, Pl. II, fig. 8.

A description will be given here, summarising moreover the characteristics as described in the papers mentioned above. Corallum irregularly noduliform or branched. Diameter of branches irregular ellips-shaped or round. Surface

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EXPLANATION OF PLATES.

Pl. I.

Fig. 1—2. Fungia sibogae Van der Horst, natural size; loc. 93B, 260.

Fig. 3—4. Fungia sommervillei Gardiner, natural size; loc. 93B 498, no. 260 B.

Fig. 5—6. Fungia costulata Orthm, natural size: loc. 93B 498, 260 B.

Fig. 7—8. Seriatopora micrommata Felix, var. javana nov. var. × 3; fig. 7, loc. 99B no. 9 (Putjangan beds), fig. 8. loc. 93B no. 258 (Kalibeng beds).

Pl. II.

Fig. 9—10. Stylophora solida nov. spec., × 3, loc. 93B no. 253; fig. 10 upper view, fig. 9 lateral view.

Fig. 11—12. Pavona microstoma Umbgrove, loc. 93B, 373, no. 252; fig. 11, × 5; fig. 12, natural size.

Fig. 13. Leptastraea transversa (Klunz.), loc. 93B, no. 260, × 3.

J. H. F. UMBGROVE: *Corals from the Upper Kalibeng beds (Upper Pliocene) of Java.*

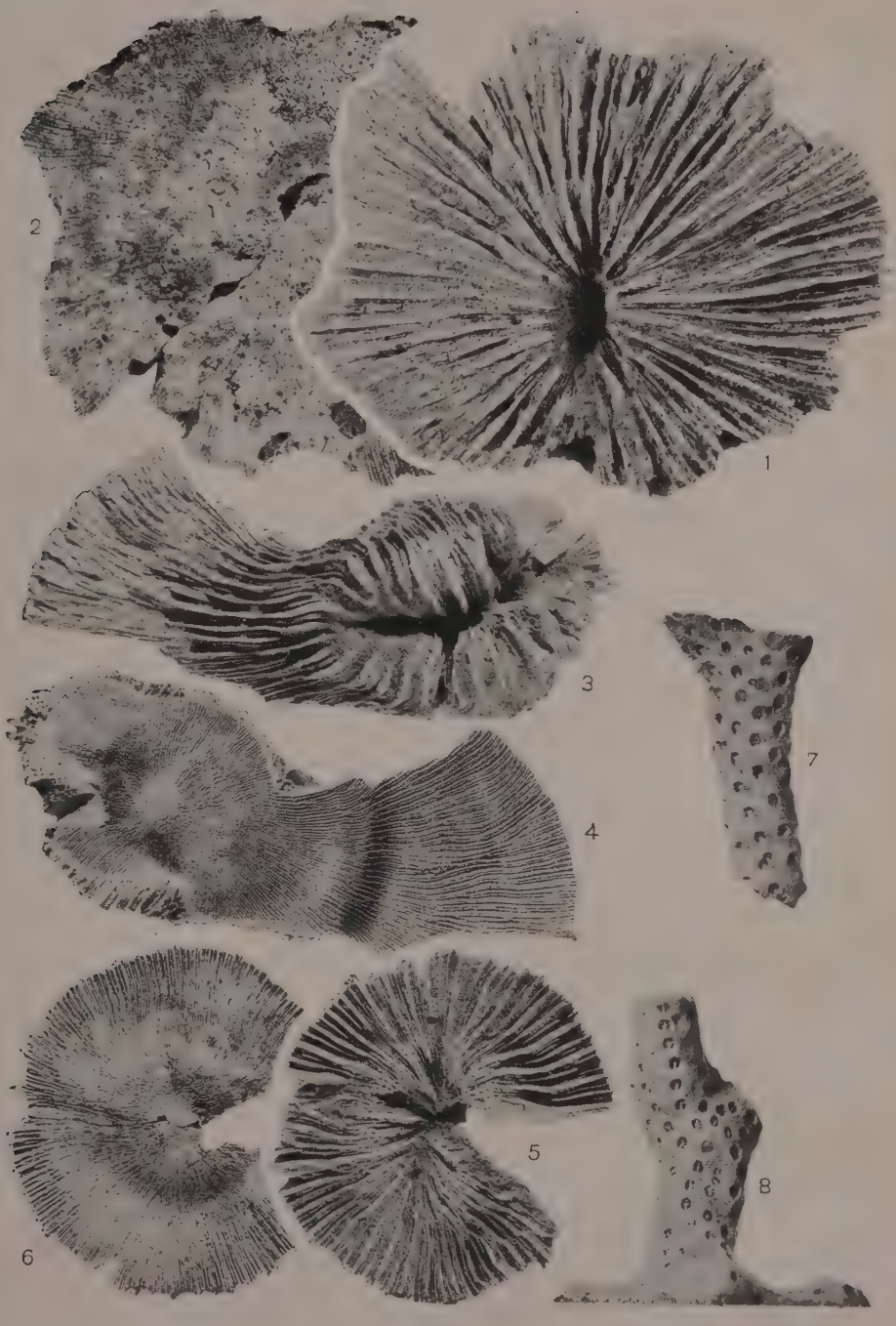
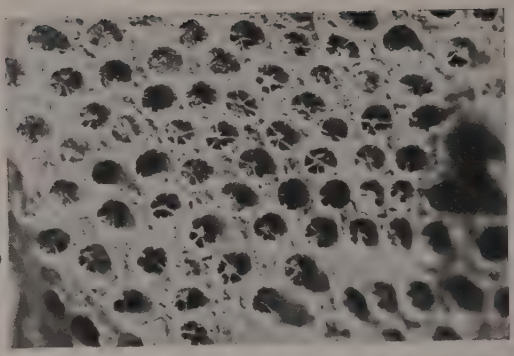


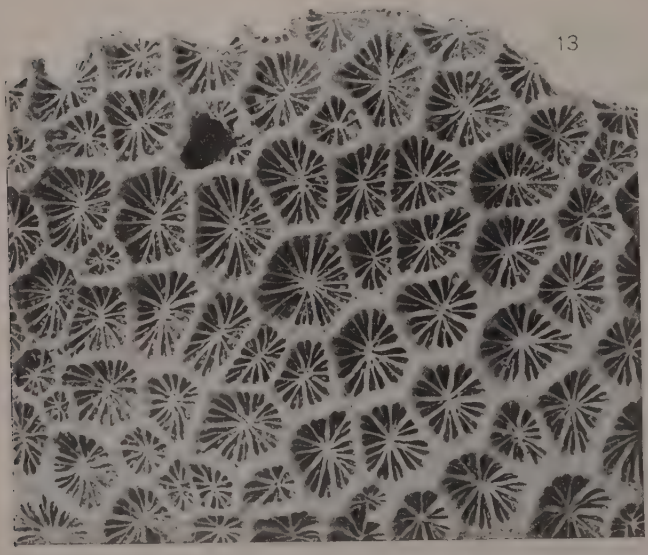
PLATE I.

9



10

11



13

12

PLATE II.

crowded with small calices. Calicular openings $\frac{1}{2}$ mm in diameter. Distance between calicular centra varying from 1—2.5 mm.

Septa mostly in two complete cycles, sometimes the second cycle is incomplete. Septa locally developed alternately more strongly and more weakly, elswhere subequal in size. Collines flattened, (no crests) crossed by the septcostae, which are continuous between the calices and often slightly gyrose. Primary septa joining the columella deep in the calice, columella not distinct from the surface. This species can be easily distinguished from other fossil and recent species of the genus Pavona, by its very small calices.

Distribution: Plio-Pleistocene, Nias and Atcheen (holotype, North Sumatra).

Anatomy. — *The nucleus of* BELLONCI *and adjacent cell groups in Selachians.* II. *The nucleus of* BELLONCI. *The nucleus commissurae hippocampi (eminentia thalami).* By J. L. ADDENS. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of November 24, 1945.)

The nucleus of Bellonci.

In the unmyelinated part of the stria medullaris, as it reaches the front end of the ganglion habenulae, cells begin to appear, which caudally become more and more numerous, and continue into the commissura habenularum. These cells form two nuclei, separated by a small gap. The foremost of them, oblong in shape, extends caudally in the stria until it turns medially to form the commissure (figs. 1, 2, 7, 8). The hindmost nucleus is an unpaired median cell mass, lying in the commissure and filling it up almost entirely (figs. 3, 5—8).

The foremost nucleus, in accordance with BERGQUIST (1932), is interpreted by me as nucleus commissurae hippocampi or eminentia thalami (BERGQUIST uses the latter term), the hindmost unpaired nucleus as the fused nuclei of BELLONCI.

The latter, which we now shall deal with in detail, as just said, almost entirely fills up the commissura habenularum, and consists of small cells of the same size and type as the habenula cells. They are, however, far more loosely arranged (figs. 3, 5, 6). These cells have not been described before. They may be seen, however, in the figure BURCKHARDT (1907, fig. 15) gives of a sagittal section through the origin of the epiphysis of Spinax niger, and in that of KAPPERS (1920, fig. 41) of a similar section through the epithalamus of Scyllium canicula.

In Cyclostomes, where likewise it lies entirely embedded in the stria medullaris, the nucleus of BELLONCI is paired. In Petromyzonts, on the right side, it is situated as a separate cell group immediately beneath the ganglion habenulae, while, on the left side, it has fused with this ganglion to the structure generally described as left habenula (ADDENS, 1943a). Thus the nucleus of BELLONCI, paired in Cyclostomes, according to our interpretation, in Selachians, has shifted dorsomedially in the stria, right and left nucleus having fused to a single cell mass in the median plane.

It is natural to suppose that the nucleus of BELLONCI, whose cells all of them are embedded in the compact unmyelinated stria medullaris, mainly is supplied by this bundle. This is opposed to JOHNSTON's regarding the unmyelinated stria as exclusively a hippocampus commissure, since the nucleus of BELLONCI always chiefly is supplied by the stria medullaris, the tractus olfacto-habenularis lateralis especially. But above we have already set forth as our opinion that also the commissura telencephali

superior of KAPPERS, and moreover, unmyelinated fibres of the various telencephalic habenular tracts may course in the unmyelinated stria.

In addition to the unmyelinated stria, the unpaired cell mass in the commissura habenularum, interpreted by us as the fused nuclei of BELLONCI, is supplied by myelinated fibres. When analysing the stria medullaris, we reported myelinated fibres crossing in the posterior part of the unmyelinated commissura habenularum, lying partly as small bundles at its periphery, and partly as widely scattered single fibres inside this bundle (figs. 3—8). The peripheral bundles appeared to be the decussating part of the tract interpreted by JOHNSTON as tractus septo-habenularis [1]), while the internal fibres, which came forth medial to, and from the inner layers of the optic tract, were held by us to belong to the tractus olfacto-habenularis medialis. Since they pass between the most posterior cells of the nucleus of BELLONCI, it is quite possible that they synapse with these. In addition to the decussating fibres there are, however, also direct myelinated fibres of the tractus olfacto-habenularis medialis to the nucleus of BELLONCI, and very probably also an unmyelinated part of this tract joins the stria medullaris in this region. The myelinated fibres would then run scattered among these.

Something must be said here yet about the part of the tractus olfacto-habenularis medialis, already mentioned in the first part of the paper, which runs forward and joins the tractus strio- and amygdalo-habenularis. There is a separate medial bundle of it, which traverses the crescent-shaped bundle depicted, but not labelled in figs. 1—3 which JOHNSTON calls optic radiation in his figures 9 and 10 [1]) (Scyllium canicula), and also in his figure 23 (Scyllium stellare). This bundle of the tractus shaped bundle depicted, but not labelled in figs. 1—3, which JOHNSTON designates in his figure 44 (Acanthias vulgaris) as tractus septo-habenularis.

At first we have interpreted the crossed and direct myelinated fibres of the tractus olfacto-habenularis medialis going to the nucleus of BELLONCI in a different way. We regarded them as the homologues of collaterals of the so-called fibrae tectales nervi optici of KRAUSE (1898) in Teleosts. JANSEN (1929), confirmed by CHARLTON (1933) and MEADER (1934), demonstrated these fibrae tectales to be a component of the supraoptic-commissure.

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[1]) The interpretation of this tract is by no means sure. The fibre bundle JOHNSTON depicts in his figure 44 (Acanthias vulgaris) as arising from the medial forebrain bundle, probably is a part of the tractus olfacto-habenularis medialis mentioned below. The fine fibre bundles described by BÄCKSTRÖM as given off by the medial forebrain bundle to the stria medullaris (see Part I), more probably belong to the combined tractus strio- and amygdalo-habenularis and olfacto-habenularis medialis. In Acanthias vulgaris the decussation of the so-called tractus septo-habenularis is much stronger developed than in Scyllium canicula, in that here also fibres of the tractus olfacto-habenularis medialis partake in the formation of the decussation.

[2]) In this figure the designations tractus septo-habenularis and optic radiation are interchanged.

Now JEENER (1930) described in Idus idus horizontal myelinated fibres, already observed by B. HALLER (1898) and SHELDON (1912), connecting the cell group he interprets as eminentia thalami with the fibrae tectales. He suggests these horizontal fibres to be collaterals of the latter. As I at first supposed that the eminentia thalami of JEENER might be in reality the nucleus of BELLONCI, I was led to homologize the fibres coming forth medial to, and from the inner layers of the optic tract, in Scyllium, with the collaterals of JEENER. MEADER (1934) calls the fibrae tectales commissura intertectalis ventralis. Hence the abbreviation c.i.v. in our figures.

JANSEN (1929) has come to about the same view we originally held, since he homologizes the rostral part of the dorsal thalamus of Teleosts with the nucleus of BELLONCI of Amphibians. In this he bases himself principally on the statement of HOLMGREN (1920) that collaterals of the medial optic tract end in the eminentia thalami [1]) of Teleosts. Probably these collaterals are the same as the horizontal fibres of B. HALLER, SHELDON and JEENER. On their possible significance we shall return in the section on the nucleus commissurae hippocampi.

The myelinated fibres of the tractus olfacto-habenularis medialis to the nucleus of BELLONCI have not been specifically described. They are depicted, however, in the figure KAPPERS (1920, fig. 40) gives of a sagittal section through the epithalamus of Scyllium canicula. The tractus olfacto-habenularis medialis as a whole, probably combined with the tractus strio- and amygdalo-thalamicus, has been several times observed by the older authors. It is the accessory optic root from the ganglia habenulae described by VIAULT (1876) in Trygon, as is apparent from his fig. 21. Further it is the tractus descendens ganglii habenulae of EDINGER (1892) and KAPPERS (1906), of which HALLER (1898) maintains that part of its fibres go into the optic nerve. It must be mentioned here that DUMÉRIL (1865) generally is misquoted as having stated that, in Selachians, the optic root, partly or even entirely, arose from the ganglia habenulae. He regarded these ganglia themselves as uncrossed optic roots (his "racine postérieure ou supérieure et principale du nerf optique"), arising, like the crossed roots (racine antérieure), from the optic tectum and joining the latter.

It may be pointed out that the optic fibres from the habenula of VIAULT and HALLER have nothing to do with the small bundle of the tractus olfacto-habenularis lateralis, closely resembling an optic bundle, we described in the first part of the paper. This lies farther forward.

The supply of the nucleus of BELLONCI, in Selachians, exclusively by habenulo-petal tracts, is at variance with the current view that the nucleus in question receives an additional supply by optic fibres. I will not give now a survey of the supply of the nucleus throughout Vertebrates, but will only point out that also in Mammals (1938) and Cyclostomes (1943, '43a) I could find no optic fibres going to the nucleus. In Reptiles (1938) and Birds (1940) I must leave this point undecided.

Closing the discussion of the nucleus of BELLONCI and its connections, it must be remarked that no efferent tracts could be found in the preparations at our disposal.

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[1]) It must be remarked that HOLMGREN uses the term eminentia thalami in a much wider sense than usual (cf. KUHLENBECK, 1929; JEENER, 1930).

Nucleus commissurae hippocampi (eminentia thalami).

The foremost of the two nuclei embedded in the stria medullaris, the nucleus commissurae hippocampi, begins, as aforesaid, approximately at the same level as the ganglion habenulae and extends caudally in the stria until it turns medially to form the commissure (figs. 1, 2, 7, 8). In cross-sections through its hindmost part, close to its medial side, the nucleus of BELLONCI begins to appear, always, however, separated from it by a small gap.

Like those of the nucleus of BELLONCI the cells of the nucleus commissurae hippocampi are of the same size and type as the habenula cells. They lie very scattered, however, still more so than those of the nucleus of BELLONCI. Thus the nucleus is less well circumscribed than the latter, it especially being difficult to say where it begins exactly. Moreover, part of its cells lie outside the stria, ventral to it, and these are not easily differentiated from their surroundings. In the reconstructions it is not tried to indicate this, the nucleus, for the sake of clearness, being represented as lying entirely inside the stria.

As with the nucleus of BELLONCI we may safely assume that the nucleus commissurae hippocampi, embedded for a large part in the unmyelinated stria, chiefly is supplied by this bundle. This, however, is no proof of JOHNSTON's contention that the unmyelinated stria is exclusively a hippocampus commissure. According to HERRICK (1933), in Urodeles, the nucleus commissurae hippocampi, besides from the commissura hippocampi, which here has freed itself from the commissura habenularum, also receives impulses from the stria medullaris s. str.

There is still another connection of the nucleus commissurae hippocampi. When describing the tractus olfacto-habenularis lateralis in the first part of the paper, we mentioned in a note that, in addition to the principal tract, there is present a separate posterior part of this tract. The principal tract ascends to the stria medullaris just in front of the ganglia habenulae. The posterior part lies at the level of the foremost end of these ganglia. Its fibres come forth from the outer layers of the optic tract and, bending horizontally, pass beneath the unmyelinated stria. Here they cross the ascending fibres of the tractus olfacto-habenularis medialis. How they end could not be made out with certainty, but possibly they do so medial to the stria medullaris, in the region of the nucleus commissurae hippocampi. These fibres may be seen, but are not labelled in fig. 1. They closely resemble the horizontal fibres connecting, in Teleosts, the eminentia thalami with the so-called fibrae tectales nervi optici, and may be their homologues. As the horizontal fibres of Teleosts, according to B. HALLER and SHELDON, are efferent, it is more probable that the fibres considered by us as a posterior part of the tractus olfacto-habenularis lateralis are not such, but an efferent tract of the nucleus commissurae hippocampi (eminentia thalami).

In Petromyzonts the nucleus commissurae hippocampi was described by us (1943, '43a) as a cell plate, lying at the boundary between tel- and diencephalon and separated from the ventricle by the primordium hippocampi and nucleus of BELLONCI (see especially our figure of a horizontal section: 1943a, fig. 1). According to HERRICK and OBENCHAIN (1913), and BERGQUIST (1932), who also described this nucleus in Petromyzonts, it is, however, adjacent to the ventricle, being intercalated between the primordium hippocampi in front and the nucleus of BELLONCI behind. In Myxinoids, where the nucleus also is present as a well-defined structure, it lies medial to the ventral thalamus, between this and the median plane (1943a: fig. 4).

Comparing the reconstructions of Scyllium (figs. 7, 8) with those of Cyclostomes and Necturus (1943: figs. 6—8; '43a: fig. 2), it appears that the nucleus commissurae hippocampi, as also the nucleus of BELLONCI, occupy the same level in Petromyzonts, Selachians and Urodeles, the former lying always close in front of the latter. In Myxinoids, on the contrary, the nucleus commissurae hippocampi lies at the level of the nucleus of BELLONCI of the other groups and the latter on that of the nucleus commissurae hippocampi of these, the relationship of the two nuclei being reversed. No doubt this is a secondary condition, but how it has been brought about, I am not prepared to say.

Although it lies outside the scope of this paper, something may be said here about the two sulci designated in the figures as sulcus telo-diencephalicus internus and sulcus praeopticus (figs. 5—8).

Our sulcus telo-diencephalicus internus is the sulcus intra-encephalicus anterior of VON KUPFFER (1906). This is a vertical groove extending from the preoptic recess dorsally into the region of the velum transversum. It is present in all Vertebrate classes, in the embryonic state at least (VON KUPFFER, 1906; V. HALLER, 1922, '29, '34; KUHLENBECK, 1929; BERGQUIST, 1932; HERRICK, 1933).

I have been led to name this groove sulcus telo-diencephalicus internus in the following way. In my figures of adult Petromyzon fluviatilis (1943, '43a) I designated as sulcus telo-diencephalicus internus the groove between the primordium hippocampi and nucleus of BELLONCI, considering it as the internal boundary between telencephalon and diencephalon. It is the sulcus limitans hippocampi of JOHNSTON (1912), the first thalamic sulcus of HERRICK and OBENCHAIN (1913) and the sulcus thalami anterior of BERGQUIST (1932). There can be no doubt that this groove between the primordium hippocampi and nucleus of BELLONCI in the adult Petromyzon is the dorsal part of the groove which in the Ammocoetes is called by VON KUPFFER (1906) and V. HALLER (1922) sulcus intra-encephalicus anterior.

The sulcus praeopticus is a vertical sulcus of the telencephalon medium, lying at its beginning, considerably in front of the sulcus telo-

diencephalicus internus (figs. 5—8). It has not been reported before in Selachians and is homologized by us with the similar sulcus described under this name by KUHLENBECK (1929) in Amphibians.

Summary.

A description and discussion are given of the ganglia habenulae, the nucleus of BELLONCI and the nucleus commissurae hippocampi (eminentia thalami) of Scyllium canicula, which, in essence, apply to Selachians generally, of all of whose three groups, Sharks, Rays and Holocephali, representatives were examined.

Contrary to the prevailing notion, the right ganglion habenulae is larger than the left, though only to a small extent, and gives rise to a stronger and more myelinated fasciculus retroflexus.

Right as well as left ganglion habenulae are divided into a medial and a lateral nucleus, of which the latter is subdivided into a pars dorsalis and a pars ventralis. The pars dorsalis of the right lateral nucleus consists of large, loosely arranged cells, embedded in a dense myelinated network. The pars dorsalis of the left lateral nucleus is only differentiated from the rest of the ganglion by the possession of scarce myelinated fibres.

The myelinated part of both right and left fasciculus retroflexus exclusively arises from the pars dorsalis of the lateral habenular nucleus.

An analysis is given of the stria medullaris and commissura habenularum. The existence of a tractus tecto-habenularis is confirmed and, as a new finding for Vertebrates generally, a tractus tegmento-habenularis is reported.

The nucleus of BELLONCI was demonstrated. Right and left nucleus have fused to a median cell mass, lying in the commissura habenularum. No optic connections of the nucleus could be found.

The nucleus commissurae hippocampi (eminentia thalami) lies on either side, in front of the nucleus of BELLONCI, partly embedded in the stria medullaris and partly ventral to it. A fibre tract, probably efferent, connected with the nucleus commissurae hippocampi is described, which comes forth from the outer layers of the optic tract and is homologized with the horizontal fibres connecting, in Teleosts, the nucleus commissurae hippocampi (eminentia thalami) with the so-called fibrae tectales nervi optici of KRAUSE (commissura intertectalis ventralis MEADER).

The sulcus intra-encephalicus anterior of VON KUPFFER is named sulcus telo-diencephalicus internus, as it is homologized with the groove so designated by us in Petromyzonts.

A sulcus praeopticus is reported for Selachians.

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Zoology. — *The osmotic properties of the egg of Limnaea stagnalis L.* By CHR. P. RAVEN and H. KLOMP. (From the Zoological Laboratory, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of November 24, 1945.)

It has been observed by one of us (RAVEN 1945) that the eggs of *Limnaea stagnalis* in the 4—5 hours between oviposition and 1st cleavage show a considerable increase in volume. Increases of 36—56 % (average: ± 45 %) have been observed in different egg-masses. When volume is plotted against time, one gets no quite regular curve (l.c., fig. 6). A rapid rise during the formation of the 1st polar body is followed by a nearly horizontal part, in some cases even a slight drop, shortly afterwards; similar irregularities are found during the formation of the 2nd polar body and (to a less extent) about one hour before 1st cleavage. Though a similar course has been found in many experiments, these fluctuations have, perhaps, no real significance, and must be attributed to systematic errors in the volume determinations. The eggs show, shortly after the extrusion of either of the polar bodies, some amoeboid activity, by which the outline of the egg becomes rather irregular; therefore, the calculation of volume from the measured longest and shortest diameter (using the equation $V = {}^1/_6 \pi ab^2$) will yield, at this moment, too high results.

The swelling of the egg is due, apparently, to the absorption of substances, especially water, from the surrounding egg capsule fluid. It seemed important, therefore, to study the osmotic properties of the *Limnaea* egg.

1. *The swelling in distilled water.*

When the eggs are prepared out of their capsules shortly after oviposition and transferred to distilled water, a considerable swelling, much exceeding the normal increase in volume in the egg capsule fluid, takes place. Fig. 1 gives an example.

Experiment PD. Oviposition at 9 h. 25. 10 Minutes later 6 control eggs (Co), in the egg capsule, were measured; average volume = 653.000 μ^3. Then, 12 eggs were prepared out of the capsules and transferred to distilled water at 9.52 (A). The first measurement of this sample took place at 10.01', then at regular intervals of 6½—7½ minutes measurements were taken till 10.22; at 10.26 the controls were measured again. The further course of the experiment may be taken from the figure. The controls reached after 145 minutes a volume of 1.009.000 μ^3 (swelling: 55 %); the eggs in distilled water after 190 minutes a volume of 1.336.000 μ^3 (swelling: 105 %); in the controls at this moment cleavage had begun.

The further development of the eggs in distilled water is as follows: The first cleavage takes place nearly always, be it often somewhat delayed as compared with the controls. It shows a number of peculiarities

which will be treated below. In some cases development comes to a standstill at the 2-cell stage; mostly, however, the 2d cleavage takes

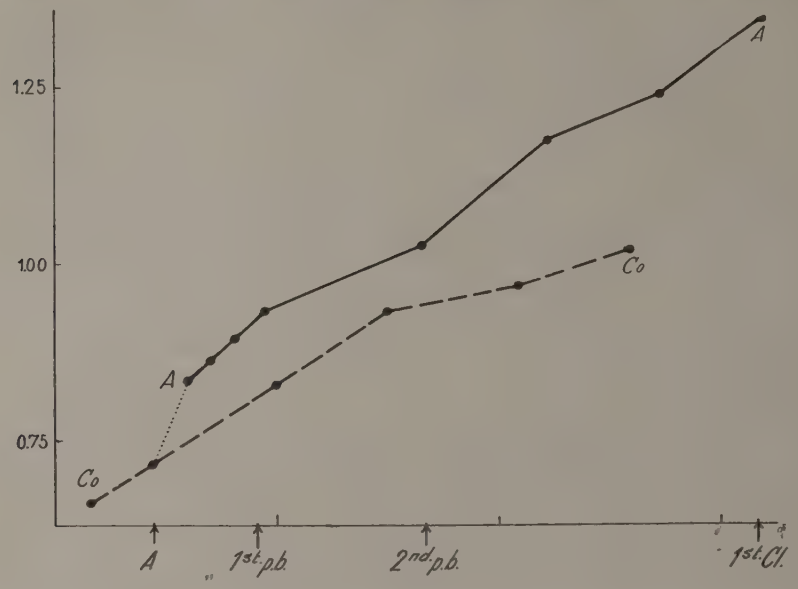

Fig 1.

Experiment PD. Swelling of *Limnaea* eggs in distilled water. Abscissae: time. Ordinates: average volume of eggs in $10^6 \mu^3$. A: eggs in distilled water. Co: controls in egg capsule. 1st p.b., 2nd p.b., 1st cl.: time of 1st polar body, 2nd polar body, 1st cleavage.

place also. Development does not proceed beyond the 4-cell stage, even when the eggs are transferred to distilled water after the 1st or even the 2d cleavage. Apparently, the susceptibility of the egg to hypotonicity is great at the 4-cell stage.

The blastomeres swell further until cytolysis begins. The moment of cytolysis is very unequal even in the blastomeres of one egg; it varied from 8—28 hours after oviposition. When cytolysis occurs late, vacuoles become visible in the egg, which can protrude far from the egg surface. Immediately before cytolysis sets in, they sink somewhat back into the egg, probably because one of the vacuoles has broken through to the exterior and expelled its fluid contents; soon afterwards, the yolk granules begin to pour out. This vacuolization process reminds that described by LEITCH (1936) in eggs of *Echinometra lucunter* swelling in dilute seawater, which occurred also shortly before the eggs underwent cytolysis.

In eggs, which have been transferred to distilled water with a slight addition of $CaCl_2$, and, therefore, show a normal cleavage with formation of a cleavage cavity (cf. below), these vacuoles are not formed. This may be due to the active fluid output by the periodic contraction of the egg, with expulsion of the fluid filling the cleavage cavity (cf. COMANDON and DE FONBRUNE 1935).

2. *The osmotic pressure in the egg immediately after oviposition.*

In order to determine the osmotic pressure in the recently laid eggs they were transferred to solutions of sucrose and urea of different concentrations. The equilibrium volumes, reached after some hours, were found to depend only upon the molar concentration of the solution; in isotonic solutions of urea and sucrose, equilibrium volumes were equal. This shows that the swelling of the egg is an osmotic process.

In solutions above 0.10 M the eggs shrink; in solutions below 0.09 M they swell; at the concentrations of 0.09 M and 0.10 M the volume remains nearly constant. It may be concluded that the recently laid eggs are isotonic with 0.09—0.10 M solutions of non-electrolytes. (The average internal osmotic pressure of the egg, as calculated from our experiments, equals that of a 0.093 M solution; cf. fig. 3).

In solutions of 0.03 M, the rapidity of swelling equals that of the controls in the egg capsule fluid. This observation does not prove, of course, that this fluid is isotonic with a 0.03 M solution. We have to reckon, in fact, with an influence of the composition of the external medium upon the permeability of the eggs; therefore, the rapidity of swelling can be no measure of the osmotic pressure of the medium. The equilibrium volume was not reached in these solutions, as the swelling continues till the entrance of 1st cleavage, after which an accurate volume determination is no longer possible.

3. *The non-solvent volume.*

Fig. 2 gives an example of an experiment, in which eggs of one egg-mass have been transferred to various concentrations of urea.

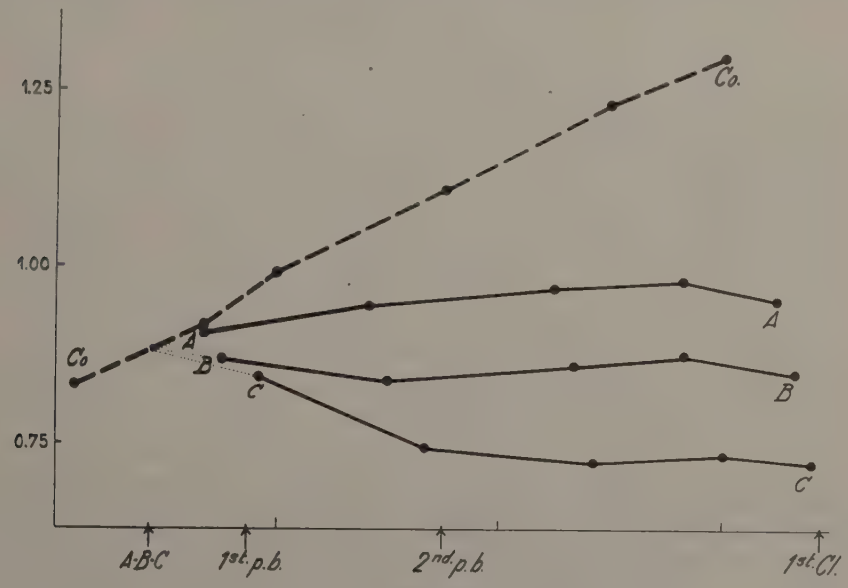

Fig. 2.

Experiment OR. Swelling of *Limnaea* eggs in 0.08 M (A), 0.10 M (B) and 0.15 M (C) urea solutions. Explanation cf. fig. 1.

Experiment OR. Oviposition at 14 h. 10. After 5 minutes volume determination of 6 controls in the egg capsule (Co): 833.000 μ^3. The controls show normal swelling: 56 % in 3 hours. A number of eggs was prepared out of the capsules, and transferred at 14.35 to 0.08 M. (*A*: 4 eggs), 0.10 M. (*B*: 6 eggs) and 0.15 M. (*C*: 11 eggs) solutions of urea. The equilibrium volumes after about 3 hours were: $A = 967.000\ \mu^3$, $B = 862.000\ \mu^3$, $C = 728.000\ \mu^3$.

These figures show that the product of volume and external osmotic pressure is by no means constant; the swelling and shrinking are less than inversely proportional to the osmotic pressure. This must be attributed to the presence in the egg of substances which are osmotically inert. By means of the equation $P(V—b) = P_0(V_0—b)$, in whicht P is the osmotic pressure, V the total egg volume and b the volume occupied by these osmotically inactive substances ("non-solvent volume"), the latter can be calculated (cf. BROOKS & BROOKS 1941). In the above-mentioned experiment the non-solvent volume, as determined in this way, is 453.000 μ^3, constituting 54 % of the original egg volume.

In other experiments, similar values have been obtained. In fig. 3, the observed equilibrium volumes (reduced to percentage of original volume) have been plotted against the reciprocal of the relative pressures; a straight line is fitted to the points and produced to the ordinate at $1/M = 0$,

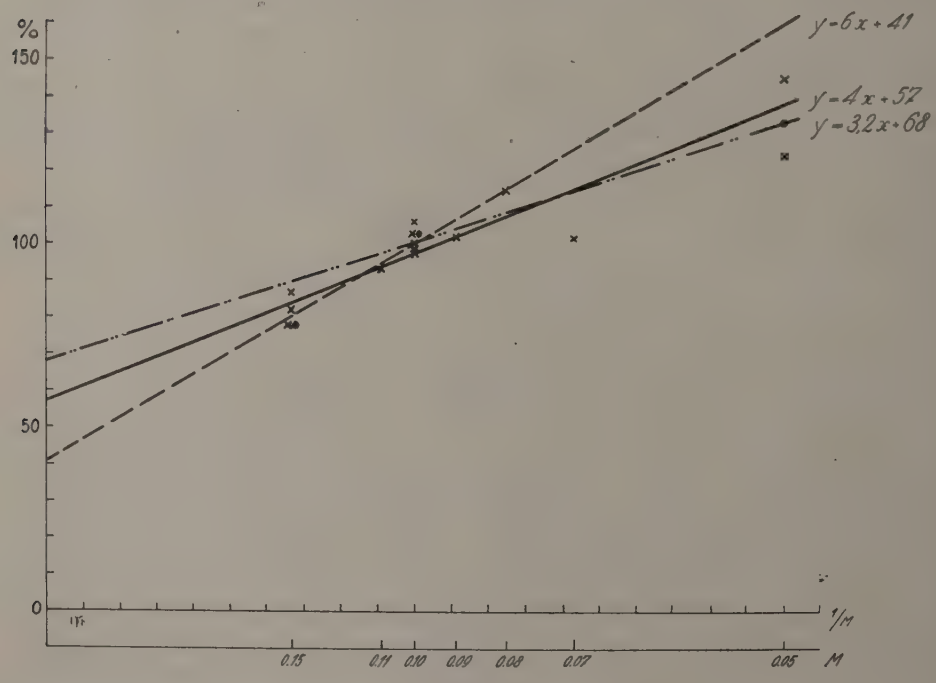

Fig. 3.

Graph representing relation between egg volume and osmotic pressure. Abscissae: reciprocal of molar concentration. Ordinates: equilibrium volumes in percentage of original volume. Crosses: urea experiments. Dots: sucrose experiments.

—..—..—..— swelling. — — — — — shrinking. ———— both combined.

where the average non-solvent volume can be read off (Mc.Cutcheon. Lucké and Hartline 1931). It is found to equal 57 % of the original egg volume.

This non-solvent volume is exceptionally high. In marine eggs, the osmotically determined non-solvent volumes range between 0 % an 48 % (Brooks & Brooks, p. 68); the higher of these values are already unduly high, as compared with the analytically determined dry weights, which range from 13 % to 30 % with a distinct crowding between 18 % and 23 %. Although the latter determinations concern only echinoderms and worms, it seems very improbable that the volume of dry matter in the egg of *Limnaea* will much exceed these values. Therefore, the unduly small osmotic volume changes of this egg must be due to other factors. In this connection, it is of interest to note that in our experiments the calculated non-solvent volume for swelling alone (68 %) exceeds considerably that for shrinking (41 %; cf. fig. 3); this means that the eggs swell less than would be expected on the basis of shrinking experiments. This may be due to a resistance of the egg surface or vitelline membrane to stretching when swelling occurs. Possibly, the vitelline membrane offers, in a less degree, also some resistance to shrinking; that it is rather inelastic in these solutions is shown by the fact that it is thrown into folds in shrunken eggs.

4. *The permeability constant.*

The permeability constant for swelling in distilled water has been calculated using the formula

$$P = \frac{dV}{dt} \cdot \frac{1}{S_0 (a_i - a_e)_0}$$

in which the quantities have been expressed in the standard units of GM.cm^{-2}.sec^{-1}. (GM.l^{-1})$^{-1}$ (cf. Brooks & Brooks 1941).

In the case of fig. 1, for the period between the first two measurements (9—15½ min. after the eggs have been transferred to distilled water) the increase of volume $dV = 31.200\ \mu^3 = 31.2 \times 10^{-9}$ cm^3 $= 1.73 \times 10^{-9}$ GM; the time $dt = 6½$ minutes $= 390$ sec. The surface S_0, as calculated from the diameter of the eggs, is 428×10^{-6} cm^2, while the activity gradient $(a_i - a_e)_0$ equals the internal osmotic pressure of the egg $=$ 0.093 GM/l. This yields for the permeability constant P the value 1.13×10^{-7}. In another experiment, a permeability constant for swelling in distilled water $P = 2.14 \times 10^{-7}$ was determined. This suffices to show that the permeability constant in *Limnaea* is of the same order of magnitude as in various marine eggs (mostly echinoderms) ,where most of the values lie close to 2×10^{-7} (Brooks & Brooks 1941, p. 86). In view of the difference in systematical position and natural external medium the agreement is rather remarkable.

5. *Development of the eggs in urea and sucrose.*

In a 0.15 M solution of urea, the eggs extrude one or both polar bodies, but do not cleave. In a 0.10 M solution, however, the eggs undergo the 1st and, as a rule, also the 2d cleavage; the 3d cleavage can be indicated in some of the blastomeres, but is not finished. Therefore, in most eggs development stops at the 4-cell stage; soon, cytolysis sets in. In all weaker solutions of urea employed (0.03—0.10 M), the same behavior of the eggs has been observed: cytolysis at the 4-cell stage. Apparently, the susceptibility of the egg to the action of urea is great at the 4-cell stage. It has been shown above that at this stage there is also an increased susceptibility to hypotonicity. The influence of weak urea solutions cannot be attributed, however, to their hypotonicity. In the first place, the eggs behave in the same way in a 0.10 M urea solution, which is nearly isotonic to the egg or even slightly hypertonic. Secondly, the natural external medium of the eggs, the egg capsule fluid, is, evidently, also strongly hypotonic. Finally, the behavior of the eggs in sucrose solutions is quite different.

In a 0.15 M sucrose solution, both polar bodies are formed, but, as a rule, the eggs do not cleave; only in some cases a 1st cleavage groove appears. With falling concentration development improves; in 0.07—0.08 M solutions, on an average, a stage of about 30 blastomeres is reached; then, with still weaker solutions development gets worse again (end stage in 0.03 M solutions: ± 8 blastomeres). Hence, the injurious action of sucrose on the eggs differs in its effects from that of urea; probably, the chemical properties of both substances play a part.

6. *The influence of Ca^{++}-ions on cleavage.*

Both in distilled water and in sucrose and urea solutions, the eggs show a remarkable deviation of the normal course of cleavage.

In normal cleavage, the cleavage groove, beginning at the animal pole, extends around the egg to the vegetative side; then, it deepens from all sides, the blastomeres pull apart, until they are entirely separated, remaining in contact in one point only. After that, the blastomeres begin to flatten against each other, with formation of a cell boundary between them; finally, the cleavage groove disappears altogether, the outline of the egg becoming nearly spherical again (fig. 4). Soon thereafter, a lens-shaped cleavage cavity becomes visible between the two blastomeres; its contents is extruded periodically to the exterior, as described by COMANDON & DE FONBRUNE (1935). At the next cleavages, this cycle of events is repeated: deepening of the cleavage grooves, pulling-apart and rounding off of the blastomeres; then, flattening of the blastomeres and disappearance of the cleavage grooves, followed by formation of a wide cleavage cavity.

In eggs cleaving in distilled water or in solutions of sucrose or urea,

the first phases of the division process proceed normally: deepening of the cleavage groove, pulling-apart of the blastomeres. After the daughter

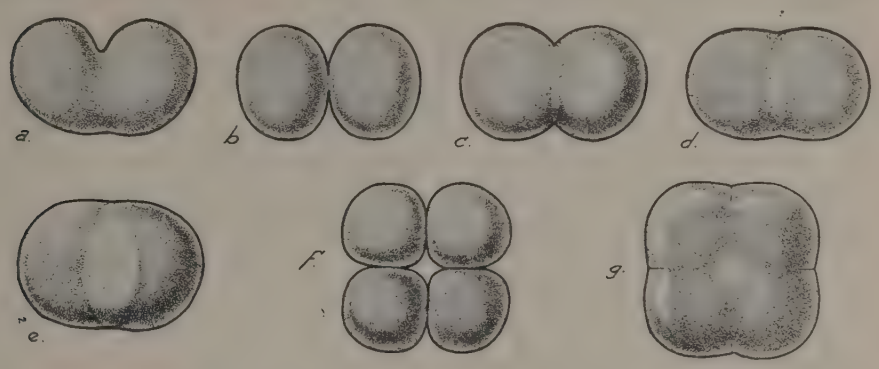

Fig. 4.
Normal cleavage in *Limnaea stagnalis*. *e.*—*c.* First cleavage. *d.*—*e.* Formation of cleavage cavity. *f.* Second cleavage. *g.* Reappearance of cleavage cavity.

cells have maximally rounded off, they beginn to flatten against each other; however, this process stops soon, and the blastomeres remain connected by a rather narrow stalk, the whole egg being dumb-bell-shaped (fig. 5). No cleavage cavity is formed. This state of things lasts

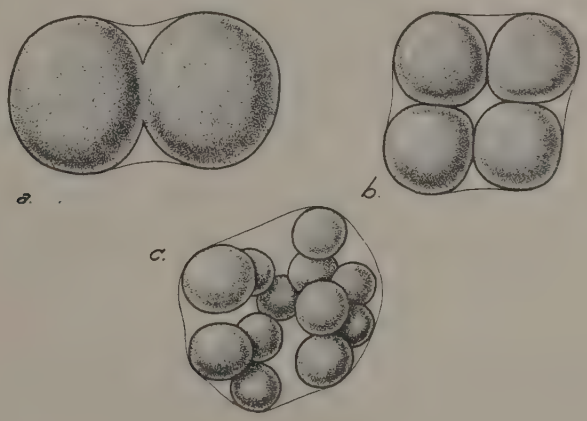

Fig. 5.
Cleavage in $Ca^{\bullet\bullet}$-free media. *a.* 2-cell stage. *b.* 4-cell stage. *c.* Later cleavage stage.

till the next division, which leads to the formation of four contiguous spherical cells, soon losing every connection. When cleavage advances further (in sucrose solutions), each of these cells divides independently in the same way; finally, a loose aggregate of spherical cells is formed.

This mode of cleavage is reminiscent of the abnormal cleavage of sea urchin eggs in $Ca^{\cdot\cdot}$-free sea water, as first described by HERBST (1900). It seemed probable, therefore, that the lack of $Ca^{\cdot\cdot}$-ions in the external medium was the cause of this deviation of cleavage in *Limnaea*, too.

This was proved by the fact that the addition of a small quantity of $CaCl_2$ to the medium leads to normal cleavage. In distilled water with $CaCl_2$, the swelling of the eggs is like that in pure water. Cleavage proceeds normally, the blastomeres flatten against each other, and a cleavage cavity is formed, which is wider than in control eggs in egg capsule fluid. Development stops at the 4-cell stage, but cytolysis is somewhat delayed as compared with eggs in pure distilled water.

In sucrose solutions with addition of $CaCl_2$, the eggs die at the same stage as in the corresponding solutions without $CaCl_2$; but cleavage, as far as it occurs, is normal; the blastomeres remain in connection and cleavage cavities are formed. The minimum concentration of $CaCl_2$, which had an appreciable effect, was 0.005 %; in 0.01—0.08 % $CaCl_2$ the effect is clearly observable.

Apparently, the lack of $Ca^{\cdot\cdot}$-ions brings about a change of the egg surface, by which the coherence of the blastomeres is seriously disturbed. Our observations give some indications as to the point of attack of this influence.

The uncleaved egg is surrounded by a fine plasmatic lamella, the vitelline membrane. In normal cleavage, this membrane remains in contact with the egg surface throughout; it folds inwards with the cleavage grooves.

In eggs cleaving in $Ca^{\cdot\cdot}$-free media, already at the 1st cleavage the vitelline membrane loses its contact with the egg surface in the region of the cleavage groove; it does not fold inwards, but bridges the cleavage groove from one blastomere to the other (fig. 5). At the 4-cell stage, the membrane surrounds tightly the spherical blastomeres. At later stages, it detaches itself still more from the blastomeres, at last forming a loose envelope surrounding them.

It is probable, therefore, that a change in the properties of the vitelline membrane is the primary cause of the abnormal cleavage in $Ca^{\cdot\cdot}$-free media.

Summary.

1. The eggs of *Limnaea stagnalis*, in their egg capsules, show a swelling of 36—56 % between oviposition and 1st cleavage.

2. In distilled water, swelling exceeds considerably that in egg capsule fluid. The eggs reach the 4-cell stage, then they cytolyse.

3. In urea and sucrose solutions, the equilibrium volumes depend upon the molar concentration of the solution. The recently laid eggs are, on an average, isotonic with 0.093 M solutions of non-electrolytes.

4. The average non-solvent volume, as calculated from the osmotic volume changes, is 57 % of the original egg volume. This high value is, probably, due to a resistance to swelling and shrinking by the vitelline membrane.

5. The permeability constant for swelling in distilled water lies close to 2×10^{-7}.

6. In 0.03—0.10 M urea solutions, development stops at the 4-cell stage. With sucrose, in 0.07—0.08 M solutions a stage of about 30 blastomeres is reached; in stronger and weaker solutions, development comes to a standstill at an earlier stage.

7. In Ca^{++}-free media, cleavage has an abnormal course; the blastomeres remain spherical in shape and lose connection at an early stage. Addition of a small quantity of $CaCl_2$ leads to normal cleavage. The primary action of the lack of Ca^{++}-ions seems to be a change in properties of the vitelline membrane, which loses its contact with the egg surface in Ca^{++}-free solutions.

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Physics. — *Hyperboloïdical cooling towers.* By F. K. TH. VAN ITERSON.

(Communicated at the meeting of January 26, 1946.)

Recourse is taken to cooling towers at industrial plants, mainly electric-power stations, where river water in sufficient quantity is deficient. Cooling towers are used for transmitting the heat of the water to the atmosphere so that the water may circulate over and over again.

For this purpose it is pumped on hurdles or fillings of other type, built in the lower part of a wide chimney provided with openings near the bottom. Natural draught is obtained by the warming of the air and the difference (five eights) in specific weight between vapour and air. (Figure 1.)

In this communication we shall not deal with the theory of heat transmission, the physical basis of the apparatus, we shall mainly describe the principles which lead to the adoption of a hyperboloïdical shell for the chimney, but in order to arouse the interest of the reader in these huge engineering structures, we briefly mention their utility.

As long as the use of atomatic energy remains outside his scope of action, the conversion of the combustion heat of coal into power, remains the chief task of the mechanical engineer. The efficiency of this process although much improved during two centuries is still low. The highest theoretical efficiency could be obtained by the CARNOT cycle and is $\eta_c = \frac{T_1 - T_2}{T_1}$ where T_1 is the highest temperature to which the acting medium can be heated and T_2 the temperature of the waste heat. But we generally work with steam engines under the RANKINE cycle which limits the efficiency to $\eta_R = \frac{H_3 - H_2}{H_3 - H_2''}$ in which H_3 is the total heat (enthalpy) of the superheated steam, H_2 the total heat of steam after adiabatic expansion to condensor temperature and H_2'' the heat in the condensed water. (By extracting steam from the turbine and progressive heating of the feed water, the RANKINE cycle may be somewhat improved.)

Working at a pressure of 20 atmospheres and 700° Kelvin (absolute temperature) of the superheated steam and 300° Kelvin in the condensor, the theoretical efficiencies are

$$\eta_c = 57 \ \% \text{ at the CARNOT cycle}$$
$$\eta_R = 34{,}4 \ \% \text{ at the RANKINE cycle}$$

This latter figure would be raised to $\eta_R = 37\ \%$ in summer if, in-stead of working with cooling towers, we should dispose of an ample supply of natural cooling water with about 10° centigrade lower temperature. In winter the difference in temperature is even about 20°. Therefore the use

of cooling towers seems to be prejudicial. But in engineering practice things are often complicated by events of economic order. It is extremely costly to use steam of low temperature and the corresponding high vacuum. If we

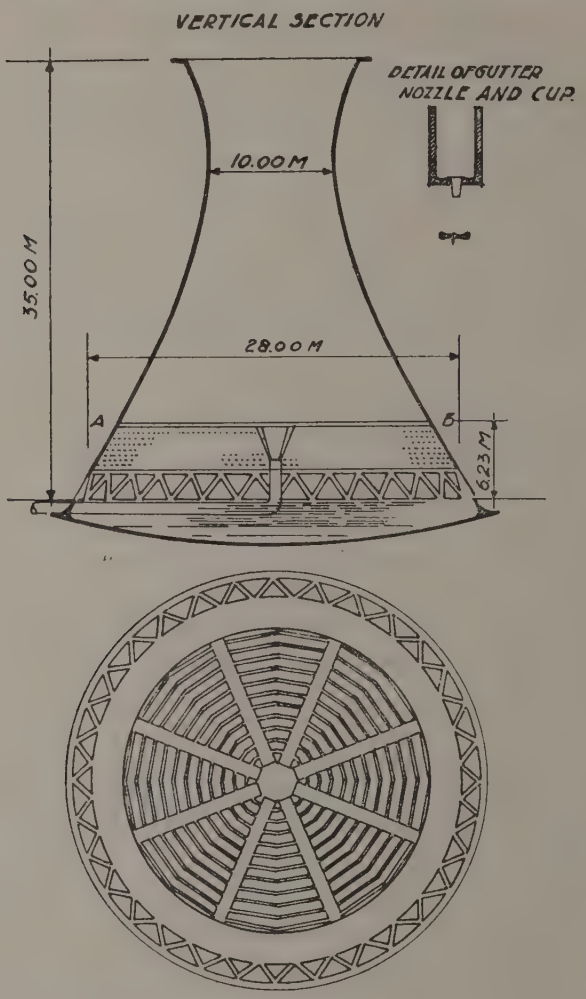

Fig. 1.

reach 20° C, in-stead as for instance 40° C, the volume of the steam increases threefold and the density drops to one third. The economic consequences on the construction and running of the steam turbine are:

1⁰. Some large turbine discs must be added to the low pressure part of the turbine. As the circumferencial velocity of the turbine blades is prescribed by the criterium of safety, the result is, that at a given weight and floor space a greater output can be obtained with warm cooling water than with cold; a rather paradoxical achievement.

2⁰. The largest turbine parts, namely exhaust connection and

condensor are much reduced in size when cooling towers are used. The feed water is warmer.

3°. Mechanical efficiency is improved and blade erosion much lessened.

However, the use of cold cooling water remains an advantage. If possible large electric power stations ought to be errected at lake-sides, river or canal banks. In summer the water temperature may then be about 25° C in stead of 35° to 40° with cooling towers. In practice with cooling towers the steam and coal consumption are about 2 % higher and it depends on factors of quite another magnitude whether preponderance must be given to a dry site for the erection of a power plant. Cheap supply of fuel as on coal or lignite fields and better and cheaper building space are often decisive. Experience in England, for instance Ham's Hall Power Station near Birmingham, has shown that electricity with cooling towers may be produced at the same price or cheaper than with natural water. This has been realized by our invention. Formerly cooling towers were built from timber. This is not only more expensive than concrete, but the cooling effect suffers by leakage and warping. In consequence of putrification many cooling towers were destroyed by a gale within 15 years after their erection. The conditions of humidity and warmth to which these structures are subjected are most deleterious to their preservation. During the many and long periods of inaction for repairs the enormous risk of an unextinguishable fire was so prominent that for this reason alone the application had to be excluded.

We shall now deal with the incentives which led to the construction of the unifoil concrete cooling towers, which in the first place were invented on behalf of the exigencies prevailing at the Dutch State Mines, where the building site is subjected to subsidences. Notwithstanding the preservation of a substantial shaftssafety pillar, the possibility of an irregular subsidence of the surface near the coal pits had to be taken into account.

From experience and theoretical investigations on gasholder frames and other three dimensional frame-work we were aware that mantlelike frames open at the top may follow uneven subsidences of their foundation and we soon were conscious that the same freedom of coactions exists in thin walled unifoil structures open at the top, as we had long ago demonstrated with gasholder tanks.

We experimented with paper and tin-plate models and found our prediction confirmed. In our days of thorough investigation into the strength of unifoil aeroplane wings this deformability has become common knowledge. We read in FLÜGGE, Statik und Dynamik der Schalen p. 56: "is the shell open at its top, then we may prescribe two of the displacements u, v and w at one edge, or one at each edge (u is the component of displacement along a circle of latitude, v along a meridian and w the displacement normal to the shell).

That our assumptions were justified has been proved in a remarkable manner. Figure 3 shows a cooling tower at mine Emma severely deformed

by soil subsidence. It is the second cooling-tower from figure 4. As the reinforcement of the concrete can freely bend and is not subjected to pull or compression in the sence of the shell surface, the concrete is only cracked but the structure presents no danger and still resists the strongest gales.

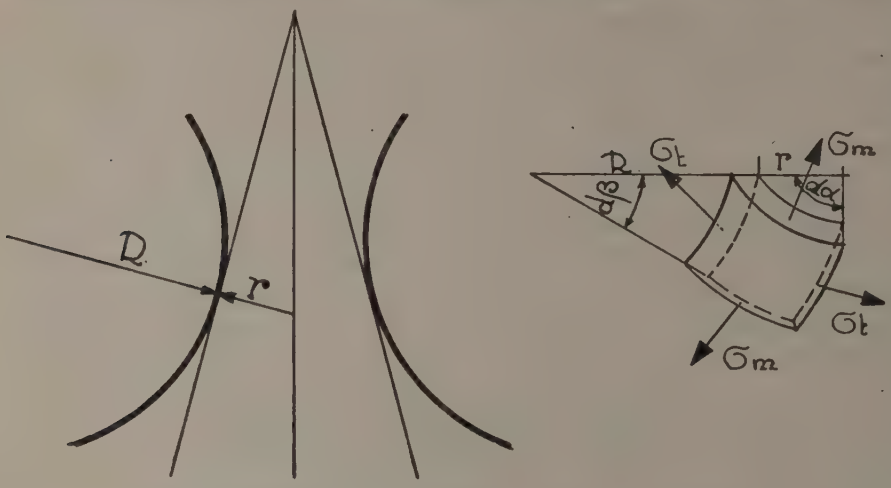

Fig. 2. Equilibrium of main tensions σm and σt in thin wall.

Another reason for the adoption of a thin shell of reinforced concrete, without buttresses, beams or ribs was our desire to realise a shrinkproof construction. In order to prevent the occurrence of shrinkage cracks large structures of reinforced concrete must be built in sections. But the unifoil shell may unevenly shrink to any amount without giving rise to shrinkage stresses.

Not enough emphasis can be laid on the fact that concrete hardening in the air shrinks about $\frac{1}{2}$ mm per m for a mortar of one part of cement on two parts of sand (see Eisenbeton 1926 Graf p. 46). The addition of gravel alleviates the evil, which is one of the gravest preoccupations of manufacturers and users of cement. It is not generally known, that the shrinkage of concrete not only can give rise to large visible cracks detrimental even to the reinforcement but quite generally causing microscopic fisures making illusory the partaking of the concrete in tensile stresses. Moreover the reinforcement cannot be utilised up to the permissible stresses in the iron. The surprising success of prestressed concrete is due to the surmounting of these defaults. But in our construction we had also already conquered these evils in the most simple and natural manner. The cooling towers are gradually built up while the concrete shrinks. The wall of double curvature is so thin, that the structure may be compared to an egg shell or a paper structure, which is stiff as a whole although not resisting to bending stresses in the wall proper. It is clear that at each circle of latitude the compressive stress along the meridian carries the overlaying weight of the structure. This compensates the shrinkage stresses and puts

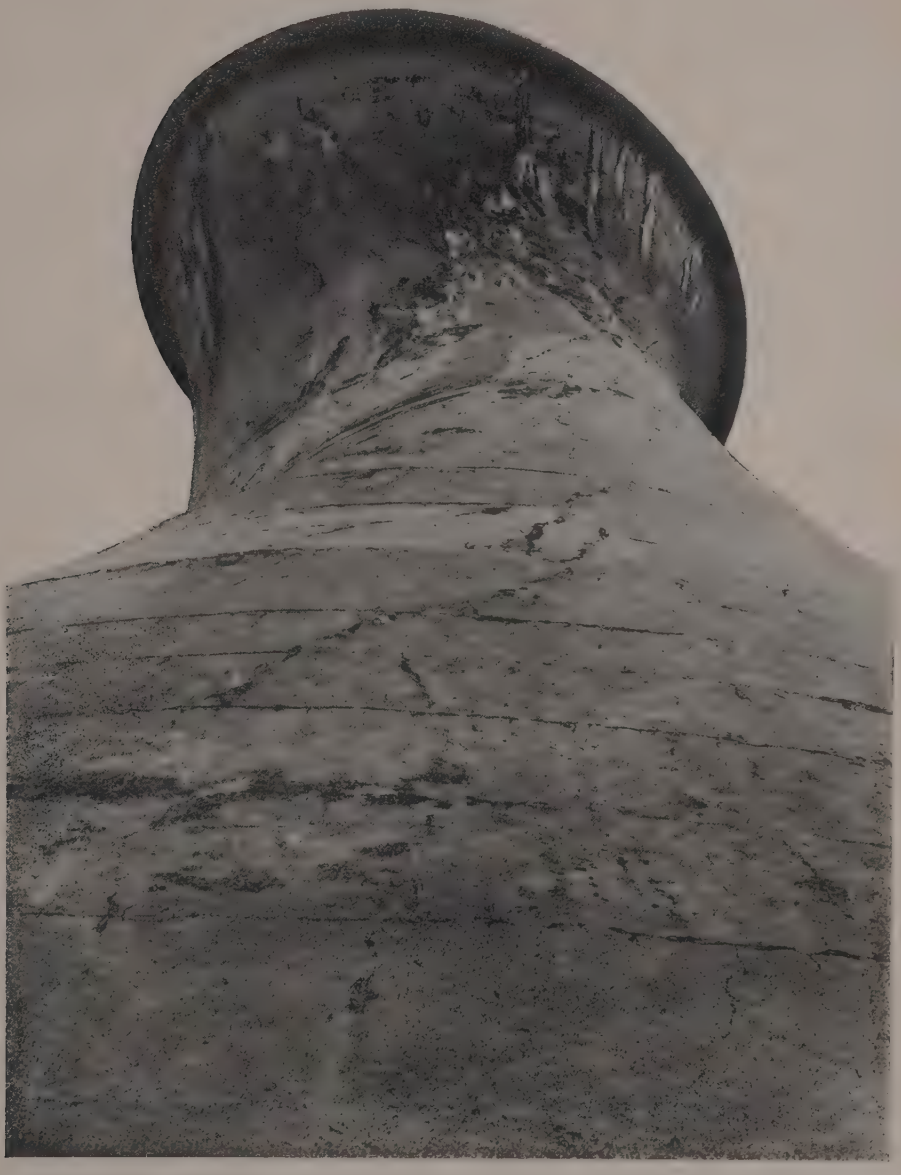

Fig. 3. Cooling tower at coal-pit Emma deformed by soil subsidence.

Fig. 4. The second cooling tower has been deformed to the shape represented in fig. 3.

more load on the reinforcement. But the most particular property of the construction is, that also in the other principal direction i.e. perpendicular to the meridian and therefore in every sense in the plane of the wall we only meet compressive stresses to which our material resists in the most remarkable way. We have now to explain this rather unique disposition of the hyperboloïdical shell.

Let r in figure 2 denote the radius of curvature of the shell normal to the meridian and R the curvature of the meridian itself. Compared to the stresses the air pressure on the wall may be neglected. Let σ_m and σ_t stand for the meridional and the tangential stress in the thin wall, then the equilibrium in the direction normal to the wall for an element $r\,d\alpha\,R\,d\varrho$ and a wall thickness t prescribes

$$\sigma_t R\,d\beta\,dat = \sigma_m\,r\,da\,d\beta\,t \text{ or } \sigma_t = \frac{r}{R}\,\sigma_m.$$

We first calculate σ_m. The main stress in the wall results from its weight above the section under consideration. The computation is done with the expression $\sigma_m \times 2\pi r \cos^2\varphi\, t = W$. φ denotes the angle between tangent and vertical. And σ_m being a compressive stress so is also σ_t. This is really a remarkable result of the hyperboloïdical shape, for in domes and egg-shaped shells σ_t and σ_m are of opposite sign, which reduces considerably the permissible load. In a spherical dome $\sigma_t = -\sigma_m$.

But our construction shows other merits. Although of minor importance we must now proceed to the consideration of the stresses by wind pressure. We have for 40 years, since the first days of aerodynamical investigations submitted our largest structures to model tests in wind-tunnels. But for our cooling towers such tests were considered as superfluous. We know enough of the repartition of the windpressure by our gasholder experiments [1]). As a first approximation we consider the windpressure as a sine function of the centerangle, i.e. pressure at luff and suction at lee. If desired we can add further terms after harmonic analysis of the distribution of the wind-pressure. All the stresses vary according to the same sine-laws.

The main outline of our construction was conceived after a study of EIFFEL's famous lecture in the French Institution des Ingénieurs Civils on his tower of 300 meters. We also applied the underlying principle in a project of masts 300 m high proposed during the first world war to be built as thin walled steel shells of circular section intended for wire-less communication with the dutch East Indies [2]). This principle may be expressed as follows (see figure 5): If the center of wind-pressure on the part of the tower above the section under consideration, is situated at the intersection of the two tangents to the meridian the section is exempt of shearing stresses because

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[1]) Winddruk op cylinders in het algemeen en op gashouders in het bijzonder. De Ingenieur 1930 No. 27. Bouw en Waterbouwkunde 10.

[2]) Stresses in thin shells of circular section Engineering Nov. 14, 1919.
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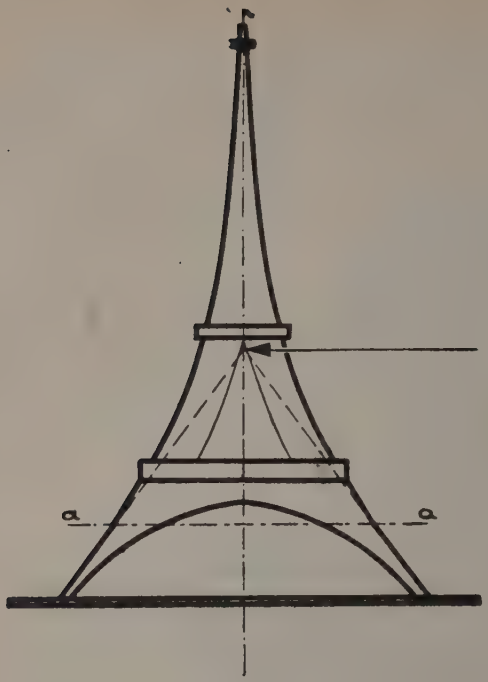

Fig. 5. EIFFEL's shearless construction.

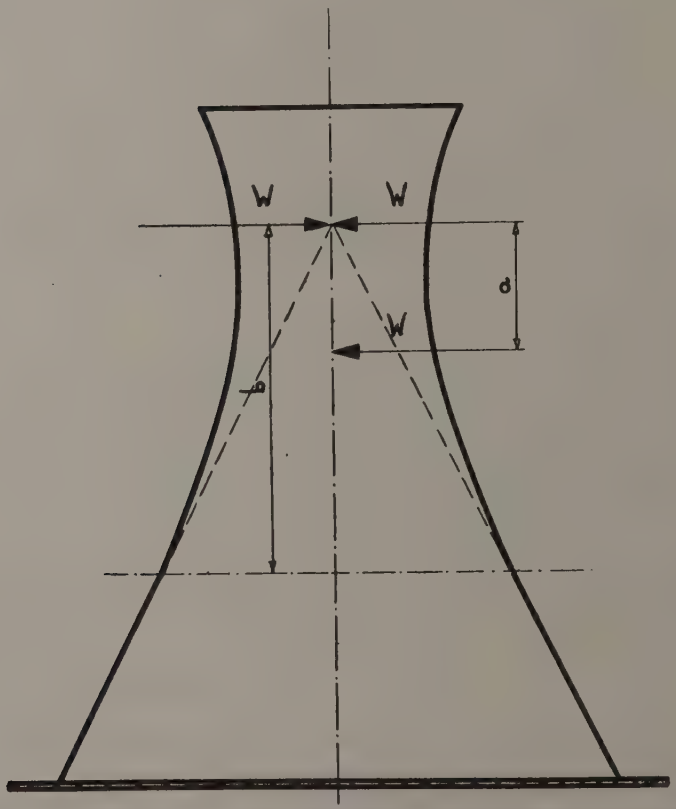

Fig. 6. Computation of meridian Tension σm and shearstress τ in thin wall.

all the meridional stresses act in this point of intersection. In the case of figure 6 representing our cooling tower the conditions are only slightly different.

The center of windpressure W is in the under part of the tower a small distance a lower than the point of intersection of the tangents. With the notations given in the figure we have to calculate with the horizontal force $W(1-\frac{a}{b})$ acting in the point of intersection and with the shearing force $W\frac{a}{b}$ acting in the section under consideration. The lower part of the shell is by far the most ponderous. Here a is almost equal to b, so that the shearing stresses are negligible. But also the circumference and the wall thickness are such that the meridional stresses which counteract the horizontal force $W(1-\frac{a}{b})$ are very small. The meridional compressive stresses due to the weight are the only charges of some importance. But as already mentioned, reinforced concrete, especially that which is built up under a growing load and compressed in both main directions is very resistant to this kind of stress. The result of the accurate strength computation of our cooling towers is that our aim is overreached. For practical reasons we cannot make the shell thin enough to make full use of the strength of the chosen material of construction. Mr. CHR. GROOTHOFF, civil engineer of the State Mines, who took a great part in the development of our cooling towers suggested to build the thin shell of masonry, and indeed this would be possible. The reason which withheld us from executing this proposition was our fear of deterioration by frost of our somewhat porous brickwork always thoroughly wetted by a continual internal shower.

With regard to the foregoing remark we must mention a peculiar consequence of the laws of similitude applied to cooling towers. At comformable enlargement of the structure the stresses due to windpressure, both tension and compression stresses remain the same. The far more important compression stresses due to weight increase in proportion to the height of the cooling tower but are independent of wall thickness and even for the highest towers they remain small. The higher these stresses are the better. The largest cooling towers, several of which have been built (1) have a capacity of 5 million gallons = 22,500 m³/h, height 320 ft = 98 m; diameter at ground level 220 ft = 67 m. These apparatus are the best in every respect and a further enlargement will be profitable. The reason for this is that the wall-thickness needs not to increase conformably and the larger units are the cheapest per million of gallons of circulation. The choice of the wall-thickness is a matter of experience. The reinforcement is only wanted in order to prevent cracking due to local irregularities in shrinkage.

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1) Engineering, Febr. 1, 1946, p. 97.

For a full understanding of the conditions to which the concrete shell is subjected, we must also pay some attention to temperature stresses. Although considered as a whole the unifoil is free of such stresses which property may be considered as its chief merit, so that it may be exposed to frost at the shadowed side and be more than handwarm on the sunny side, there are conditions for which the temperature stresses must be taken into account.

When during a severe frost the air openings become partly clogged by ice, temperature differences of the order of 30° may arise between inner- and outer surface of the wall. But the state of compression reigning in the wall is extremely beneficial in this case and it is a fact that the cost of upkeep during a quarter of a century has been nil for concrete towers.

We have hardly to mention that the reinforcement of the concrete consists of straight bars placed according to the descriptive lines of the hyperboloïd. During erection the exact shape of the shell is simply controlled by a string.

The buckling strength of cylindrical chimneys can be chequed by computation, but this theory has hitherto not been extended to double curved shells. Although rather superfluous, we took the precaution of experimenting the strength to buckling by a tin plate model, nailed to a timber scaffold and loaded by means of a strong string round its neck.

We could not produce any deflection under a load surpassing largely the highest possible buckling pressure equivalent to wind load.

The hyperboloïdical shape, the smoothness of the interior, the absence of crossbeams prevail in the performance of the apparatus. Every aero-dynamical expert will acknowledge that we have realised the conditions for potentical flow, i.e., the streamlines converge and diverge evenly. The thoroughflow in the stack is free from eddies and vortexes, there is no loss of energy.

The external smoothness reduces the windpressure to less than a half of what it should be on a cooling tower of the usual rectangular section and the same vertical area of projection.

As these large structures would predominate in the silhouette of the industrial plants for which they are intended, attention had to be paid to their becoming appearance. Our aesthetical expert, the architectural engineer W. F. FONTEIN, had to make perspective sketches of our projected power stations and everybody was pleased by the prospect. We came to the conclusion that large but simple mathematically shaped bodies like pyramids, spheres, cones, hyperboloïds give satisfaction to the eye. To day hundreds of these towers built on an ever increasing scale are erected all over the world and are generally admired. In our mining district they became an emblem of the industry.

When the idea of constructing these simple shells free to contract and to deform in case of subsidence was first presented for execution, the prominent contractors of reinforced concrete works refused to tender. They

lacked confidence in the righteousness in our contrivance for meeting shrinkage and setting stresses totally differing from their conceptions.

An enterprising and very competent friend of ours, the civil engineer G. KUYPERS without hesitation built the first one in 1918 and took an active part in the development of design, shuttering, scaffolding and other details of execution. His assistent and later successor Ir. H. H. KUPERS is still foremost in this branch of engineering in our country. KUYPERS in later years was worried by the patent law cases fought in England against infringement on of our patents under an absurd and costly system of procedure. In that country it was the firm L. G. MOUCHEL and Partners, who in the early stage of their application have shown full confidence in the exactness of our assertions. They enjoyed the benefit of their enterprise and built not only the many hyperboloïds which are caracteristic of the English inland power stations, but in our days they charm us by pictures of large scale application of our invention in France, Belgium, India, Germany, Egypt, Manchuria, Iran, Italy, Morocco, Rumania, South Africa and by the proofs of their lasting friendship.

Appendix.

As a posthumus homage to Mr. G. KUYPERS civil engineer, the talented designer and contractor of reinforced-concrete constructions we want to divulge his amendment to our above mentioned formula.

We had neglected the proper weight of the element in computing the equilibration of stresses. If we take this into account calling the specific weight of the reinforced concrete γ, we find for the equilibrium in the direction of the normal to the element (see figure 7).

$$\sigma_m t r d\alpha d\beta - \sigma_t t R d\beta d\alpha + \gamma t r d\alpha d\beta \sin \varphi = 0$$

or

$$\sigma_t = \sigma_m \frac{r}{R} + \gamma r \sin \varphi$$

for the part of the cooling below the neck and

$$\sigma_t = \sigma_m \frac{r}{R} - \gamma r \sin \varphi$$

for the part above the neck. At the top of the cooling tower, where σ_m is unimportant we find a very small tension stress in stead of pressure in every direction. And also when R becomes very large compared to r it is clear that the correction, although small, surpasses the first term of σ_t.

The same story may be told in regard with local windpressure w per unit of surface. Let σ_m be the meredional stress which can be easily calculated as exposed before.

Then

$$\sigma_t = \sigma_m \frac{r}{R} \pm w \frac{r}{t}.$$

The additional term although very small may be larger than the first. But compared to the incalculable shrinkage, subsidency and temperature stresses it is insignificant.

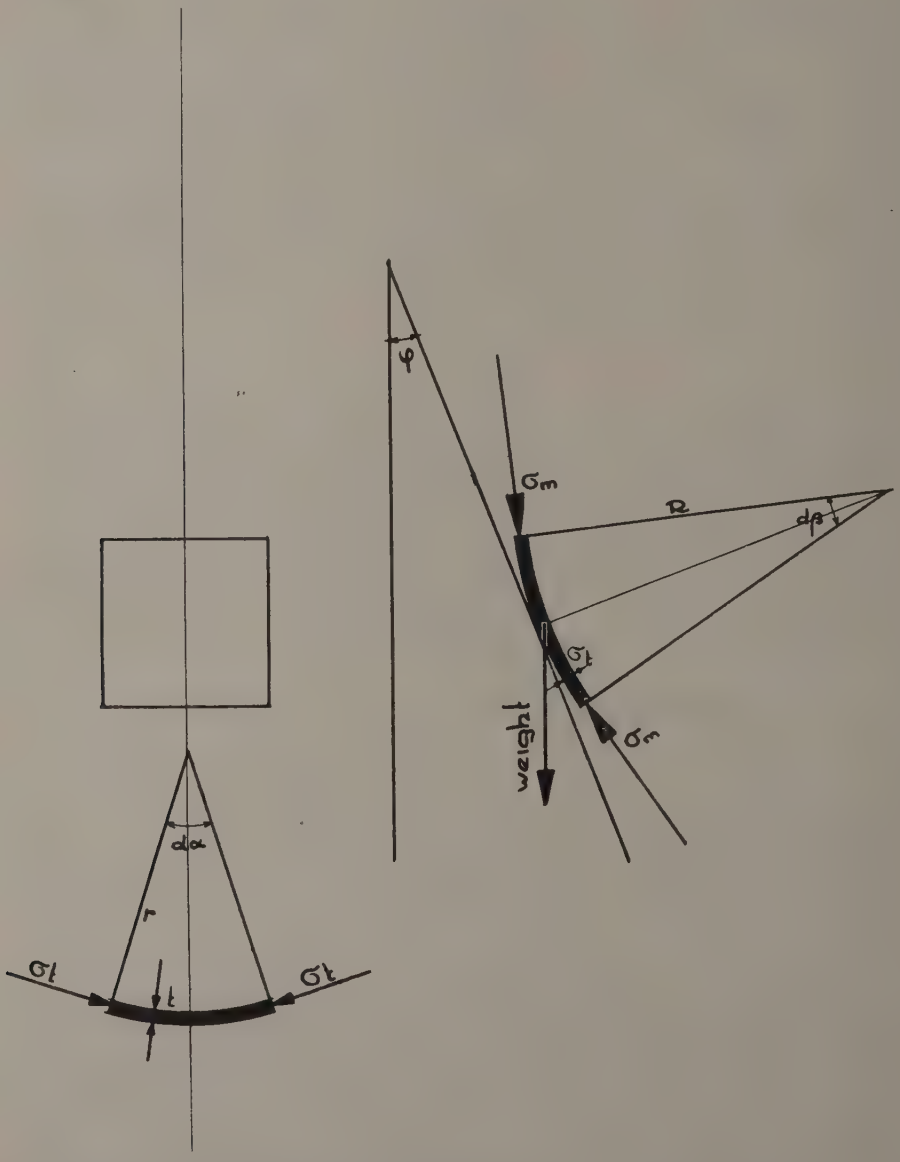

Fig. 7. Equilibrium of stresses and weight for an element of tower-wall.

Mathematics. — *On Unities and Dimensions.* I. By H. B. DORGELO and J. A. SCHOUTEN.

(Communicated at the meeting of January 26, 1946.)

§ 1. *Introduction.*

In the following we deal with some questions, discussed often in the literature of the last decennaries and in several committees:

I. Is it allowed to look at electric field strength $\overline{F}$ and electrical displacement $\overline{D}$ and in the same way at magnetic fieldstrength $\overline{H}$ and magnetic induction $\overline{B}$ as "essentially identical" quantities?

II. Can it be recommended to call B the magnetic fieldstrength and to look upon H (in the same way as $\overline{D}$) only as an auxiliary quantity for calculating purposes?

III. Has the rationalized or the not rationalized form of GIORGI's system of units to be propagated?

IV. Have the dimension formulas to be expressed in 3, 4 or 5 fundamental units and which fundamental units are most favourable?

After some preliminary considerations on physical and geometric objects and their absolute and relative dimensions in § 2 and § 3, in § 4 the electromagnetic equations are formulated independent of the choice of the fundamental units and these general equations are specialized for the four usual systems. We will see, that there exist only two methods of rationalization and that one of them (the method of GAUSS) leads to units unfit for use. In § 5 the dimension formulas are formulated independent of the choice of the electromagnetical fundamental units and these formulas are specialized for the four usual systems. From these general formulas we will see that, using 8 fundamental units, it is always possible to formulate dimension formulas valid for every possible system, but that it is practically not profitable to use more than 4 fundamental units. Some examples illustrate the relation between the relative dimensions and the possibilities of measuring. In § 6 the equations are made independent of the coordinate-system and this leads to the geometrical representations of $\overline{F}$, $\overline{D}$, $\overline{H}$ and $\overline{B}$ and to their absolute dimensions. We illustrate with a number of examples how these absolute dimensions are related to the possibilities of the construction of the geometric images. In § 7 the equations are brought on a fourdimensional invariant form. In this form the absolute dimensions only contain electric charge and quantum. § 8 contains a survey of the answers on the questions put in § 1 [1]).

[1]) Cf. for the historical development of electric and magnetic systems of units and the names of electric and magnetic units E. BODEA, Das Kalantaroff-Giorgische Maszsystem (Ch. III); JULIUS WALLOT, „Die Physik in regelmässigen Berichten", 1, 5 and 10; H. B. DORGELO, Handleiding Electriciteit, third edition 1944 (Ch. V).

§ 2. *Physical and geometric objects and their dimensions* [2]).

If physical phenomena are described in a quantitative way, there occur numbers as results of measurings and calculations. To get these numbers it is in general necessary to introduce a *coordinatesystem* and a system of objects for comparison, called *units*. If we have to deal with scalar quantities no coordinatesystem is necessary, and if the measuring can be performed by counting only (f.i. the counting of electrons) no unit has to be introduced. A unit has to be of the same kind as the object to be measured. But it is possible to relate different units by introducing a certain number of *fundamental units*. Then the other units are called *derived units*. For instance the unit of velocity can be defined as the velocity of a point advancing over the unit of length in the unit of time. The choice of the fundamental units is free. Usually the units of mass, length and time are used and often also the unit of electric charge [3]). The numbers obtained in general change if the coordinatesystem or the units or both are changed. Then we see that there exist always sets of numbers with the property that the change of every number of the set depends on the values of the other numbers of the set but not on the values of the numbers of some other set (f.i. the three components of a vector with rotations of the coordinatesystem). The existence of these sets gives rise to the notion of *physical object*, defined as follows:

By a physical object we mean in the following an adjoining of N numbers (the components of the object) to *every* allowable coordinatesystem and *every* system of fundamental units, satisfying the condition that for every choice of two systems of coordinates and fundamental units the numbers belonging to one system can be computed using only the numbers belonging to the other system and the formulas of transformation of the coordinates and the fundamental units. Two physical objects are said to have the same *kind of transformation* if their transformations depend in the same way on the transformation of coordinates and fundamental units.

In the following we use only objects, coordinates and transformations satisfying the conditions:

1. every fundamental unit gets a constant factor, i.e. a factor not depending on the coordinates. This transformation will be called a *transformation of units*;

2. the components of an object can always be chosen in such a way that with a transformation of units each component gets the same constant factor and that this factor is a product of powers of the constant factors occurring in the transformation of units.

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[2]) For the greatest part taken from J. A. SCHOUTEN, Tensoranalysis for physicists, lectures Delft 39—40, opalographed.

[3]) If thermal quantities are considered it is necessary to introduce a fundamental unit of temperature. We do not consider this unit here.

The following remarks will be useful:

1. A set of components can constitute a physical object with respect to certain coordinate transformations and not with respect to other coordinate transformations. For instance the three components of the electric fieldstrength form a geometric object with respect to transformations in x, y, z. But with respect to transformations in x, y, z, t only the six components of electric fieldstrength and magnetic induction together form a physical object.

2. If the fundamental units are not changed we get an adjoining of N numbers to *every* allowable coordinatesystem satisfying the condition that for every choice of two systems of coordinates the numbers belonging to one system can be computed using the numbers belonging to the other system and the formulas of transformation of the coordinates. But this is just the usual definition of a *geometric object*. Hence to every physical object belongs a geometric object whose components get a constant factor with a transformation of units. This geometric object is called the *geometric image* of the physical object. A physical object is not wholly characterized by its geometric image and its behaviour with transformation of units. F.i., using orthogonal coordinates x, y, z and the system of fundamental units of GAUSS, $\bar{D}$ and $\bar{H}$ have the same geometric image and the same transformation of this image with transformation of units but they are essentially different in a physical sense.

3. If p, q and r are components of a geometric object, f.i. $\log p$, q^2 and $\cos r$ are components as well, though they transform in quite another way. This freedom in the choice of the components can be used to choose the components in such a way that their transformation is as simple as possible.

4. If two geometric objects transform in different ways with certain allowable coordinate transformations, it may occur that, after a limitation of the allowable coordinate transformations (f.i. passing from general linear transformations to orthogonal transformations), it becomes possible to choose the components of the two objects in such a way that they have the same transformation. This opens the possibility of *identification* of these geometric objects. This possibility of identification is much (often to much) used. At one hand it gives a simplification but at the other hand many details can get lost.

5. Using n general coordinates $\xi^\varkappa$; $\varkappa = 1, \ldots, n$ in an n-dimensional space the most general transformation of coordinates has the form

$$\xi^{\varkappa'} = \xi^{\varkappa'}(\xi^\varkappa) \; ; \; Det\left(\frac{\partial \xi^{\varkappa'}}{\partial \xi^\varkappa}\right) \neq 0 \; ; \; \varkappa = 1, \ldots, n; \varkappa' = 1', \ldots, n'. \quad . \quad (1)$$

Because of

$$d\xi^{\varkappa'} = A^{\varkappa'}_{\varkappa}(\xi^\varkappa)\, d\xi^\varkappa \; ; \; A^{\varkappa'}_{\varkappa} \overset{\text{def}}{=} \frac{\partial \xi^{\varkappa'}}{\partial \xi^\varkappa}; \; \varkappa = 1, \ldots, n; \varkappa' = 1', \ldots, n' \quad . \quad (2)$$

the transformation of the differentials $d\xi^\varkappa$ is always linear homogeneous.

The simplest geometric objects, defined in any point of the n-dimensional space are those, whose components with respect to (x') are functions of the components with respect to (x) and of the $A^{x'}_{x}$, satisfying the conditions:

a. the functions are linear homogeneous in the old components;

b. they are homogeneous algebraical of a certain degree in the $A^{x'}_{x}$.

Geometric objects of this kind are called *geometric quantities*. If the image of a physical object is a geometric quantity the object is called a *physical quantity*. Most physical objects are physical quantities. The gravitational field, given by the CHRISTOFFEL symbols $\left\{ \begin{matrix} x \\ \mu\lambda \end{matrix} \right\}$ is a specimen of a physical object not being a physical quantity, because in the transformation of the $\left\{ \begin{matrix} x \\ \mu\lambda \end{matrix} \right\}$ not only the $A^{x'}_{x}$ but also their derivatives occur. In the following we deal only with physical quantities.

6. If the fundamental units M, L, T, Q, ... of mass, length, time, electric charge, etc. transform into

$$M' = m^{-1} M; \quad L' = l^{-1} L; \quad T' = t^{-1} T; \quad Q' = q^{-1} Q; \ldots \quad . \quad (3)$$

and if a component of a physical object gets a factor $m^{\alpha} l^{\beta} t^{\gamma} q^{\delta} \ldots$ by this transformation, we say that the *dimension* of this component is $[m^{\alpha} l^{\beta} t^{\gamma} q^{\delta} \ldots]$. The dimension of the components may depend on the choice of the allowable coordinatesystems because it is possible that these coordinatesystems themselves depend on the fundamental unit of length, f.i. if we use ordinary orthogonal coordinatesystems.

The dimension of the components of a physical quantity as defined under (5) is the same for all components if the allowable coordinatesystems are *independent* of the choice of the fundamental units. In fact, the $A^{x'}_{x}$ do not depend on the choice of the fundamental units and interchanging of coordinates is an allowable coordinate transformation. This mutual dimension is called the *absolute dimension* of the physical quantity. It may be called also the *dimension of the geometric image* because it indicates the factor this image gets with transformation of units *).

If orthogonal transformations in x, y, z or in x, y, z, ct (LORENTZ transformations) are used, the dimensions of the components of a physical quantity are all the same (the $A^{x'}_{x}$ not depending on the choice of the fundamental units) but they may differ from the absolute dimension. This mutual dimension is called the *relative dimension* or shortly *dimension* of the physical quantity. For instance the relative dimension of the radiusvector is $[l]$, its absolute dimension $[1]$. In fact, the radiusvector is its own geometric image. The relative dimension of the velocity vector is $[lt^{-1}]$ but its absolute dimension is $[t^{-1}]$. In fact, the geometric image is the way travelled over in the unit of time, provided with an arrow, and this image does not change if the unit of length is changed.

*) Mr. D. VAN DANTZIG informs us that he had already about 1938 the idea to distinguish the dimension of a physical quantity from that of its components.

Every dimension formula, relative or absolute, expresses a relation between the units of different physical quantities and, therefore, gives an indication as to the possibilities of measuring. If the dimension of a component is $[m^\alpha l^\beta t^\gamma q^\delta \ldots]$, this component can be determined if a unit of the same kind, having also the dimension $[m^\alpha l^\beta t^\gamma q^\delta \ldots]$, is given. But such a unit can be constructed if we have at our disposal a certain number of units whose dimensions have $[m^\alpha l^\beta t^\gamma q^\delta \ldots]$ as their product. Now nature itself furnishes certain *natural units*, for instance the velocity of light, dimension $[lt^{-1}]$ or shortly $[\tilde{c}]$, the elementary quantum, dimension $[ml^2t^{-1}]$ or shortly $[\tilde{h}]$ [4]), the charge of the electron, dimension $[q]$, etc. If this natural units are used it is possible to use less artificial fundamental units. For instance an energy, dimension $[ml^2t^{-2}]$ can be measured by using the fundamental units of mass, length and time, or by using the fundamental units of mass and the derived unit of velocity. But because $[ml^2t^{-2}] = [t][E][I]$ it can also be measured by using the fundamental unit of time and the derived units $[E]$ of electromotoric force and $[I]$ of electric current. Because of $[ml^2t^{-2}] = [\tilde{h}t^{-1}]$ it is also possible to use only the fundamental unit of time if the elementary quantum is known. In fact the energy steps in an atom are measured by means of the frequence of the light emitted, dimension $[t^{-1}]$. Because of $[ml^2t^{-2}] = [m\tilde{c}^2]$ it is also possible to use only the fundamental unit of mass, the velocity of light being known. In fact the energy produced by the composition of an atomic nucleus from other nuclei is measured by the mass defect. Because of $[ml^2t^{-2}] = [qE]$ it is also possible to use only the derived unit of electromotoric force, the charge of an electron being known. In fact the energy of α-radiation is often expressed in electron-volts. There are many other examples. Of course from the dimension formula only the theoretical possibility of a certain method of measurement can be deduced, not the practical possibility. For instance it will not be easy to construct an apparatus for the measurement of kilowatthours based on the measurement of a frequence or a mass defect. The absolute dimension, not depending on any coordinatesystem, indicates the units, necessary for the construction of the geometric image of a physical quantity without using any coordinate system. For instance a watch is the only thing necessary to construct the geometric image of a velocity of a point moving without acceleration. Hence the absolute dimension represents an essential property of the physical quantity, existing independent of any system of coordinates.

7. Components with non linear transformation, for instance the polar coordinates of a point, can have different dimensions. These dimensions can easily be derived if the dimensions of the ordinary components are known. They will not be discussed here after.

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[4]) We use $\tilde{c}$ and $\tilde{h}$ instead of c and h to avoid ambiguity later on.

§ 3. *The simplest geometric objects in ordinary space.*

We start from general linear homogeneous transformations of coordinates in ordinary space [5])

$$\xi^{\varkappa'} = A^{\varkappa'}_{\varkappa}\,\xi^{\varkappa}\ ;\ \xi^{\varkappa} = A^{\varkappa}_{\varkappa'}\,\xi^{\varkappa'};\ \ Det\,(A^{\varkappa'}_{\varkappa}) \neq 0;\ \varkappa = 1, 2, 3;\ \varkappa' = 1', 2', 3' \quad (4)$$

with *constant* coefficients $A^{\varkappa'}_{\varkappa}$ and $A^{\varkappa}_{\varkappa'}$. Every coordinatesystem consists of a system of three (oblique) axes through O, each of which is provided with a measuring unit. These measuring units have nothing to do with the unit of length. The measuring unit on the 1-axis is the segment between $\xi^1 = 0$ and $\xi^1 = 1$. In each of the three coordinateplanes there is a measuring unit for areas, the parallelogram determined by the measuring units on the two axes. The measuring unit for volumes belonging to the coordinatesystem is the parallelepiped of the three measuring units on the axes.

The simplest geometric quantities are [6]):

I. A *scalar* s with one component, invariant with (4).

II. A *contravariant vector* $v^{\varkappa}$ with the transformation

$$v^{\varkappa'} = A^{\varkappa'}_{\varkappa}\,v^{\varkappa}\ ;\ \varkappa = 1, 2, 3;\ \varkappa' = 1', 2', 3'. \quad . \quad . \quad . \quad . \quad (5)$$

The figure is an arrow, the components are the projections on the axes, measured by the measuring units. A component f.i. v^1 is $+$ if the projected arrow indicates the $+$-sense of the 1-axis.

III. A *covariant vector* w_{λ} with the transformation

$$w_{\lambda'} = A^{\lambda}_{\lambda'}\,w_{\lambda}\ ;\ \lambda = 1, 2, 3;\ \lambda' = 1', 2', 3'. \quad . \quad . \quad . \quad . \quad (6)$$

The figure is a set of two parallel planes with a (not necessary straight) arrow *through* them. The components are 1: the segments cut out on the axes, measured by the measuring units. A component, f.i. w_1, is $+$ if the sense of the arrow corresponds to the $+$-sense of the 1-axis.

IV. A contravariant bivector $f^{\varkappa\lambda} = f^{[\varkappa\lambda]}$ [7]) with the transformation

$$f^{\varkappa'\lambda'} = A^{\varkappa'}_{\varkappa}\,A^{\lambda'}_{\lambda}\,f^{\varkappa\lambda};\quad \varkappa, \lambda = 1, 2, 3;\ \varkappa', \lambda' = 1', 2', 3'. \quad . \quad . \quad . \quad (7)$$

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[5]) If curvilinear coordinates are used the transformation of the $d\xi^{\varkappa}$ is always linear homogeneous. Hence the definitions of this section hold also if the more general transformations of curvilinear coordinates are used, but they have only significance in every point apart.

[6]) Cf. J. A. SCHOUTEN and D. J. STRUIK, Einführung in die neueren Methoden des Differentialgeometrie, Part I; J. A. SCHOUTEN, Über die geometrische Deutung von gewöhnlichen p-Vektoren und W-p-Vektoren und den korrespondierenden Dichten, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, XLI, 709—716 (1938); J. A. SCHOUTEN and D. v. DANTZIG, On ordinary quantities and W-quantities, Comp. Mathem. VII, 447—473 (1940).

[7]) The symbol [ ] indicates "alternation", f.i.

$$f^{[\varkappa\lambda]} = {}^1\!/_2\,(f^{\varkappa\lambda} - f^{\lambda\varkappa})$$

$$P^{[\varkappa\lambda\mu]} = {}^1\!/_6\,(P^{\varkappa\lambda\mu} + P^{\lambda\mu\varkappa} + P^{\mu\varkappa\lambda} - P^{\varkappa\mu\lambda} - P^{\mu\lambda\varkappa} - P^{\lambda\varkappa\mu}).$$

There are three independent components f^{23}, f^{31}, f^{12}. The figure is a part of a plane with a sense of rotation *in* it. The components are the projections of the figure on the coordinateplanes, measured by the measuring units for areas. A component f.i. f^{23} is + if the projected sense of rotation correspondents to the sense from 2 to 3.

V. A *covariant bivector* $h_{\lambda\varkappa} = h_{[\lambda\varkappa]}$ with the transformation

$$h_{\lambda'\varkappa'} = A^{\lambda}_{\lambda'} A^{\varkappa}_{\varkappa'} h_{\lambda\varkappa} \; ; \; \varkappa, \lambda = 1, 2, 3; \varkappa', \lambda' = 1', 2', 3'. \quad . \; . \quad (8)$$

There are three independent components h_{23}, h_{31}, h_{12}. The figure is a straight tube of infinite length with a sense of rotation *round* it. The components are 1: the sections of the tube with the coordinateplanes, measured by the measuring units for areas. A component, f.i. h_{23} is + if the sense of rotation corresponds to the sense from 2 to 3.

VI. A *contravariant trivector* $p^{\varkappa\lambda\mu} = p^{[\varkappa\lambda\mu]}$ with the transformation

$$p^{\varkappa'\lambda'\mu'} = A^{\varkappa'}_{\varkappa} A^{\lambda'}_{\lambda} A^{\mu'}_{\mu} \, p^{\varkappa\lambda\mu} \; ; \; \varkappa, \lambda, \mu = 1, 2, 3; \varkappa', \lambda', \mu' = 1', 2', 3'. \quad (9)$$

There is only one independent component, p^{123} with the transformation

$$p^{1'2'3'} = \Delta \, p^{123} \; ; \; \Delta = Det\,(A^{\varkappa'}_{\varkappa}). \quad . \; . \; . \; . \; . \; . \quad (10)$$

The figure is a part of space with a screw sense. $|p^{123}|$ is the volume, measured bij the measuring unit for volumes. p^{123} is + if the screw sense is the same as the screw sense fixed by the axes 1, 2, 3 in this order.

VII. A covariant trivector $q_{\mu\lambda\varkappa} = q_{[\mu\lambda\varkappa]}$ with the transformation

$$q_{\mu'\lambda'\varkappa'} = A^{\mu}_{\mu'} A^{\lambda}_{\lambda'} A^{\varkappa}_{\varkappa'} \, q_{\mu\lambda\varkappa} \; ; \; \varkappa, \lambda, \mu = 1, \ldots, 3; \varkappa', \lambda', \mu' = 1', 2', 3'. \quad (11)$$

There is only one independent component q_{123} with the transformation

$$q_{1'2'3'} = \Delta^{-1} q_{123}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

The figure is a part of space with a screwsense. $|\, q_{123} \,|$ is 1: the volume, measured as above. q_{123} is + if the screwsense is the same as the screwsense fixed by the axes 1, 2, 3 in this order.

VII'. A (WEYLian) *density* $\mathfrak{q}$ *of weight* + 1 with the transformation

$$\overset{(\varkappa')}{\mathfrak{q}} = |\Delta|^{-1} \overset{(\varkappa)}{\mathfrak{q}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

The figure is a part of space *without* a screwsense but with a +- or —-sign. $|\overset{\varkappa}{\mathfrak{q}}|$ is 1: the volume measured as above. The sign of $\mathfrak{q}$ is the sign of the quantity.

VI'. A *density* $\mathfrak{p}$ of weight — 1 with the transformation

$$\overset{(\varkappa')}{\mathfrak{p}} = |\Delta| \overset{(\varkappa)}{\mathfrak{p}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

The figure is a part of space *without* a screw sense but with a +- or

—-sign. $|\overset{x}{\mathfrak{p}}|$ is the volume measured as above. The sign of $\mathfrak{p}$ is the sign of the quantity.

V'. A *contravariant vectordensity* $\mathfrak{h}^x$ *of weight* + 1 with the transformation

$$\mathfrak{h}^{x'} = |\Delta|^{-1} A^{x'}_{x} \mathfrak{h}^x \; ; \; x = 1, 2, 3; x' = 1', 2', 3' \quad . \quad . \quad . \quad (15)$$

The figure is a straight tube of infinite length with a sense *in* the direction of the tube. The components are determined as under V. A component, f.i. $\mathfrak{h}^1$, is + if the sense corresponds to the +-sense of the 1-axis.

IV'. A *covariant vectordensity* $\mathfrak{f}_\lambda$ *of weight* — 1 with the transformation

$$\mathfrak{f}_{\lambda'} = |\Delta| A^{\lambda}_{\lambda'} \mathfrak{f}_\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The figure is a part of a plane with a (not necessary straight) arrow *through* it. The components are the projections of the figure on the coordinate planes, measured as under IV. A component f.i. $\mathfrak{f}_1$ is + if the sense of the arrow corresponds to the +-sense of the 1-axis.

III'. A *contravariant bivector density* $\mathfrak{w}^{x\lambda} = \mathfrak{w}^{[x\lambda]}$ *of weight* + 1 with the transformation

$$\mathfrak{w}^{x'\lambda'} = |\Delta|^{-1} A^{x'}_{x} A^{\lambda'}_{\lambda} \mathfrak{w}^{x\lambda} \; ; \; x, \lambda = 1, 2, 3; x', \lambda' = 1', 2', 3'. \quad . \quad (17)$$

There are three independent components, $\mathfrak{w}^{23}$, $\mathfrak{w}^{31}$, $\mathfrak{w}^{12}$. The figure is a set of two parallel planes with a sense of rotation *in* them. The components are 1: the segments cut out on the axes, the segments being measured as under III. A component, f.i. $\mathfrak{w}^{23}$ is + if the sense of rotation corresponds to the sense from 2 to 3.

II'. A *covariant bivector density* $\mathfrak{v}_{\lambda x} = \mathfrak{v}_{[\lambda x]}$ of weight — 1 with the transformation

$$\mathfrak{v}_{\lambda' x'} = |\Delta| A^{\lambda}_{\lambda'} A^{x}_{x'} \mathfrak{v}_{\lambda x} \; ; \; x, \lambda = 1, 2, 3; x', \lambda' = 1', 2', 3'. \quad . \quad (18)$$

There are three independent components, $\mathfrak{v}_{23}$, $\mathfrak{v}_{31}$, $\mathfrak{v}_{12}$. The figure is a segment of a straight line with a sense of rotation *round it*. The components are the projections on the axes, measured as under II. A component, f.i. $\mathfrak{v}_{23}$ is + if the sense of rotation corresponds to the sense from 2 to 3.

I'. A *contravariant trivector density* $\mathfrak{S}^{x\lambda\mu} = \mathfrak{S}^{[x\lambda\mu]}$ of weight + 1 and a covariant trivector density $\mathfrak{s}_{\mu\lambda x} = \mathfrak{s}_{[\mu\lambda x]}$ of weight — 1 and the transformation

$$\mathfrak{S}^{x'\lambda'\mu'} = |\Delta|^{-1} A^{x'}_{x} A^{\lambda'}_{\lambda} A^{\mu'}_{\mu} \mathfrak{S}^{x\lambda\mu} \quad ; \quad x, \lambda, \mu = 1, 2, 3; \quad . \quad . \quad (19)$$

$$\mathfrak{s}_{\mu'\lambda' x'} = |\Delta| A^{\mu}_{\mu'} A^{\lambda}_{\lambda'} A^{x}_{x'} \mathfrak{s}_{\mu\lambda x} \quad ; \quad x', \lambda', \mu' = 1', 2', 3' \quad . \quad . \quad (20)$$

There is only one independent component, $\mathfrak{S}^{123}$ or $\mathfrak{s}_{123}$ respectively, with the transformation

$$\mathfrak{S}^{1'2'3'} = \Delta |\Delta|^{-1} \mathfrak{S}^{123} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\mathfrak{s}_{1'2'3'} = \Delta^{-1} |\Delta| \mathfrak{s}_{123}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

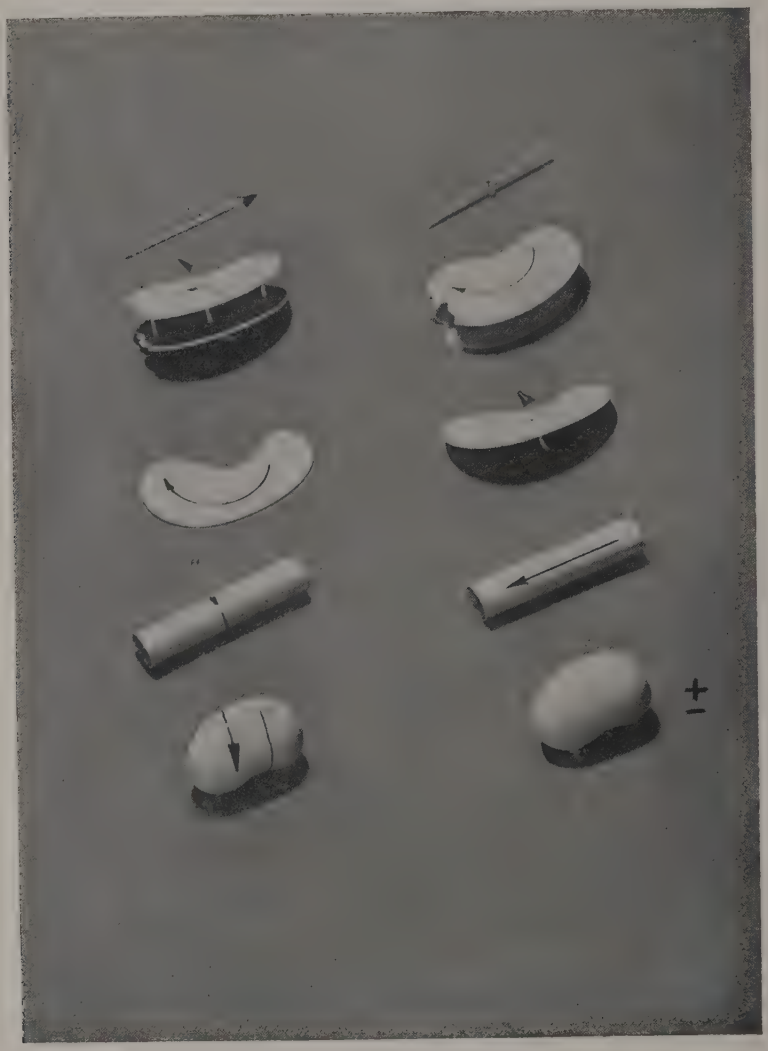
II $v^{\varkappa}$

III w_{λ}

IV $f^{\varkappa\lambda}$

V $h_{\lambda\varkappa}$

VI $p^{\varkappa\lambda\mu}$
VII $q_{\mu\lambda\varkappa}$



$\mathfrak{v}_{\lambda\varkappa}$ II′

$\mathfrak{w}^{\varkappa\lambda}$ III′

$\mathfrak{f}_{\lambda}$ IV′

$\mathfrak{h}^{\varkappa}$ V′

$\mathfrak{p}$ VI′
$\mathfrak{q}$ VII′

Fig. 1.

For real transformations we have $|\Delta|^2 = \Delta^2$ and that implies that the transformations (21) and (22) are the same and that they express that the component only changes if a right system is transformed into a left system and v.v. and that in that case the component gets only a factor -1. Such a quantity is often called a *pseudoscalar*. The figure is a screwsense with a number. The pseudoscalar with the value $+1$ for right systems and -1 for left systems may be denoted by $\tilde{\omega}$.

From fig. 1 we see that the quantities II, III', IV, V', VI and VII have an *inner* orientation and II', III, IV', V, VI' and VII' an *outer orientation*. If the quantities to the left are multiplied with $\tilde{\omega}$ we get algebraically another kind of quantities $\tilde{v}^x$, $\tilde{w}_\lambda$, $\tilde{f}^{x\lambda}$, $\tilde{h}_{\lambda x}$, $\tilde{p}^{x\lambda\mu}$ and $\tilde{q}_{\mu\lambda\nu}$ having in their transformation formulas an additional factor $|\Delta|\Delta^{-1}$. We call them *pseudovectors, -bivectors* and *-trivectors*. Geometrically the multiplication with $\tilde{\omega}$ means the change of an inner orientation to an outer one and vice versa. Hence we have got no new quantities but only another kind of notation for the quantities to the right. In the same way by multiplying the quantities to the right with $\tilde{\omega}$ we get the quantities $\tilde{\mathfrak{v}}_{\lambda x}$, $\tilde{\mathfrak{w}}^{x\lambda}$, $\tilde{\mathfrak{f}}_\lambda$, $\tilde{\mathfrak{h}}^x$, $\tilde{\mathfrak{q}}$ and $\tilde{\mathfrak{p}}$ having in their transformation formulas Δ instead of $|\Delta|$. We call them *pseudo-densities* (non-WEYLian densities). They represent another kind of notation of the quantities to the left. The pseudo trivector densities $\tilde{\mathfrak{E}}^{x\lambda\mu}$, $\tilde{\mathfrak{e}}_{\mu\lambda x}$ of weight $+1$ and -1 respectively, defined by the equations

$$\tilde{\mathfrak{E}}^{123} = 1 \quad ; \quad \tilde{\mathfrak{e}}_{123} = 1 \qquad \ldots\ldots\ldots \quad (23)$$

with respect to *every* coordinatesystem, have geometrically the character of the scalar 1. They can be used to pass from one notation for the components of a quantity to the other notation. For instance the contravariant pseudovector $\tilde{v}^x$ with the transformation

$$\tilde{v}^{x'} = |\Delta|\,\Delta^{-1} A^{x'}_x \tilde{v}^x \qquad \ldots\ldots\ldots \quad (24)$$

passes by transvection with $\tilde{\mathfrak{e}}_{\mu\lambda x}$ into a covariant bivector density of weight -1:

$$\mathfrak{v}_{\mu\lambda} = \tilde{\mathfrak{e}}_{\mu\lambda x} \tilde{v}^x \qquad \ldots\ldots\ldots \quad (25)$$

According to (23) this equation expresses that the components of $\mathfrak{v}_{\mu\lambda}$ and $\tilde{v}^x$ are numerically the same: $\mathfrak{v}_{23} = \tilde{v}^1$ [8]).

[8]) It is often convenient to use pseudovectors instead of densities f.i. in the theory of the wavefunctions of the mesonfield. Cf. J. DE BOER, Quantummechanica der vrije deeltjes, Ned. Tijdschr. v. Nat. XI (1944), p. 122.

(To be continued.)

Medicine. — *Clinical and pathological-anatomical recovery from meningitis.* By A. DE KLEYN.

(Communicated at the meeting of January 26, 1946.)

In the manuals and in the communications about meningitis it is stated more than once that in this disease there is often no parallelism between the subjective complaints and the objective symptoms during life on the one hand and the pathological-anatomical anomalies on the leptominges on the other hand. It is meant then that sometimes small pathological-anatomical anomalies are seen in patients who had very serious symptoms when alive and inversely, that sometimes serious pathological-anatomical disorders can be attended with remarkably few complaints during life. It is therefore very doubtful whether in patients who seem to have recovered and so in whom the subjective and objective meningeal complaints and disorders have disappeared, one may also speak of a pathological-anatomical recovery.

Only a great accident can give the clinical investigator an opportunity to answer this question to a certain extent. For this is the case if a patient, who seems just to have recovered from a meningitis, suddenly dies of another illness, if obduction can be carried out in this patient and the disease which had death at a consequence cannot itself exert influence on the condition of the cerebral membranes.

Such a situation presented itself in the Amsterdam Otological clinic. A female patient of 79 got an otogeneous meningitis, from which it seemed that she had completely recovered. When she was already sitting up outside her bed, she suddenly became oppressed and died after about 10 minutes. At post-mortem examination it appeared that death had been caused by a pulmonary embolus, which had loosed itself from a thrombus in the vena iliaca and vena femoralis. Dissection of the brain was possible, so that the cerebral membranes could be examined more in detail microscopically. The clinical history of this patient runs as follows:

Patient A., 79 years old, was admitted into our clinic on March 27th, 1941. She had already been nursed there in December 1940. Then for 8 days she had had a pain in the right ear without discharge, without a feeling of giddiness and without fever.

Though at the examination only a light congestion of the right tympanic membrane could be found, the whispering voice on this side was only perceived ad concham. The tuning-fork picture pointed to a combined middle ear-labyrinth affection: raising of lower and regression of upper tone-limit, c 5 heard strongly shortened, RINNE negative, SCHWABACH shortened, WEBER localisation to the right, diseased side.

On the left the whispering voice was heard at 4.5 m. Here the tuning-fork picture pointed to a pure labyrinthine anomaly.

Internally no anomalies were found, nor at the examination of urine, blood, etc. Only the sedimentation rate of the blood was increased: 80 mm in the first hour. The fundi looked normal.

On January 10th 1941 patient went home at her own request, as she felt completely recovered. There, 8 days before her second admission, she got again a pain in the right ear, which began to discharge considerably. She also became giddy when getting up, but she was not troubled with it when lying in bed. She had vomited once at home. She had also complained lately of a pain felt when walking.

Internal examination: Here no anomalies were found. The blood-pressure was 150/100, the urine contained a trace of albumen, no reduction. The blood-picture was quite normal.

Neurological examination: Here no special anomalies came to light either, viz. the reflex of KERNIG, neck-stiffness and abnormal reflexes were absent.

Oto-rhino-laryngological examination: The right tympanic membrane showed a perforation down in front; there was a distinct fetid secretion from that ear. The left tympanic membrane was only a little retracted. Rhino- and laryngologically no anomalies were found.

Acoustic: The acoustic picture had not changed much since December, only the hearing on the left had dropped to 0,5 and the experiment of WEBER showed now lateralisation to the left ear.

Vestibular: At the vestibular examination no fistula-symptom was found, but there was a spontaneous nystagmus horizontally to the left, both on looking to left and on looking straight forward (nystagmus IId degree).

The two labyrinths could be stimulated both with cold and with hot water, i.e. that when the right auditory canal was syringed with cold and the left with hot water the spontaneous nystagmus was distinctly strengthened, while when the right ear was syringed with hot and the left with cold water the direction of the spontaneous nystagmus changed, so that a horizontal nystagmus to the right was observed.

On March 28th, 1941, with high temperature, a mastoid operation was carried out under local anaesthesia. The cells of the mastoid appeared to be filled with pus and granulations, the sinus, which just came to lie bare, looked normal.

At the bacteriological examination the pneumococcus mucosus was cultivated from the pus.

The following days the temperature remained high (39—40° C). For the rest patient was in a good condition. She only complained of a pain in both legs, of which no cause was found and which was only explained at the obduction.

Repeated neurological examinations (clinic Prof. BROUWER) gave no single indication of a developing meningitis.

On April 1st a light neck-stiffness was stated for the first time, patient was a little dull-headed, the reflexes were somewhat low and it appeared that calorically the two labyrinths could no longer be stimulated, not even with ice-water. In connection with this a lumbar puncture was carried out. The result of the BROUWER clinic ran: reactions of PANDY and NONNE and on glucose positive; cells 1856/3. At microscopic examination of the liquor and by cultivation the pneumococcus mucocus was shown.

After administering dagénan and after the regular carrying out of lumber punctures patient's condition improved quickly. As early as from April 4th pneumococci could no longer be shown in the lumbar liquid or cultivated from this liquid. The number of cells in the liquid constantly dropped and on April 7th amounted to 154/3. Now the temperature was quite normal.

On April 14th it seemed as if the woman had completely recovered. The auditory canal was dry. All subjective complaints had disappeared, only she still felt a little tired, which could be understood very well in connection with her age and the serious illness just gone through. Objectively not a single meningeal symptom could be found any more.

April 21st. For some days already patient had asked to be allowed to get up a moment.

Also in order to interrupt the continual bed's rest, in connection with her age, she was allowed to pass some time outside her bed on a restchair. She had scarcely lain on this, however, when she collapsed, became strongly cyanotic and died within 10 minutes without having regained conscience.

As the cause of death at the obduction an embolus was found in the arteria pulmonalis coming from a large, obstructing thrombus in the left vena iliaca and vena femoralis. This thrombus had probably been the cause of the pains of which patient had complained already before her illness, which she felt especially when walking.

At the obduction of the skull the roof appeared to be remarkably thick. Below the dura there was much clear liquid. The brains had the normal form and consistency. The leptominges were a little turbid and thickened. Below these there was no pus anywhere, as far as could be seen macroscopically, except below the leptominges of the left frontal lobe, where a little pus was found. The ventricles contained much clear liquid.

The labyrinths had been opened at dissection by mistake. The left labyrinth showed no anomalies microscopically; the right, however, contained pus, as could also be stated later by microscopic examination. After fixation no anomalies could be stated macroscopically on the whole surface of the brain.

At the microscopic examination the leptominges of the various parts of the hemispheres and also of the cerebellum were examined. *It appeared that everywhere there was still a marked leptomeningitis.* The soft membranes were moderately thickened by fibrin. There was an infiltration with polynuclear leucocytes and many lymphocytes and plasma-cells. Especially round the veins were many infiltrations (fig. 1). In the cerebellum and the cerebrum itself generally speaking no anomalies were observed; only in one single place we found an infiltration consisting of small cells in the hemisphere itself (fig. 2).

In this communication we chiefly meant to throw some light on our responsibility in nursing meningitis-patients. When we observe the focuses of inflammation scattered all over the hemispheres and cerebellum in this old woman, who thought herself that she had already recovered from her meningitis and whom we ourselves also considered as such, it is very probable that in many cases we allow our patients to get up much too early after they have gone through a meningitis and during the recovery permit them liberties much too early, among other things as regards sitting up. But it is difficult to go on considering patients who no longer have any complaints and in whom we ourselves can no longer show any anomalies as ill and to think of the fact that the recovery can be only apparent and that the meningitis-process has not at all calmed down yet.

It may be quite possible that also part of the recidivisms which we can sometimes observe suddenly after a meningitis has been gone through well are only caused by the fact that we permitted more to the patients than could be allowed in view of the real condition.

And in conclusion, might not the complaints of head-aches, tiredness etc., often existing a rather long time after a so-called cured meningitis be partly explained by the above? For one may be sure that it will take a long time before we can speak of a really cured meningitis, not only clinically, but also from a pathological-anatomical point of view.

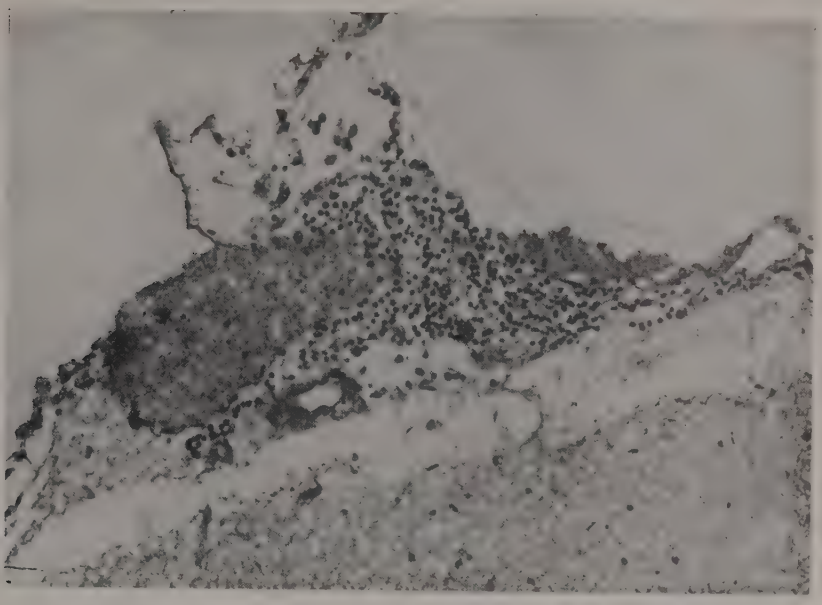


Fig. 1

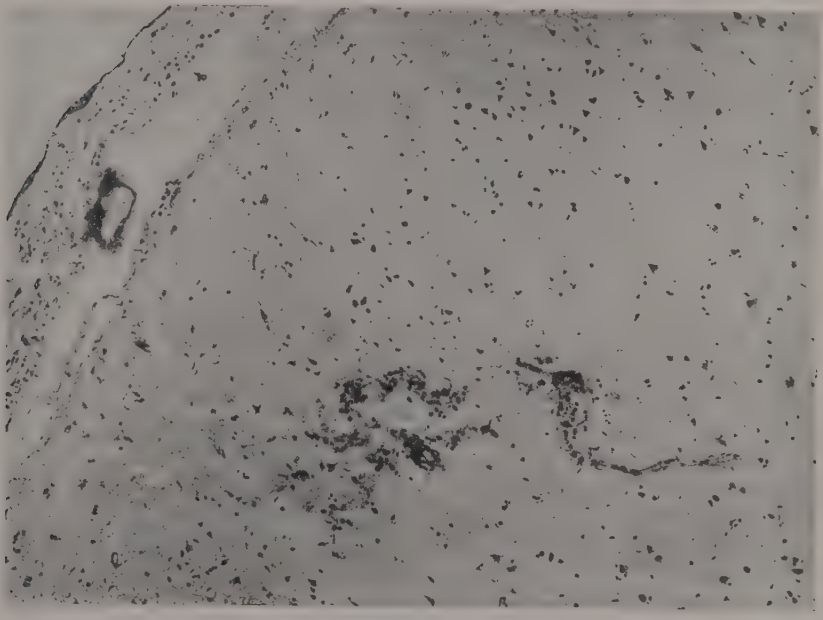


Fig. 2

Medicine. — *Reflex influences occurring with cardiospasm.* By A. DE KLEYN.

(Communicated at the meeting of January 26, 1946.)

The so-called idiopathic cardiospasm or idiopathic achalasia is characterised by the combination of a more or less strong dilatation of the esophagus and the fact that the cardial opening-reflex is absent when food is swallowed. The cause of this idiopathic form is still unknown, in contrast with the symptomatic one, where processes in or round the esophagus cause the same complex of symptoms to arise.

Many clinical observators have objected to the name cardiospasm because the use of this word would imply that the existence of a sphincter cardiae is considered as an established fact. But even if it is admitted that anatomically speaking no real sphincter cardiae is present [1]), there always remains the possibility that circumscript contractions of the circular muscle-fibres bring about the same effect as a real sphincter, which, for that matter, has already been known for a long time for the stomach, under the influence of poisons or excitation of the vagus (MAGNUS [2]), KLEE [3])).

The fact that in most cases of cardiospasm bougies pass very easily is, according to me, certainly no sufficient argument to deny the existence of a sphincter-spasm in any form. The observations of thin bougies being not seldom clasped and that during an operation for cardiospasm the finger of V. EISELSBERG [4]) introduced into the cardia from the stomach was clearly clasped, rather support a contrary opinion!

In all these considerations it is often forgotten that the circumstances in bouginage are not the same as those which exist when food is swallowed. This appears very clearly from an observation made by LOTHEISEN [5]), who at the moment when patients suffering from cardiospasm swallowed barium-pap, saw such a spasm appear in the esophagus that the bougie, which before that time had been introduced without any difficulty, was clasped. When bougies provided with little holes were used, by means of which the pap could easily flow down to the stomach, the spasm disappeared much more quickly than when solid ones were used.

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1) See o.a. BROWN KELLY, Jnl. of Laryng. **221**, 1927 (Semon-lecture).

2) MAGNUS, R. Münch. med. Wochenschr., 1907, nr. 29.

3) KLEE, PH. Plügers Arch., **145**, 557 (1912).

4) V. EISELSBERG, Wiener klin. Wochenschr. **20**, 811 (1907): „In ganz charakteristischer Weise trat ein Kardiospasmus mit derartiger Vehemenz auf, dasz mein Daumen fast umklammert wurde und ich in denselben ein ganz taubes Gefühl bekam".

5) LOTHEISEN, G. Chirurgie der Speiseröhre, Neue Dt. Chir. v. P. BRUNS, **34**, 250 (1926).

The saying of BROWN KELLY [6]): "the spasm is little more than a state of non-relaxation" finally represents an opinion, which comes closely to the opinion of those clinical investigators who, in the so-called cardiospasm, deny every spasm and suppose that this clinical picture is caused by an insufficiency of the longitudinal muscle-fibres of the stomach, thus preventing the normal active opening of the cardia by these fibres (ZAAYER [7]) a.o.). However this may be and whatever theory one may adhere to about the origin of the cardiospasm, it is certain that in patiens who suffer from this affection the normal opening-reflex of the cardia when food is taken does not come into being.

It is not known with certainty which path this reflex takes. Especially about the question in how far the vagus, sympathicus and plexus of AUERBACH are implicated in the reflex and whether perhaps the three of them play a part here, there still exists a great difference of opinion [8]).

As early as 1889 v. OPENCHOWSKY [9]), when experimenting on animals, could observe that by stimulation of the N. ischiadicus and of various organs the cardia could be opened in a reflectory manner [10]) while in clinical literature cases were described where after operation of the gall-bladder, appendix and stomach an existing cardiospasm was cured entirely [11]) or for the greater part, so that also in those cases a reflectory spasm was assumed [12]). The following, certainly reflectory influence in a typical case of cardiospasm, which could be demonstrated again and again with great regularity, has not yet been described, as far as I know and may therefore be mentioned here in brief.

Patient C., a chauffeur of 21 years old, had had difficulties in eating from this 10th year. He clearly felt that the food would not go down normally. Sometimes he vomited it quite undigested and then it did not taste sour. As a child already he took food with difficulty, but no neurotic anomalies could be observed in him. The difficulties, it is true, increased when he was getting nervous. For the rest he had always been in good health nor did the family-anamnesis furnish anything special except the fact that his mother suffered from tuberculosis of the kidneys.

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6) BROWN KELLY, A. Brit. med. Jnl., 1912, Oct. 19th.

7) ZAAYER, J. H. Acta oto-laryng, 2, 188 (1920).

8) See e.g. IMMINK, E. A. M. Thesis Amsterdam, 1928; LOTHEISEN l.c. MAGNUS, R.: Die Bewegungen des Verdauungsrohres, Handb. d. phys. Methodik von R. TIGERSTEDT, 1908; ROELFSEMA, E. J., Thesis Leiden 1937; WORMS, M. G., in TERRACOL, J. Les maladies de l'oesophage, Masson, Paris 1938, p. 159.

9) OPENCHOWSKY, TH. V., Dt. med. Wochenschr., **15**, 717, 1889.

10) „Endlich möchte ich noch erwähnen, dass man bei Thieren, deren sämmtliche Nervenbahnen noch erhalten sind, durch Reizung von Nieren, Blase, Uterus, Ischiadicus eine Öffnung der Cardia erziehen kann."

11) See literature in MOERSCH, H. J., Annals of Otology, **43**, 1165, 1934.

12) The experiments of V. OPENCHOWSKY, however, did not refer to cardiospasm and were made on normal animals, while as regards the clinical observations just mentioned the post, but not at all the propter is established.

Some years ago the patient had noticed that as soon as his food stuck, he only had to press on a certain place medially just above the manubrium sterni to get relief and make the food go down. If he exerted this pressure when taking food, he could complete his meal without further disturbances. For that reason he himself did not consider it necessary to consult a doctor, but his fiancée as can be understood, objected to this way of eating and sent him to our polyclinic. From there he was admitted into the clinic. When a general internal and oto-rhino-laryngological examination was made, no anomalies were found. The number of leucocytes amounted to 5400, the number of erythrocytes to 5.260000, the haemogoblin content was 94 %, while the sedimentation rate amounted to 1 mM in the first and to 3 mM in the second hour. The blood-picture was also completely normal.

The nutritious condition of the patient was very good, contrary to what we mostly observe in sufferers from cardiospasm. This must probably be ascribed to the trick which the patient always applied when eating.

When a röntgenological examination of the esophagus was made, the fluoroscope showed and on the photograph was found the typical picture of a cardiospasm. The swallowed bismuth-pap remained sticking in the strongly widened esophagus at the height of the diaphragm. Sometimes the esophagus was, as it were, quite obstructed, sometimes it ended in a little canal, the so-called mouse-tail (fig. 1). It was only with great intervals that a little pap went into the stomach, in spite of the fact that rather strong peristaltics could be seen. At the oesophagoscopy, which took place a few days later, it appeared that much saliva was found in the strongly widened esophagus. After this had been sucked away, the hyperaemic mucous membrane became visible. The esophagus-tube could be introduced without difficulty and easily passed through the cardia, which was also the case with the several esophagus- and mercury-bougies. At last a röntgenological examination of the patient was made, while he was requested to exert the above-mentioned pressure above the manubrium sterni. The consequence of this was surprising. Fig. 2 represents the picture without, resp. with pressure on any part of the neck, while fig. 3 shows a photo taken after the patient had just exerted a pressure medially above the manubrium sterni. It is seen on the photograph how after this the cardia opened, which the fluoroscope showed even better. This phenomenon could be shown to others repeatedly and on different days, without the reflex ever failing. Only some minutes must have elapsed before a new pressure could be applied effectively.

We have not systematically examined whether this reflectory opening of the cardia of our patient could also be brought about by pressure on other parts of the body. As has already been said, we were not successful when exerting pressure on other parts of the neck-region.

Finally we must answer the question which nerve is excited when

pressure is exerted above the manubrium sterni. Colleague WOERDEMAN also thought that here only the N. recurrens vagi, which in this region liest against the trachea, can be considered. The N. vagus itself lies in this place much too lateral for this.

Summarizing one can therefore say that the cardia of the above-mentioned patient who suffered from cardiospasm opened in a reflectory manner, if there was food in the esophagus as soon as pressure was exerted medially above the manubrium sterni and this very probably as a consequence of an excitation caused by pressure of the N. recurrens vagi. Without this pressure the food did not flow to the stomach or only very slowly. The reflex which played a part here could be brought about regularly if the pressure was not exerted too quickly in succession.

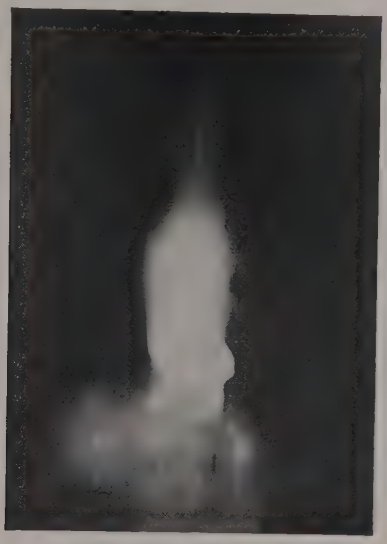

Fig. 1.

Fig. 2.

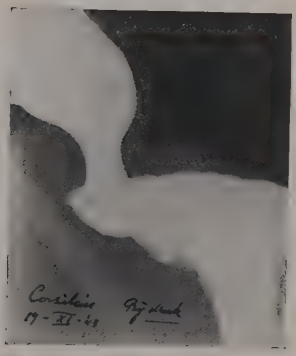

Fig. 3.

Applied Mechanics. — *On the non-linear deflection of a semi-circular ring, clamped at both ends.* By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of January 26, 1946.)

1. *Introduction.* In a remarkable paper on the buckling of cylindrical shells [1]), TH VON KÁRMÁN, L. G. DUNN and HSUE-SHEN TSIEN suggest that a longitudinal strip of a cylindrical shell may be considered as an elastically supported column. However, it follows both from physical considerations and from experimental observations, that the actual elastic support cannot be linear dependent on the local deflection. Therefore the explanation for the discrepancies observed between the experimental values of the buckling load of cylinders and the values obtained theoretically must be found in some characteristic property of a non-linear support. It was therefore judged to be of general interest to investigate experimentally the effect on the load-deflection relation of a column-supported by a non-linear elastic element. To this end a thin semi-circular ring as shown in fig. 1,

Fig. 1

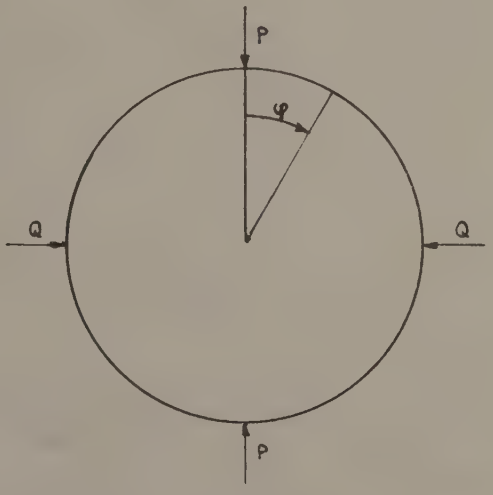

Fig. 2

was used as the elastic suport, but its elastic behaviour, viz. its force-deflection relation, has by the authors been determined only experimentally. In the following sections this relation will be deduced in a theoretical way.

Instead of the semi-circular ring, clamped at both ends we first consider

1) Comp. TH. VON KÁRMÁN, L. G. DUNN and HSUE-SHEN TSIEN, The Influence of Curvature on the Buckling Characteristics of Structures, Journal of the Aeronautical Sciences, Vol. 7, 1940, p. 276—289.

a complete ring loaded by two pairs of diametral forces P and Q, placed at right angles, and calculate, for a number of values of the ratio P/Q, viz. $\frac{P}{Q} = 1,\ 1{,}05,\ 1{,}10,\ \ldots,\ 1{,}20$ and their reciprocals the radial deflections u at $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. These calculations are performed after a method, developed in two preceding papers I and II, to which the reader may be referred [2]). It will be roughly indicated for the two cases $P/Q = 1$ and 1,2, whereas for the other ratio's only results will be given. From each of these results (relating to a distinct value of the ratio P/Q), the absolute value of P will be deduced for which the deflection $u_{\pi/2}$ at $\varphi = \frac{\pi}{2}$ becomes zero, and by substituting the smallest of these values into the corresponding expression of u_0, the deflection at the point $\varphi = 0$ can be calculated. The results thus obtained will be compared with the experimental data of VON KÁRMÁN c.s.

2. *The ring compressed by two equal pairs of diametral forces P, placed at right angles.* As has been shown in paper I, the four forces P can be replaced by two systems (A) and (B):

$$(A)\begin{cases} q^* = -\frac{2P}{\pi r} + \frac{4P}{\pi r}\sum_{k=1}^{\infty}\frac{\cos 4k\varphi}{16k^2-1}; \\ t^* = \frac{4P}{\pi r}\sum_{k=1}^{\infty}\frac{4k\sin 4k\varphi}{16k^2-1}; \end{cases} \qquad (B)\begin{cases} q^{**} = -\frac{4P}{\pi r}\sum_{k=1}^{\infty}\frac{16k^2}{16k^2-1}\cos 4k\varphi \\ t^{**} = -\frac{4P}{\pi r}\sum_{k=1}^{\infty}\frac{4k}{16k^2-1}\sin 4k\varphi \end{cases} \qquad (1)$$

the first of which represents a truly "compressive" loadsystem, the second one a pure "bending" loadsystem. The internal normal fore N_0 in the cross-section (φ), due to the (A)-system is given by

$$N_0 = -\frac{2P}{\pi} + \frac{4P}{\pi}\sum_{k=1}^{\infty}\frac{\cos 4k\varphi}{16k^2-1} = -\frac{2P}{\pi}\left[1 + \sum_{k=1}^{\infty}\varepsilon_{2k}\cos 2k\varphi\right] \Bigg\} \qquad (2)$$
$$\varepsilon_{2k} = -\frac{2}{16k^2-1} \Bigg\}$$

the bending moment M_B due to the (B)-system by

$$\left.\begin{aligned} M_B &= -\frac{4Pr}{\pi}\sum_{k=1}^{\infty}\frac{\cos 4k\varphi}{16k^2-1} = \\ &= -\frac{4Pr}{\pi}[0{,}06667\cos 4\varphi + 0{,}01587\cos 8\varphi + 0{,}00699\cos 12\varphi + \\ &\quad + 0{,}00392\cos 16\varphi + 0{,}00251\cos 20\varphi + 0{,}00174\cos 24\varphi] \end{aligned}\right\} \qquad (3)$$

[2]) I. C. B. BIEZENO and J. J. KOCH, The generalized buckling problem of the circular ring. These Proc. Vol. XLVIII, p. 447.

II. C. B. BIEZENO and J. J. KOCH. The circular ring under the combined action of compressive and bending loads. These Proc. XLIX, p. 3.

It follows from the expression (2), — comp. I) — that the characteristic functions belonging to the (A)-system break up into two separate classes, distinguished in that the first class is built up of terms of the type cos $(2 + 4p)\varphi$ only, the second one of terms of the types cos $4p\varphi$ only (p being integer). The first two characteristic numbers λ of both classes and the corresponding characteristic functions of the second class are given here without any further comment (comp. for the required computations paper I):

$$\left.\begin{aligned} \lambda_1 &= 1{,}071\,\frac{3EI}{r^2}\,\frac{\pi}{2P} & \lambda_2 &= 1{,}01432\,\frac{15EI}{r^2}\,\frac{\pi}{2P} \\ \lambda_3 &= 11{,}8082\,\frac{3EI}{r^2}\,\frac{\pi}{2P} & \lambda_4 &= 4{,}22990\,\frac{15EI}{r^2}\,\frac{\pi}{2P} \end{aligned}\right\} \quad . \quad . \quad (4)$$

$$\left.\begin{aligned} U_2/r = {} & 0{,}92924\cos 4\varphi - 0{,}02170\cos 8\varphi - 0{,}00200\cos 12\varphi - \\ & - 0{,}00052\cos 16\varphi - 0{,}00020\cos 20\varphi - 0{,}00009\cos 24\varphi \\ U_4/r = {} & 0{,}1571\cos 4\varphi + 1{,}6230\cos 8\varphi - 0{,}0956\cos 12\varphi - \\ & - 0{,}0071\cos 16\varphi - 0{,}0023\cos 20\varphi - 0{,}0009\cos 24\varphi \end{aligned}\right\} . \quad (5)$$

The characteristic functions U_1 and U_3 of the first class need not be computed as will directly be seen.

In the expansion of $-M_B r^2/EI$ into a series of the characteristic functions (comp. II)

$$-\frac{M_B r^2}{EI} = b_1 U_1 + b_2 U_2 + b_3 U_3 + b_4 U_4 \ldots \quad . \quad . \quad . \quad (6)$$

the coefficients b_2 and b_4 prove to have the following values:

$$b_2 = 0{,}06969\,\frac{4Pr^2}{EI} \qquad b_4 = 0{,}00999\,\frac{4Pr^2}{EI} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whereas:

$$b_1 = b_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as can be predicted from the expression (3) which does not contain terms of the type cos $(2+4p)\varphi$. The deflections u_i of the ring corresponding to the components $-\frac{EI}{r^2} b_i U_i$ of the bending moment M_B can be calculated from the equations

$$u_i'' + u_i = b_i U_i \qquad (i = 1, 2, 3, 4). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

They are — as far as u_2 and u_4 are concerned —

$$\left.\begin{aligned} u_2 = {} & [-0{,}004317\cos 4\varphi + 0{,}000024\cos 8\varphi + \\ & + 0{,}000001\cos 12\varphi + \ldots]\,\frac{4Pr^3}{\pi EI} \\ u_4 = {} & [-0{,}000105\cos 4\varphi - 0{,}000257\cos 8\varphi + \\ & + 0{,}000007\cos 12\varphi + \ldots]\,\frac{4Pr^3}{\pi EI} \end{aligned}\right\} . \quad . \quad . \quad (10)$$

The deflection u as caused by the (B)-system can either be found by integrating the equation

$$u'' + u = -\frac{M_B r^2}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which M_B has to be replaced by the expression (3) or by solving the problem, illustrated by fig. 2, by one of the direct methods of solution (f.i. by CASTIGLIANO's method).

The first way leads to:

$$u = \left[-\frac{0{,}06667}{15} \cos 4\varphi - \frac{0{,}01587}{63} \cos 8\varphi - \right.$$
$$\left. \frac{0{,}00699}{143} \cos 12\varphi - \frac{0{,}00392}{255} \cos 16\varphi - \ldots \right] \frac{4 P r^3}{\pi EI} \quad . \quad . \quad (12)$$

If all deflections u_i were known, we also could have written $u = \sum_{i=1}^{\infty} u$. The total distortion u^*, as caused by the simultaneous action of the (A)- and (B)-load is represented by (comp. II)

$$u^* = \sum_{i=1}^{\infty} \frac{\lambda_i u_i}{\lambda_i - 1} = \sum_{i=1}^{\infty} u_i + \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i - 1} = u + \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i - 1} \quad . \quad . \quad . \quad (13)$$

(All functions u_i corresponding to an odd value of the index i are zero, because the coefficient b_i — not the function U_i from which u_i is derived! — is zero).

It must be remembered now, that we look for such values of the two pairs of equal forces P, for which the radial displacement at $\varphi = \frac{\pi}{2}$ should be zero.

With regard to (13) it would appear that for every value of P, for which $u_{\varphi=\pi/2} = 0$, simultaneously the deflection at $\varphi = 0$ becomes zero. It must however be brought into mind, that (for the *first* time) the displacement at $\varphi = \frac{\pi}{2}$ can be made zero, (without annulling the displacement at $\varphi = 0$) when λ_1 becomes equal to unity.

In this case an unknown multiple of the first characteristic deflection u_1 may be added to u^* (13), which itself does *not* contain the component u_1. This multiple of u_1 (say pu_1) evidently can also be chosen in such a way that the total deflection at $\varphi = \frac{\pi}{2}$ becomes zero, and it is an easy matter to calculate the deflection, which in that case occurs at $\varphi = 0$. To this end we remark that, $u^*_{\varphi=0} = u^*_{\varphi=\pi/2}$, whereas at the other hand $u_{1(\varphi=0)} = -u_{1(\varphi=\pi/2)}$.

The multiple p has to be chosen such that

$$p u_{1(\varphi=\pi/2)} = -u_{\varphi=\pi/2}.$$

And the deflection at $\varphi = 0$ consequently becomes:

$$u^*_{\varphi=0} + pu_{1\,(\varphi=0)} = u^*_{\varphi=0} - pu_{1\,(\varphi=\pi/2)} = u^*_{\varphi=0} + u^*_{\varphi=\pi/2} = 2\,u^*_{\varphi=0}.$$

Restricting ourselves to u_2 and u_4, we find by the aid of (13) for the resulting deflection at $\varphi = 0$

$$2\left[u + \frac{u_2}{\lambda_2 - 1} + \frac{u_4}{\lambda_4 - 1}\right] \qquad \varphi = 0;\ \lambda_1 = 1.$$

A simple computation leads — with the use of (4), (10), (12) — to the deflection 0,0764 r.

3. *The ring compressed by two pairs of diametral forces of the magnitude 1,2 P (at $\varphi = 0$ and $\varphi = \pi$), and P $\left(\text{at } \varphi = \pm \frac{\pi}{2}\right)$.*

The A and B-loads are represented here — as easily can be checked — by

$$(A)\begin{cases} q^* = -\dfrac{2{,}2\,P}{\pi r} + \dfrac{P}{\pi r}\sum\limits_{k=1}^{\infty}\dfrac{a_{2k}\cos 2k\varphi}{4k^2-1}; \\ t^* = \dfrac{P}{r}\sum\limits_{k=1}^{\infty}\dfrac{2\,k a_{2k}\sin 2k\varphi}{4k^2-1}; \end{cases} \qquad (B)\begin{cases} q^{**} = -\dfrac{P}{r}\sum\limits_{k=1}^{\infty}\dfrac{4\,k^2 a_{2k}\cos 2k\varphi}{4k^2-1} \\ t^{**} = -\dfrac{P}{r}\sum\limits_{k=1}^{\infty}\dfrac{2\,k a_{2k}\sin 2k\varphi}{4k^2-1}. \end{cases} \qquad (14)$$

$$a_{2k} = 0{,}4\ (k = \text{odd}) \qquad a_{2k} = 4{,}4\ (k = \text{even})$$

whereas the normal force N_0 acting upon the crosssection (φ) (and caused by the A-system) and the bending moment M_B belonging to the B-system are given by

$$N_0 = -\frac{2{,}2P}{\pi}\left[1 + \sum_{k=1}^{\infty}\varepsilon_{2k}\cos 2k\varphi\right] \qquad \varepsilon_{2k} = -\frac{a_{2k}}{2{,}2\,(4k^2-1)} \qquad (15)$$

$$M_B = -\frac{Pr}{\pi}\sum_{k=1}^{\infty}\frac{a_{2k}\cos 2k\varphi}{4k^2-1}. \qquad (16)$$

No different classes of characteristic functions are distinguishable here, and *all* functions contain terms of the type cos $(2 + 4p)\varphi$ *and* of the type cos $4p\varphi$. Again restricting ourselves to the first two characteristic numbers λ_1, λ_2 and the corresponding characteristic functions, it is found that

$$\lambda_1 = 1{,}07028\,\frac{3\pi EI}{2{,}2Pr^2} \qquad \lambda_2 = 5{,}0745\,\frac{3\pi EI}{2{,}2Pr^2} \qquad (17)$$

$$\left.\begin{aligned} U_1/r \backsim V_6^* &= 0{,}71036\cos 2\varphi - 0{,}00629\cos 4\varphi - 0{,}00590\cos 6\varphi - \\ &\quad - 0{,}00010\cos 8\varphi - 0{,}00053\cos 10\varphi - 0{,}00002\cos 12\varphi \\ U_2/r \backsim \overline{V}_{11}^* &= 0{,}04453\cos 2\varphi + 1{,}06722\cos 4\varphi - 0{,}02684\cos 6\varphi - \\ &\quad - 0{,}02469\cos 8\varphi - 0{,}00031\cos 10\varphi - 0{,}00231\cos 12\varphi \end{aligned}\right\}. \qquad (18)$$

The expansion of $-\frac{M_B r^2}{EI} = b_1 U_1 + b_2 U_2 + \ldots$ into a series of the characteristic functions leads — as far as the first terms are concerned — to the following values of the coefficients b_1 and b_2

$$b_1 = 0{,}04178 \frac{4Pr^2}{\pi EI} \qquad b_2 = 0{,}06680 \frac{4Pr^2}{\pi EI} . \quad . \quad . \quad . \quad (19)$$

The deflections u, u_1 and u_2 in the series

$$u^* = u + \frac{1}{\lambda_1 - 1} u_1 + \frac{1}{\lambda_2 - 1} u_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

are represented by

$$\left.\begin{aligned} u = \Bigg[&- \frac{0{,}03333}{3} \cos 2\varphi - \frac{0{,}07333}{15} \cos 4\varphi - \frac{0{,}00286}{35} \cos 6\varphi - \\ &- \frac{0{,}01746}{63} \cos 8\varphi - \frac{0{,}00101}{99} \cos 10\varphi - \frac{0{,}00769}{143} \cos 12\varphi \Bigg] \frac{4Pr^3}{\pi EI} \end{aligned}\right\} \quad (21)$$

$$u_1 = \left[- \frac{0{,}02968}{3} \cos 2\varphi + \frac{0{,}00026}{15} \cos 4\varphi + \frac{0{,}00025}{35} \cos 6\varphi + \ldots \right] \frac{4Pr^3}{\pi EI} \quad (22)$$

$$\left.\begin{aligned} u_2 = -\Bigg[&\frac{0{,}00297}{3} \cos 2\varphi - \frac{0{,}07129}{15} \cos 4\varphi + \\ &+ \frac{0{,}00179}{35} \cos 6\varphi + \frac{0{,}00265}{63} \cos 8\varphi + \ldots \Bigg] \frac{4Pr^3}{\pi EI} \end{aligned}\right\} \quad . \quad . \quad (23)$$

and from these data it is found that

$$u^*_{\varphi=0} = -\left[0{,}001642 + \frac{0{,}00986}{1{,}07028 \dfrac{3\pi EI}{2{,}2Pr^2} - 1} + \frac{0{,}00572}{5{,}0745 \dfrac{3\pi EI}{2{,}2Pr^2} - 1} \right] \frac{4Pr^3}{\pi EI} \quad (24)$$

$$u^*_{\varphi=\pi/2} = \left[0{,}00598 + \frac{0{,}00990}{1{,}07028 \dfrac{3\pi EI}{2{,}2Pr^2} - 1} - \frac{0{,}00384}{5{,}0745 \dfrac{3\pi EI}{2{,}2Pr^2} - 1} \right] \frac{4Pr^3}{EI} . \quad (25)$$

By substituting the smallest root P of

$$u^*_{\varphi=\pi/2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

into (24) we find the required corresponding displacement $u^*_{\varphi=0}$.

The numerical result is:

$$u^*_{\varphi=0} = 0{,}1216\, r.$$

4. *The spring-characteristic.* As has been stated in section 1 analogous computations have been performed for the ratio's P/Q mentioned there. (See table 1.)

TABLE I

Q/P	u	P	$-P/u$
1/1.2	+0.1216	−17.14	+140.8
1/1.1	+0.0103	− 0.92	+ 89.1
1/1.05	−0.0348	+ 2.65	+ 76.0
1	−0.0764	+ 5.05	+ 66.1
1.05	−0.1130	+ 6.62	+ 58.5
1.1	−0.1436	+ 7.69	+ 53.6
1.2	−0.1938	+ 8.88	+ 45.8
	. r	. EI/r^2	. EI/r^3

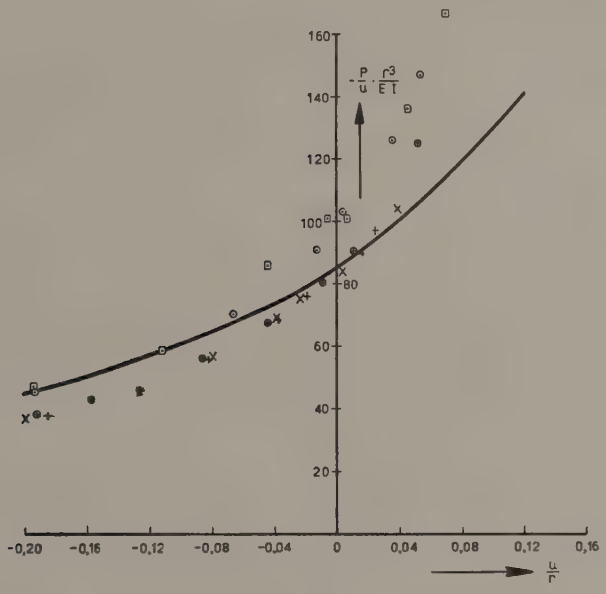

Fig. 4

The result is represented in fig. 3, where the experimental results of VAN KÁRMÁN are given too.

Physics. — *A new method of producing aspherical optical surfaces.* By H. RINIA and P. M. VAN ALPHEN. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven — Holland.) (Communicated by Prof. G. HOLST.)

(Communicated at the meeting of January 26, 1946.)

Introduction.

Hitherto almost only flat or spherical surfaces have been used in the manufacture of optical instruments. This has not been because the importance of aspherical surfaces was not realized, for DESCARTES and NEWTON have already pointed out their advantages. Rather it is because the technique of making aspherical surfaces had not progressed far enough to yield accurate results; an optical surface cannot be considered accurate until it equals the calculated curves to within a few wave lengths. The only exception were the parabolic mirrors of astronomical reflectors.

Apart from a few handground aspherical lenses, what has aroused considerable interest in this direction in recent years is the mirror system constructed by SCHMIDT in Hamburg [1]). This system consists of a spherical concave mirror with the diafragm in the center of curvature. This mirror system which is free from chromatic aberrations, coma and astigmatism, can now, moreover be made free from spherical aberration by applying in the diafragm a correction plate with a 4th degree surface *). In this way with the help of only two elements, a system is obtained which shows only curvature of field as a main optical error, whilst a relative aperture greater than 1 : 1 can easily be obtained **).

The fabrication of aspherical surfaces.

Such systems of very large aperture may constitute a great advance for many technical applications and it is therefore not surprising that in various quarters methods have been sought for cheap mass-production of these correction plates.

With SCHMIDT's method a glass plate was bent by unilateral air pressure and then ground flat [2]), but unfortunately this method is not suitable for small plates, as was proved by A. COUDER [3]).

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*) In practice one usually chooses a combination of 2nd, 4th and higher degrees.

) For large apertures it is not permissible to take the *f*-number as a measure of the brightness of the image, because then the angle deviates considerably from the sinus. It is better to adopt the conception of numerical aperture usually employed with microscope objectives, particularly because this expression is used for the object and picture distance for which the system is intended. See also BOUWERS and VAN HEEL, Physica **10, 714 (1943).

Various research-workers tried to overcome this difficulty by reverting to the spherical surface construction of the correction element.

Most investigations however, are based on the idea of making the aspherical correction element from an optically transparant material pressed or cast in a mould; the material used not to be so much glass but rather one of the new plastics, such as polystrene and the like.

Drawbacks of pressing or casting in moulds.

For this method it is necessary to have a mould of the desired shape. The shaping of such a mould has to be done for instance by turning on a special lathe to an accuracy of about 1 μ and the surface must show optical finish. Moreover the mould must retain these properties during the pressing process, so that the choice of material for the mould itself is limited; sometimes the right shape is obtained bij hand-finishing. Further the optical material for the correction element must remain optically homogeneous and not suffer from double refraction, or subsequently show elastic flow, which is difficult to avoid with transparant artificial resins. Finally the plastic used must take the optical surface quality of the mould and must easily loosen from the mould without any deformation or damage either to the casting or to the mould, so as to avoid the necessity of any further finishing or repairing.

Thus it will be seen that it would be very difficult to obtain castings of glass answering these requirements.

Gelatin for optical surfaces.

Our research has shown that this problem can be solved by using glass plates coated with a thin layer of gelatin. When certain precautions are taken it is possible to let a gelatin gel dry in such a way that the original surface remains unaltered, though on a reduced scale. If, for instance, a 10 % gelatin solution is used the mould must be such as to give a section of the ultimate desired form enlarged 10 times in thickness. The absolute accuracy of this mould therefore may be 10 times less, say 10 μ. After the gel has dried, the errors are reduced to one-tenth. Owing to the strong adhesive power of gelatin to glass, there will only be shrinkage, after drying, in thickness. The final correction plate therefore consists for the greater part of optical glass, with a thin coating of gelatin. Double refraction or elastic flow does not occur, the gelatin layer is no thicker than is required for aspherical moulding and is too thin to cause much trouble as a result of internal imperfections in the gelatin.

It is possible to apply these layers also to lenses, so that in this manner lenses of different dispersions and refractive index can also be made with an aspherical surface.

Method of fabrication.

The carefully turned and polished mould *) can be heated by means of a continuous flow of water. A 10 to 20 % gelatin solution is poured on the mould and the lens or glass plate is laid on it, care being taken to avoid air bubbles. By means of three set screws the lens or glass plate is kept at a small distance from the mould surface. Cold water is then pumped through the mould until the whole is cooled off and the solution is gelatinized to a stiff gel. The lens or glass plate is then removed from the mould with the aid of the set screws and owing to its strong adhesion to glass the gelatin readily loosens from the mould surface, showing in all the smallest details a perfect casting of the mould.

The gelatin coating is then hardened in formalin vapour and afterwards dried uniformly. It shrinks only in the direction of thickness, being held laterally by the glass plate. The surprising feature about this shrinking is, that it proceeds so uniformly, that the ultimate shape is optically correct. Even minor imperfections of the mould surface, such as extremely fine scratches are greatly reduced.

An advantage is that different kinds of correction plates can be made with the same mould. If the SCHMIDT-camera is not used for infinity, but for a shorter distance an optically stronger correction plate has to be used. It can be proved that with an image magnification N the power of the plate must be increased by a factor $\left(\frac{N+1}{N-1}\right)^2$ in respect to the power for infinity **), whilst the surface is approximately uniform. This higher power is easily obtained by using a gelatin solution of higher concentration. After drying the plate has a proportionally higher power. Adaptations are also possible in this manner for small differences in focal distance.

Stability of the gelatin layer.

The lenses or glass plates treated in this way bear a great similarity to photographic plates without bromide of silver. They are very stable, and hardened gelatin is a very tough and strong substance. Nevertheless it is preferable to apply the gelatin layer on the inner surface of lenses, and to protect the correction plates with a second flat plate of glass, because wet fingers or rain drops are apt to leave marks on the gelatin.

Applications.

The most important technical application is as a projection system

*) The moulds were made by L. LEBLANS, who also took a considerable part in overcoming the further difficulties.

**) For $N = 1$ the power of the plate has become ∞. This seems to be strange because when the image is in the centre of curvature, the spherical aberration is just nil. It has to be born in mind however, that also the height of incidence has become nil, and this makes the greater optical power necessary for short distances.

in television sets. The SCHMIDT-camera is adaptable for this by interposing between mirror and correction plate at an angle of 45° a flat mirror with a hole in it, into which can be inserted the spherically ground head of the small high tension cathode ray tube. All deflection coils and the tube leads are placed behind the 45° mirror. The front face of the mirror is coated with aluminium, and the light cast back from the concave mirror is reflected on to the correction plate. In this manner with a small cathode ray tube a sufficiently sharp and flat picture of adequate dimensions can be obtained.

Although developed for television projection it was found to be of such excellent quality that a SCHMIDT-system produced in this manner can also be applied for photographic reproduction. Owing to the very large f.-number (e.g. 1 : 0.7, n.a. 0,58) the depth of focus is very small and such apertures are therefore practically only of use for flat objects. In consequence of the curvature of field it is necessary to use photographic film with a not too thin celluloid layer and to press this against a spherical object. This has been found possible not only for small detached pieces of film, but also for a series of pictures in a strip of film.

In this way it has been possible to make a camera for serial *X*-ray screen photography, which produced snapshots of 24 × 36 mm or 40 × 40 mm of the fluorescent image on the *X*-ray screen.

A similar sort of camera used for the cathode ray tube of an oscillograph made it possible to reproduce spot velocities on the tube of 60000 km/sec (2 % of the light velocity).

An astronomic photograph gave a sharply defined picture right to the very edges in spite of the short focal distance ($f = 8$ cm) over a field of 25°.

Cameras have been made in this way with a focal distance of 26 to 3.5 cm and a f.-number of 1 : 1,4 to 1 : 0,6 (n.a. 0,34—0,63). Even when fine-graine positive film is used the smallest perceptible detail is still determined by the granulation of the film.

Some experiments with lenses provided with this aspherical gelatin surface likewise yielded good results.

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Physics. — *New possibilities for the air engine.* By H. RINIA [1]). (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven — Holland.) (Communicated by Prof. G. HOLST.)

(Communicated at the meeting of January 26, 1946.)

Introduction.

The main types of caloric engines that have hitherto been found in technical use are summarized in the table below, where they are divided into two groups, viz. *piston engines* and *piston-less engines*. In the first group the potential energy of the expanding working medium is converted direct into energy of the moving parts of the engine, whereas in the second group this conversion takes place via a phase in which the energy is at first present in the form of kinetic energy of the working medium.

A further distinction has been made according to the method of heating.

Summary of the principal types of caloric engines.

Method of heating	Piston engines	Piston-less engines
Internal with oxygen from the air	Internal combustion engines	Gas turbines
Internal without oxygen from the air	Normal fire-arms	Rockets
External, the working medium passing through two phases during the process	Piston steam engines	Steam turbines
External, the working medium remaining in the same phase during the whole process	Air engines	Aerodynamic turbines

The types of engines in the right-hand column are usually more suited for high power, whilst piston engines are for smaller power, and the steam and air engines are also for very small powers.

Several of the types summarized here have only been recently developed.

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1) Research in the field of air engines has been undertaken by a group of collaborators of the Philips Physical Laboratory, Eindhoven, under the guidance of the author. This group comprised Ir. H. DE BREY, Ir. F. L. VAN WEENEN, W. H. STIGTER, Ir. W. J. VAN HEECKEREN, the late Ir. Dr. P. H. CLAY, Dr. F. K. DU PRÉ, H.H. METTIVIER MEYER, Dr. B. H. SCHULZ, Ir. A. J. J. LAMBEEK, Ir. A. HOROWITZ, Dr. G. W. RATHENAU. Many others have also contributed by additional research in regard to materials, burners, etc.

because it has not before been technically possible to realize ideas, which often are very old already. In the case of the air engine the position is somewhat different, for this had already reached technical development to a certain extent, but that subsequently came to be discarded. Nevertheless it has for a long time already been realized that the process applied in this type of engine was a very attractive one theoretically. It has now been found that, partly for the same reasons as obtain for the other recent developments, with the present-day technique it is possible to bring the theoretical promises of the air engine to practical realization.

After its invention by STIRLING in 1817 the air engine enjoyed a brief period of success in the middle of the last century, but it failed to stand up against the development of the steam engine and, especially later on, the internal combustion engine.

The principle of the hot-air process.

Fundamentally the air engine comprises two cylinders filled with a gaseous medium, usually air. For the sake of simplicity we will assume that the medium behaves like an ideal gas. The two cylinders are connected together via a regenerator. One has a high temperature T_w, the other a low temperature T_k; further we assume — as is approximately correct — that the temperature of the gas in both cylinders is always the same as that of the cylinder wall.

The volume under the piston in each of the cylinders can be varied from nil to a certain value by moving the pistons. Again assuming ideal conditions, the process brought about by the working medium is as follows:

1st Phase: At the start the volume in the hot cylinder is nil, and that in the cold cylinder at a maximum. The medium in the cold cylinder space is compressed (isothermally) by moving the piston inwards — from now on for the sake of brevity we shall refer to this as the cold piston, while that in the other cylinder will be called the hot piston. The heat of compression thereby developed is carried off. In this phase therefore a certain amount of energy is taken up by the engine.

2nd Phase: Both pistons are moved simultaneously in such a way that the volume of the medium is kept constant, and this motion is continued until the volume of the cold space has been reduced to nil and the gas is consequently all in the hot space (for the sake of simplicity the volume of the connections between the two cylinders and in particular that of the regenerator is disregarded). As a result of the higher temperature prevailing in the hot cylinder the gas there will be under higher pressure. During this phase no external work is performed.

3rd Phase: The hot piston is drawn outwards until the volume equals that of the cold space at the beginning of the first phase. By this motion work is performed which, as a consequence of the higher pressure, is

greater than the work done in the first phase. For this (isothermic) expansion heat is required. This heat is applied to the medium externally through the wall of the hot cylinder.

4th Phase: The two cylinders are then again moved in such a way that the total volume of gas remains constant, while the medium is now wholly transferred back to the cold space, thereby completing the cycle. During this phase the external work is again nil.

The net result of the whole cycle is a surplus of energy which is applied by the gas to the pistons. The process described is very difficult to realise. This is approximated in practice by causing the two pistons to move sinusoidally with a certain phase difference. This has to be arranged in such a way that the hot piston in its movement is advanced about 90° in phase in respect to the movement of the cold piston.

The efficiency of the hot-air process.

From the foregoing it is seen that the hot-air process yields mechanical energy. We will now consider what the efficiency is. First of all attention has to be paid to the regenerator. If the cold space and the hot space were connected directly, then during the second phase a considerable quantity of heat would be applied to the gas to heat it from T_k to T_w, and in the fourth phase this heat would be transferred to the cold cylinder. Thus a flow of heat arises, which means a loss. The function of the regenerator, which may consist of a mass of fine wire, is to reduce this loss by accumulating the heat that is contained in the gas when leaving the hot space (4th phase) and giving it off again to the medium as it flows back (2nd phase). Given an ideal action of the regenerator this heat is returned without loss, whilst everywhere in the regenerator the exchange of heat with the gas takes place with an infinitesimal difference in temperature.

The function of the regenerator is of the utmost importance, because the quantities of heat interchanged in it are about three times as large as that which has to be supplied for the actual process. As a matter of fact if there were no regenerator the efficiency would be reduced 4 times.

When the action of the regenerator is ideal and the gas, as already assumed, maintains the temperature T_w in the hot space and T_k in the cold space (so that there too the heat exchange takes place with an infinitesimal difference in temperature) then, with an ideal gas as medium, the process is entirely reversible and the efficiency, according to a known theorem in thermodynamics, is equal to $1—T_k/T_w$, the same as in a CARNOT process and the theoretical maximum possible with the given temperature ratio. Where the high temperature is 900° K and the low temperature 340° K, the efficiency is therefore 62 %.

In reality compression and expansion do not take place isothermically. because the temperature exchange with the wall is too slow. Consequently

the efficiency is reduced, but due to the small compression ratio, i. e. the ratio between the highest and the lowest pressure of the air engine (1 to 2—2.5), this reduction of efficiency does not amount to more than about 10 %, so that on this account and under the conditions given above the efficiency drops to 56 %.

It is also on account of the negligable temperature exchange with the cylinder wall that the heat required for the process is fed to and drawn from the gas in a special heater and cooler respectively, the former being interposed between the hot space and the regenerator and the latter between the regenerator and the cold space.

Differences between the modern and the old constructions of the air engine.

The reasons why full advantage has not hitherto been taken of the theoretical principles of the hot air process were mainly:

a. unfavourable temperature ratio, no suitable heat-resisting alloys being known at that time;

b. unsatisfactory heat transfer, involving voluminous machines with low mechanical efficiency and large heat losses;

c. inadequate regeneration, or even none at all.

It has now been found possible to surmount these difficulties for the greater part. Modern grades of steel allow of the application of a sufficiently high maximum temperature.

Heaters and coolers can now be constructed which in a limited volume and at high speed of the engine perform their functions satisfactorily without involving too large flow resistances. It is essential to limit the volume of this apparatus as much as possible because it forms part of the so-called dead space; this adversely affects the compression ratio and consequently also the output capacity.

It has proved possible to produce regenerators with efficiencies of 95 % and more, which means that they are capable of returning to the gas on its way back 95 and more per cent of the heat given off by the gas on its way to the cold space.

Further, it will be evident that the work performed is proportional to the number of revolutions the engine makes per minute and to the mean pressure of the medium. Both these factors must be high to attain a specific power output corresponding to that of internal combustion engines. At the same time the losses caused by conductance and radiation are low.

Thanks to the improvements described above it has become possible to construct engines which are worked at 3000 revolutions per minute and at a mean pressure of 10 atm. and with an efficiency and an overall output exceeding those of petrol engines of corresponding power.

Finally mention is to be made of a new form of construction for a multi-cylinder engine. It is possible to use four double-acting pistons with

mutual phase differences of 90°. The cold bottom end of one cylinder is connected with the hot top end of the next cylinder via a cooler, regenerator and heater, and so on. In this way only one piston per cycle is needed. Mechanically this arrangement is very simple and has a high mechanical efficiency. Moreover it is excellently suitable for combination with a wobbler plate mechanism for converting the rectilineal motion of the pistons into the rotary motion of a shaft. Thanks to the favourable shape of the pressure diagram, which shows no peaks, the difficulties encountered with wobbler plate mechanisms in petrol engines do not arise here.

Characteristic features of the air engine.

In conclusion the main characteristics of the air engine are summarized point for point:

1. The power is in principle proportional to the product of the mean pressure of the working medium and the number of revolutions. It is limited by the amount of heat given off by the heater and cooler; with low number of revolutions a high pressure and a correspondingly high torque, or with high number of revolutions a low pressure and a smaller torque are therefore possible.
   Maximum pressure is limited by the mechanical strength, and maximum number of revolutions by the occurring flow losses.
   It is possible to construct the engine in such a way that the maximum power can be produced over a wide range of speed. This gives the engine a characteristic analogous to that of a series wound electric-motor, which is of great importance for traction. The minimum number of revolutions is only limited by the piston leakage and is very low.
2. The torque is controlled by varying the mean pressure of the medium (in fact by varying the quantity of the medium). As the heater is kept at the same temperature the thermal efficiency does not vary with the load.
3. The engine is noiseless, there being no valves and no exhaust noise.
4. As heating is applied externally no special requirements are made of the fuel.
5. The favourable pressure diagram results in a uniform torque (in a 4-cylinder engine the variation is only about 8 %).
6. There is very little wear, as there are no hot running surfaces and neither dust nor corrosive gases inside the cylinder.
7. For the same reasons as under 6. the lubricating oil consumption is very low.
8. It is possible to burn the fuel completely, so that there is no CO in the exhaust gas.
9. Starting (at room temperature) takes more time than is the case with combustion engines, because the heater has first to be warmed

up, but once hot, the starting is most reliable and independent of the surrounding temperature.

10. Cooling requirements are higher than for internal combustion engines, because as the cooling temperature rises the efficiency drops. The amount of heat given off is about the same as that with an internal combustion engine.
11. In the multi-cylinder construction the direction of rotation can be reversed while running by reversing the order in which the cold and hot spaces of the successive cylinders are connected.

Hot-air process applied to refrigeration.

As already mentioned, the hot-air process, in principle at least, is reversible. This means that it can be applied in such a way that a certain amount of heat is drawn from a low temperature reservoir and, while simultaneously applying external work, imparted to a high temperature reservoir.

In this reversed process the same high thermodynamic efficiency is attained. By giving the right form to the elements corresponding to the heater, regenerator and cooler it has been possible to make efficient refrigerators. An experimental test with hydrogen as working medium showed that a temperature of 80° K could be reached in one stage. This process is in fact excellently suitable for cooling to temperatures which are not easily attainable by the usual methods.

Mathematics. — *Space groups and their axioms.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of January 26, 1946.)

Notations. Small letters denote a point of a set or an element of a group, but also the set consisting of one point or one element.

Capitals denote sets, while 0 indicates the vacuous set.

$U(p)$ is a neighbourhood of p.

$A \subset B$ or $B \supset A$ means: all points of A belong to B.

$A \cup B$ is the sum of A and B.

$\bigcup_{i=1}^{\infty} A_i$ is the sum of the sets $A_1, A_2, \ldots$, etc.

$A \cap B$ is the intersection of A and B.

$A - B$ is the difference of A and B. This is only defined for $A \subset B$.

$A \cdot B$ or AB is the set of all products ab (the product indicates the relation in a group) with $a \subset A$ and $b \subset B$.

The notions: group, continuous and topological mapping, open and closed sets are supposed to be known.

1. In topology a general neighbourhood-space may be defined as a (not-vacuous) set satisfying both (A) and (A'):

(A) To each point p of the set belongs at least one subset as a neighbourhood. $U(p)$ always contains p. The system of all neighbourhoods is denoted by $\{U\}$.

(A') The HAUSDORFF *criterion of equivalence*. If in a set are introduced two systems of neighbourhoods $\{U\}$ and $\{V\}$ the corresponding neighbourhood-spaces will be considered as topologically not-different (or equivalent) if: each $U(p)$ of an arbitrary point p contains a $V(p)$ and, conversely, each $V(p)$ contains a $U(p)$.

If the neighbourhood-space G is defined by the system of neighbourhoods $\{U\}$, then the sets out of equivalent systems $\{V\}, \{W\}, \ldots$ are also called neighbourhoods in G.

A neighbourhood of a subset $A \subset G$ is defined as the sum of certain neighbourhoods $U(a)$, where a runs through the set A.

Further usually the following axioms are, starting from a general neighbourhood-space, successively introduced:

(B) The intersection of two neighbourhoods of one point is another neighbourhood of that point.

(C) Among the equivalent systems of neighbourhoods there is at least one system consisting of open sets.

(D') Of every two different points one has a neighbourhood which does not contain the other point (and conversely).

(D) Every two different points have disjunct neighbourhoods.

(E) Two disjunct closed sets of which at least one consists of only one point have disjunct neighbourhoods.

(F) Two disjunct closed sets have disjunct neighbourhoods.

(G) There is a system of neighbourhoods consisting of enumerably many sets.

Often the following names are used:

if for a general neighbourhood-space hold true the axioms

(B), (C),	it is called a *topological space*;
(B), (C), (D'),	„ „ „ „ RIESZ *space*;
(B), (C), (D'), (D),	„ „ „ „ HAUSDORFF *space*;
(B), (C), (D'), (D), (E),	„ „ „ „ *regular space*;
(B), (C), (D'), (D), (E), (F),	„ „ „ „ *normal space*;
(B), (C), (D'), (D), (E), (F), (G),	„ „ „ „ *separable space*.

2. By a *space group* R we shall mean a set R which

1°. is an (abstract) *group*,

2°. is a (general) *neighbourhood-space*; so the axioms (A) and (A') are satisfied,

3°. satisfies both *axioms of continuity* (indicating the connection between group and space), i.e.

a. for every two elements a and b of R and for each neighbourhood $U(ab)$ there may be found a $U(a)$ and a $U(b)$ with

$$U(a) \cdot U(b) \subset U(ab);$$

b. for every neighbourhood $U(a^{-1})$ of an arbitrary element a^{-1} of R may be found a $V(a)$ with $V^{-1} \subset U$.

The axioms (B), (C), etc., which in 1. were introduced for neighbourhood-spaces, may also be introduced for space groups and to the so formed space groups we may give the corresponding names.

It now stands to reason to suppose that for space groups there is a closer connection between the topological axioms than for neighbourhood-spaces. In the following will be investigated in how far this is the case and in how far therefore simplifications offer themselves in comparison with the neighbourhood-spaces considered in a purely topological way. It will appear that the axioms (C) and (E) hold true in every space group (that in a topological group (E) holds true is already known from D. VAN DANTZIG's paper, Math. Ann. **107** (1933), p. 608). (B) and (D') may or may not hold true; (D) and (D') are equivalent in every space group. Further it will be proved, among other things, that an enumerable space group satisfying (B) and (D') is a normal space group.

3. In this number we repeat some properties, known for "topological groups", which hold true for all space groups.

If every element x of a general space group R is mapped on ax (where a is a fixed but arbitrarily chosen element of R), then in that way a topological mapping is determined of R onto itself. In the first place the mapping is one-to-one, as $ax = b$ for a given a and b has but one solution. Further the mapping is continuous in the direction $x \to ax$. For an arbitrary $U(ax)$ there may be found, according to the first axiom of continuity, a $U(a)$ and a $U(x)$ with $U(a) \cdot U(x) \subset U(ax)$, so that in any case $a \cdot U(x) \subset U(ax)$, i.e., $U(x)$ is mapped on a set belonging to $U(ax)$. In the same way it is proved that the inverse mapping $ax \to x$ or $x \to a^{-1}x$ is continuous.

Also the mappings $x \to xa$ and $x \to x^{-1}$ are topological. So, if V is a neighbourhood of the unity e, then V^{-1} also is a neighbourhood of e and Va and aV are neighbourhoods of a.

Every space group R is homogeneous, i.e., for every two elements a and b of R there exists a topological mapping of R onto itself, which maps a on b. The mapping $x \to xa^{-1}b$ fills this requirement. This homogeneity is an important property. For instance, if one wants to prove a certain (topological) property for a neighbourhood of a point p it is, because of the homogeneity, necessary and sufficient to prove this property for a neighbourhood of the unity e.

Suppose O is an open subset, G a closed subset, W an arbitrary set and p a point of a space group. Then Gp is a closed set, for the mapping $x \to xp$ is topological, so G is mapped onto a closed set Gp. In the same way one proves that Op is an open set. But then OW also is an open set, as the sum of an arbitrary number of open sets is also open. So in general: OW, O^{-1}, WO are open; G, G^{-1}, Gp and pG are closed.

4. **Theorem.** *In every space group axiom* (C) *holds true.*

Proof. Because of the homogeneity we only have to prove that within an arbitrary neighbourhood $U(e)$ of the unity e may be found an open set containing e. As $e \cdot e = e$, it is possible, according to the second axiom of continuity, to find for U two neighbourhoods $V_1^1(e)$ and $V_2^1(e)$ with

$$V_1^1 \cdot V_2^1 \subset U.$$

For the same reason there may be found for V_1^1 two neighbourhoods V_1^2 and V_2^2 with $V_1^2 \cdot V_2^2 \subset V_1^1$. Thus continuing we construct

$$V_1^{j+1} \cdot V_2^{j+1} \subset V_1^j$$

for $j = 1, 2, \ldots$.

Now consider the sum

$$V = \bigcup_{i=1}^{\infty} V_2^i \cdot V_2^{i-1} \cdot \ldots V_2^1.$$

For each term of V

$$V_2^i \cdot V_2^{i-1} \cdot \ldots V_2^1 \subset V_1^{i-1} \cdot V_2^{i-1} \cdot \ldots V_2^1 \subset V_1^{i-2} \cdot V_2^{i-2} \cdot \ldots V_2^1 \subset \ldots \subset$$
$$\subset V_1^1 \cdot V_2^1 \subset U.$$

So V belongs to U, but at the same time V is open, for an arbitrary element

$$p_2^i \cdot p_2^{i-1} \cdot \ldots \cdot p_2^1 \qquad (p_2^k \subset V_2^k)$$

of V has the neighbourhood (belonging to V)

$$V_2^{i+1} \cdot p_2^i \cdot p_2^{i-1} \cdot \ldots \cdot p_2^1.$$

Theorem. *In every space group R a point p and a closed set, disjunct with p, have disjunct neighbourhoods.*

From this follows in particular, that axiom (E) holds true in every space group.

Proof. Because of the homogeneity we may suppose that e is the point mentioned in the theorem; further let Q be a closed set which does not contain e.

Then the open set $R - Q$ is a $U(e)$. There may be found two neighbourhoods $V_1(e)$ and $V_2(e)$ with $V_1 \cdot V_2 \subset U$. V_1 is a neighbourhood of e and QV_2^{-1} is a neighbourhood of Q, as V_2^{-1} is a neighbourhood of e. V_1 and QV_2^{-1} are the required disjunct neighbourhoods. For, supposing this was not the case, there might be found in V_1, V_2^{-1} and Q the points v_1, v_2^{-1} and q respectively, with $v_1 = qv_2^{-1}$ or $q = v_1 v_2$, in contradiction with the fact that Q and $V_1 V_2$ are disjunct.

As to the validity of (B) and (D'), it will appear from the instances given in 5. that all four possibilities may offer themselves. From (D') always follows (D) and conversely.

Theorem. *In every space group (D') and (D) are equivalent.*

Proof. We only have to prove that, in case (D') holds true, e and a point $a \neq e$ have disjunct neighbourhoods. As (D') holds true there must be a $U(e)$ which does not contain a. For the rest the proof runs exactly as that of the theorem here above, if only one substitutes a for Q (we do not, however, contend that the point a — seen as a set — is closed).

It now stands to reason to introduce, starting from a general space group, the following axioms and denominations:

A space group, satisfying (B), is called a topological space group (and is indeed, considered as a space, a topological space).

A space group satisfying (B) and (D') is called a regular space group (and again, considered as an abstract space, is indeed a regular space).

A space group satisfying (B), (D') and (G) is called a separable space group (and again is a separable space).

Remark. Instances of regular space groups which are not normal are unknown to me.

The separable spaces, which are topologically speaking the most important (these are topologically identical with the subsets of the HILBERT space) are reached already by the axioms (A), (A'), (B), (C), (D'), (E) and (G). The separable space groups, which are again the most important space groups, are reached as follows from the theorems above by the axioms (A), (A'), (C), (D'), (G) and the group- and continuity-axioms.

5. In this number we investigate in particular the *space groups of finite order.*

If a space group of that kind satisfies (B) and (D') it is of course a space group consisting of isolated points, so there is no spacial relation whatever.

Each of the three remaining possibilities may offer itself however in a not trivial way, denoted by

1°. $(-B)$ [1], (D');
2°. (B), $(-D')$;
3°. $(-B)$, $(-D')$.

To prove this we use the following lemma.

Lemma. If in an abstract commutative group a system of neighbourhoods $\{V\}$ of the unity e is defined such that the axioms of continuity are satisfied as far as e is concerned (the a and b mentioned in the definition of the axioms of continuity always are e), and if one defines a system of neighbourhoods of an arbitrary but fixed element a of the group by $\{aV\}$, then the abstract group is hereby transformed into a space group.

Proof. We apparently only have to prove the axioms of continuity. Let abU be a neighbourhood of ab, then for $U = U(e)$ there may be found a $V_1(e)$ and a $V_2(e)$ with $V_1 \cdot V_2 \subset U$. Now $aV_1 \cdot bV_2 = = abV_1V_2 \subset abU$. In the same simple way one proves the second axiom of continuity.

We shall now give the required instances the correctness of which is easy to see.

1°. $(-B)$, (D'). The abstract commutative group consists of the elements e, a, b, and ab with moreover the relations $a^2 = b^2 = e$. A system of neighbourhoods of e consists by definition of the sets $\{e, a\}$ and $\{e, b\}$.

2°. (B), $(-D')$. The abstract group as in 1°. A system of neighbourhoods of e consists by definition of the set $\{e, a\}$.

3°. $(-B)$, $(-D')$. The abstract commutative group consists of the elements e, a, b, c, ab, ac, bc and abc with moreover the relations $a^2 = b^2 = c^2 = e$. A system of neighbourhoods of e consists by definition of the sets $\{e, a, b, ab\}$ and $\{e, a, c, ac\}$.

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[1] $(-B)$ means, that axiom (B) does not hold true.

6. For a space group of enumerable order we have the following

Theorem. Every space group of enumerable order, satisfying (B), also satisfies (F). From this follows in particular that *every regular-space group of enumerable order is normal.*

Proof. Let P and Q be two closed disjunct sets, consisting respectively of the points $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ (the case where at least one of the sets consists of a finite number of points is simpler and is left out of consideration). For e may be constructed a neighbourhood U so that $p_1 U$ (this is a neighbourhood of p_1) contains no point of Q and $q_1 U$ (this is a neighbourhood of q_1) contains no point of P. The possibility of this follows from the validity of (B) and of the first theorem in 4. Within U we may construct a neighbourhood $V_1(e)$ with $V_1 \cdot V_1^{-1} \subset U$. Therefore also $p_1 V_1 V_1^{-1} \cap Q = 0$ and $q_1 V_1 V_1^{-1} \cap P = 0$. Thus continuing one constructs

$$V_1 V_1^{-1} \supset V_1 \cap V_1^{-1} \supset V_2 V_2^{-1} \supset V_2 \cap V_2^{-1} \supset V_3 V_3^{-1} \supset \ldots$$

with

$$p_i V_i V_i^{-1} \cap Q = 0 \text{ and } q_i V_i V_i^{-1} \cap P = 0 \qquad (i = 1, 2, \ldots) \ . \quad (1)$$

Now

$$U(P) = \overset{\infty}{\underset{i=1}{\cup}} p_i V_i \text{ and } U(Q) = \overset{\infty}{\underset{i=1}{\cup}} q_i V_i$$

are the required disjunct neighbourhoods of P and Q: For suppose $U(P)$ and $U(Q)$ had a common point; then this would have the shape

$$p_k v_k = q_l v_l.$$

We may suppose $k \geqq 1$. Then we find

$$p_k = q_l v_l v_k^{-1} \subset q_l V_l V_k^{-1} \subset q_l V_l V_l^{-1}$$

in contradiction with (1).

Mathematics. — *Généralisation P-adique d'un théorème de* MINKOWSKI *sur les formes linéaires.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

Il s'agit du théorème suivant:

Soient données n formes linéaires

$$L_i(x) = \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, n) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

avec coefficients réels a_{ij} *et déterminant* $\Delta = |a_{ij}| \neq 0$, *et n nombres positifs* $C_1, \ldots, C_n$ *pour lesquels*

$$C_1 \ldots . C_n \geqq |\Delta| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Alors ils existent n nombres entiers rationnels $x_1, \ldots, x_n$, *non tous 0, tels que*

$$|L_i(x)| \leqq C_i \qquad (i = 1, \ldots, n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

MAHLER a généralisé ce théorème en considérant le cas où un certain nombre des formes a des coëfficients non-réels, mais appartenant à des corps P-adiques [1]). Bienque très utile pour les applications, d'un point de vue théorique cette généralisation n'est pas entièrement satisfaisante: il n'est pas permis que les coefficients des formes-à-coefficients-réels sont tous zéro — il faut même que leur déterminant est $\neq 0$ —, il ne considère donc pas le cas où il n'y a que des formes, n'ayant que des coefficients P-adiques.

Dans ce qui suit nous donnons une nouvelle généralisation. Elle a rapport à l'espace $R_n(K(P))$, dont les points sont les suites $(x_1, \ldots, x_n)$ de nombres P-adiques et que nous métrisons par la définition

$$\| x \| = \max_{1 \leqq i \leqq n} |x_i|_P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Soit S un ensemble de points de $R_n(K(P))$ avec les propriétés suivantes:

1. S soit un ensemble discret, donc S n'a pas des points d'accumulations finis. Il s'ensuit que S est dénombrable.

2. Soit $\varphi(\lambda)$ le nombre des points x de S tels que

$$\| x \| \leqq P^{\lambda} (-\infty < \lambda < +\infty).$$

Supposons que

$$\limsup_{\lambda \to \infty} \frac{P^{n\lambda}}{\varphi(\lambda)} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

est fini.

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1) K. MAHLER, Über diophantische Approximationen im Gebiete der P-adischen Zahlen. Jahresber. d. Deutschen Math. Verein. 44, 250—255 (1934).

Considérons le système d'inégalités [2])

$$|L_i(x)|_P \leqq P^{c_i} \quad (i=1,\ldots,n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

où $a_{ij}(i,j=1,\ldots,n)$ sont des nombres P-adiques; soit $\Delta = |a_{ij}| \neq 0$. Cherchons une solution de ce système de la forme $x = a - b$ où $a, b \in S$ et $a - b$ désigne le point $(a_1 - b_1, \ldots, a_n - b_n)$.

Supposons que le système (6) n'admet aucune solution $x = a - b$ avec $x \neq O = (0,\ldots,0)$. Les points x de R_n, vérifiants (6), constituent une partie bornée I_0 de l'espace. Posons pour cela

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad (i=1,\ldots,n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Pour tous les points x vérifiants (6) on a donc

$$|y_i|_P \leqq P^{c_i} \qquad (i=1,\ldots,n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

donc

$$\|y\| \leqq \max_{1 \leqq i \leqq n} P^{c_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Puisque $|a_{ij}| \neq 0$, il suit de (7)

$$x_i = \frac{1}{\Delta} \sum_{j=1}^{n} A_{ij} y_j \qquad (i=1,\ldots,n) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Donc

$$|x_i|_P \leqq \max_{1 \leqq j \leqq n} |A_{ij} y_j|_P \cdot \frac{1}{|\Delta|_P} \qquad (i=1,\ldots,n)$$

et par suite

$$\|x\| \leqq \frac{\|y\|}{|\Delta|_P} \max_{i,j} |A_{ij}|_P,$$

de sorte que l'ensemble I_0 est borné en vertu de (9).

On peut définir dans R_n une mesure avec les propriétés de la mesure de LEBESGUE dans le domaine réel: il suffit de prendre un produit de la mesure, introduite par TURKSTRA dans le corps des nombres P-adiques [3]).

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[2]) Ce n'est pas une restriction de prendre pour le membre à droite une puissance de P puisque les membres à gauches sont aussi des puissances de P.

[3]) H. TURKSTRA. Metrische bijdragen tot de theorie der diophantische approximaties in het lichaam der P-adische getallen. Groningen 1936.

Voir aussi: A. F. MONNA, Zur Theorie des Maszes in Körper der P-adischen Zahlen. Ned. Akad. v. Wetensch., Amsterdam, **45**, 978—980 (1942).

Pour l'espace $R_n(K(P))$ voir: A. F. MONNA, Zur Geometrie der P-adischen Zahlen. Ned. Akad. v. Wetensch., Amsterdam, **45**, 981—986 (1942).

Utilisons cette occasion pour remarquer que l'introduction des intégrales au § 4 de ce dernier article peut être évitée. On peut aisément montrer le lemme sans intégrales en n'utilisant que l'additivité de la mesure.

L'ensemble I_0 est mesurable; soit mI_0 la mesure; dans ce qui suit nous donnons une expression explicite de mI_0.

Considérons dans R_n l'ensemble borné I_g, déterminé par les inégalités

$$| L_i (x-g) |_P \leqq P^{c_i} \qquad (i=1,\ldots,n) \quad . \quad . \quad . \quad . \quad (11)$$

où g parcourt les points de S. I_g est déduit de I_0 par la translation

$$x_i = \bar{x}_i - g_i \qquad (i=1,\ldots,)$$

ou

$$x = \bar{x} - g,$$

de sorte que, la mesure étant invariante par translation,

$$m\, I_0 = m\, I_g.$$

Il suit de la supposition, que les inégalités (6) n'admettent aucune solution $x = a - b \neq 0$ avec a et b dans S, qu'aucun couple des I_g n'a des points communs. Supposons en effet qu'on avait $x \in I_g$, $x \in I'_g$; $g \neq g'$. On avait

$$| L_i (x-g) |_P \leqq P^{c_i},$$

$$| L_i (x-g') |_P \leqq P^{c_i},$$

et

$$L_i (g-g') |_P = | L_i (x-g') - L_i (x-g) |_P \leqq \max (| L_i (x-g') |_P, \; | L_i (x-g) |_P) \leqq P^{c_i}.$$

En contradiction avec l'hypothèse, les inégalités admettaient alors la solution $g - g' \neq 0$.

Soit C tel que I_0 est contenu dans l'intervalle n-dimensionnel

$$| x_i |_P \leqq P^C \; (i=1,\ldots,n).$$

Soit $\lambda > C$. Alors tous les I_g pour lesquels $| g_i |_P \leqq P^\lambda (i=1,\ldots,n)$ — donc $\| g \| \leq P^\lambda$ — appartiennent à l'intervalle $\| x \| \leq P^\lambda$. En effet

$$| L_i (x-g) |_P \leqq P^{c_i} \qquad x \in I_g ; x - g \in I_0$$
$$| g_i |_P \leqq P^\lambda,$$

donc

$$| x_i |_P = | x_i - g_i + g_i |_P \leqq \max (| x_i - g_i |_P , | g_i |_P)$$
$$\leqq \max (P^C, P^\lambda) = P^\lambda.$$

La mesure de l'intervalle $\| x \| \leq P^\lambda$ vaut $P^{n\lambda}$ et $\varphi(\lambda)$ est le nombre des points g de S pour lesquels $\| g \| \leq P^\lambda$. Il s'ensuit

$$\varphi(\lambda).\, m\, I_0 \leqq P^{n\lambda},$$

$$m\, I_0 \leqq \frac{P^{n\lambda}}{\varphi(\lambda)},$$

donc avec (5) si $\lambda \to \infty$,

$$m I_0 \leqq a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Il reste encore de déterminer mI_0. Partons pour cela des relations (7) et (8). Par (8) est déterminé un intervalle n-dimensionnel avec mesure

$$P^{c_1 + \ldots + c_n}.$$

Par (7) cet intervalle est transformé dans I_0. On a

$$x_i = \frac{1}{\triangle} \sum_j A_{ij}\, y_j\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

où A_{ij} est le mineur de a_{ji}. On a donc pour tous les points x de I_0

$$| x_i |_P \leqq \frac{1}{| \triangle |_P} \max_{1 \leqq j \leqq n} | A_{ij}\, y_j |_P\,,$$

ou

$$| x_i |_P \leqq \frac{1}{| \triangle |_P} \max_j P^{c_j} | A_{ij} |_P. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Cependant si $| y_j |_P \leqq P^{c_j}$, les x_i de (13) parcourent *tous* les nombres z pour lesquels

$$| z |_P \leqq \frac{1}{| \triangle |_P} \max_{1 \leqq j \leqq n} P^{c_j} | A_{ij} |_P.$$

D'abord $A_{ij} \dfrac{y_j}{\triangle}$ parcourt *tous* les nombres avec valeur P-adique $\leqq | A_{ij} \triangle^{-1} |_P P^{c_j}$; posons cette dernière valeur $= M_{ij}$. Il reste à montrer que, si $u_{ij} (j = 1, \ldots, n)$ parcourt tous les nombres avec valeur $\leqq M_{ij}$, Σu_{ij} parcourt *tous* les nombres z avec valeur $\leqq \max\limits_j M_{ij}$. Il est clair que chaque Σu_{ij} est un nombre z. Soit inversement donné un z avec

$$| z |_P \leqq \max_{1 \leqq j \leqq n} M_{ij}.$$

Si $\max\limits_j M_{ij} = M_{ij_0}$ nous posons

$$u_{ij} = 0 \quad \text{si} \quad j \neq j_0$$

$$u_{ij_0} = z$$

Alors

$$z = \sum_j u_{ij},$$

donc de la forme désirée.

Il s'ensuit que I_0 est identique avec l'ensemble des points pour lesquels on a (14). On a alors

$$m I_0 = \frac{\prod\limits_{i=1}^{n} \max\limits_{1 \leqq j \leqq n} P^{c_j} | A_{ij} |_P}{| \triangle |_P^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

La supposition que les inégalités (6) n'admettent aucune solution $x = a—b$, avec a et b dans S, autre que O, conduit donc selon (12) à la condition

$$\frac{\prod_{i=1}^{n} \max_{1 \leqq j \leqq n} P^{c_j} \mid A_{ij} \mid_P}{\mid \Delta \mid_P^n} \leqq \alpha. \quad \ldots\ldots\ldots \quad (16)$$

Donc si

$$\prod_{i=1}^{n} \max_{1 \leqq j \leqq n} P^{c_j} \mid A_{ij} \mid_P > \alpha \mid \Delta \mid_P^n,$$

le système d'inégalités

$$\mid L_i(x) \mid_P \leqq P^{c_i} \quad (i = 1, \ldots, n)$$

admet au moins une solution de la forme $x = a - b \neq O$ *avec* $a \in S$, $b \in S$.

Remarquons que dans cette généralisation on aperçoit une circonstance qu'on ne peut pas distinguer dans le cas réel: il s'agit de la solubilité d'un système d'inégalités par des nombres, pas tous zéro, dont chacun est la différence de nombres d'une suite discrète donnée. On ne voit pas cette circonstance dans le cas réel puisque là, chaque différence appartient elle même à la suite donnée, ou autrement dit, les nombres réels entiers et rationnels constituent un groupe additif, qui est discret si on considère la valorisation absolue.

Le corps des nombres P-adiques n'admet aucun sous-groupe additif discret, différent du groupe ne contenant que l'élément 0. Chaque sous-groupe contient les nombres $\ldots -2, -1, 0, 1, 2, \ldots$. Dans le valorisation P-adique ces nombres ne constituent pas une suite discrète. La suite $P, P^2, P^3, \ldots$ converge P-adiquement vers zéro et on voit que chaque nombre entier rationnel est un point d'accumulation P-adique de la suite $\ldots -1, 0, 1, \ldots$.

Plus général, on a que chaque corps avec caractéristique nulle, valorisé d'une façon non-triviale et non-archimédienne, ne contient aucun groupe additif discret. En effet, chaque sous-groupe contient les éléments ke, si e est l'élément-unité et k entier rationnel. De plus chaque valorisation non-archimédienne des nombres rationnels est une valorisation P-adique, de sorte que, d'après ce qui précède, aucun sous-groupe ne peut être discret.

Si la caractéristique du corps est $\neq 0$ ou si la valorisation est triviale, il peuvent exister des sous-groupes discrets. Dans le dernier cas l'espace correspondant au corps ne contient que des points isolés.

Apeldoorn, printemps 1945.

Mathematics. — *Zur Mass- und Integrationstheorie in Strukturen.* I. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

Die im folgenden behandelten Mass- und Integrationstheorien in Strukturen, bei welchen die Werte der auftretenden Masse nicht immer von einerlei Verzeichen zu sein brauchen, lassen sich in die von uns in Acta math. **73** (1941) behandelten Mass- und Integrationstheorien einbetten.

I. *Beschränkt additives Mass.*

§ 1. Alle hier vorkommenden *Strukturen* genügen den Axiomen 1°, 2°, 3°, 4°, 6° und 7°; jede Struktur ist aufgebaut aus Elementen, *Somen* genannt, die mit kleinen Buchstaben angedeutet werden. Der Grundbegriff „Teil von" wird durch das Zeichen $\subset$ angedeutet. Eine Struktur, welche ausserdem Axiom 5° genügt, ist *eine Boolesche Struktur* oder *eine Boolesche Algebra.*

Axiom 1°: $\alpha)$ $a \subset a$; $\beta)$ aus $a \subset b$ und $b \subset c$ folgt $a \subset c$.

Definition. $a = b$, falls $a \subset b$ und $b \subset a$.

Satz 1. $a = a$; $a = b \rightarrow b = a$; $a = b$ und $b = c \rightarrow a = c$.

Definition. Ein Soma ab oder $a \,.\, b$ wird *Produkt* des Somenpaares a, b genannt, falls: $\alpha)$ $ab \subset a$; $\beta)$ $ab \subset b$; $\gamma)$ aus $c \subset a$ und $c \subset b$ immer folgt $c \subset ab$.

Satz 2. Hat das Somenpaar a, b ein Produkt, so ist dieses eindeutig bestimmt, d.h. zwischen zwei Somen, deren jedes, gemäss letzter Definition, als Produkt von a und b betrachtet werden kann, gilt immer die Relation der Gleichheit.

Axiom 2°. Für jedes Somenpaar a, b gibt es ein Produkt ab.

Definition. Ein Soma $a + b$ wird Summe des Somenpaares a, b genannt, falls: $\alpha)$ $a \subset a + b$; $\beta)$ $b \subset a + b$; $\gamma)$ aus $a \subset c$ und $b \subset c$ immer folgt $a + b \subset c$.

Satz 2'. Hat ein Somenpaar eine Summe, so ist diese eindeutig bestimmt.

Axiom 3°. Für jedes Somenpaar a, b gibt es eine Summe $a + b$.

Satz 3, 3'. Aus $a \subset b$ folgt $ab = a$, und umgekehrt; aus $b \subset a$ folgt $a + b = a$, und umgekehrt.

Definition. Eine Struktur S besitzt ein kleinstes Soma, *leeres Soma* genannt, und angedeutet durch 0, falls $0 \subset a$ für jedes Soma a von S.

Satz 4. Gibt es ein leeres Soma, so ist dieses eindeutig bestimmt.

Axiom 4°. Es gibt ein kleinstes (leeres) Soma 0.

Definition. Zwei Somen a, b heissen *einander fremd*, falls $ab = 0$.

Satz 5. Ist $a \subset b$ und sind a und b einander fremd, so ist $a = 0$.

Satz 6. Ist $c \subset b$ und sind a und b einander fremd, so sind auch a und c einander fremd.

Definition. Eine Struktur S besitzt *ein grösstes Soma*, angedeutet durch 1, falls $a \subset 1$ für jedes Soma a von S.

Satz 4'. Gibt es ein grösstes Soma, so ist dieses eindeutig bestimmt.

Axiom 5°. Es gibt ein grösstes Soma 1.

Definition. Ist $a \subset b$, so wird *komplementäres Soma erster Art* genannt und angedeutet durch $b - a$ jedes zu a fremde Element x mit

$$a + x = b.$$

Definition. Ein Soma $1 - c$ wird *Komplement von* c genannt und durch c' angedeutet.

Axiom 6°. $ac + bc = (a + b)c$.

Axiom 7°. Zu jedem Paar von Somen a und b, mit $a \subset b$, gibt es ein komplementäres Soma erster Art, $b - a$.

Satz 7. Unter der Annahme $a \subset b$ ist $b - a$ in einer den Axiomen $1° - 4°$, $6°$, $7°$ genügenden Struktur eindeutig bestimmt; in einer Booleschen Algebra ist sodann $b - a = a'b$.

Satz 8. Aus $c \subset a + b$ und c und b einander fremd folgt, unter Annahme der Axiome $1° - 4°$, $6°$, $7°$, $c \subset a$.

Satz 9, 9'. In einer Booleschen Struktur ist

$$(a + b)' = a'\,b' \quad \text{und} \quad (ab)' = a' + b'.$$

In einer Booleschen Struktur gilt folgendes Dualitätsprinzip: zu jedem in ihr ableitbaren Satz gibt es einen dualen, welchen man erhält, wenn im ursprünglichen Theorem überall: $\alpha)$ $x \subset y$ durch $y \subset x$, und somit: $\beta)$ xy durch $x + y$, und umgekehrt, ersetzt wird, daneben: $\gamma)$ 0 durch 1, und umgekehrt, ersetzt wird; lässt man $\gamma)$ fort, so erhält man ein in einer den Axiomen $1° - 4°$, $6°$, $7°$ genügenden Struktur gültiges Dualitätsprinzip.

§ 2. In diesem Paragraphen betrachten wir eine Struktur S, welche Axiom 5° nicht zu erfüllen braucht. K sei eine Klasse von Somen von S, welche folgende Eigenschaften hat:

I. es gibt eine nicht leere (echte oder unechte) Teilklasse K_1 von K derart, dass ein Soma x zu K gehört, falls aus $a \in K_1$ immer folgt $a - ax \in K$; umgekehrt soll aus $x \in K$, $a \in K_1$ immer folgen $a - ax \in K$;

II. aus $x \in K$, $y \in K$ folgt $x + y \in K$; und:

III. aus $a \in K_1$, $b \in K_1$ folgt $a + b \in K_1$.

Aus I. folgt $0 \in K$, und, falls Axiom 5° erfüllt wird, auch $1 \in K$.

Satz 10. Aus $x \in K$, $y \in K$ folgt $xy \in K$ und $x - xy \in K$; die Somen von K bilden eine den Axiomen $1° - 4°$, $6°$, $7°$ genügende Teilstruktur $S(K)$ von S [1]).

Umgekehrt bilden die Somen einer den Axiomen $1° - 4°$, $5°$, $6°$, $7°$ genügenden Teilstruktur eine die Bedingungen I., II., III. erfüllende Klasse K; als Teilklasse K_1 kann man dabei K selbst nehmen, oder auch die aus dem grössten Soma von $S(K)$ allein bestehenden Klasse.

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[1]) Beim Beweise wird die Bedingung III. nicht benutzt.

Definition. $f(x)$ sei eine reellwertige Somenfunktion, definiert für alle und nur alle diejenigen Somen $(x) \in K$, zu deren jedem ein Soma $a(x) \in K_1$ gehört mit $x \subset a(x)$, und nicht identisch Null. Sie soll *beschränkt additiv* heissen, falls:

1°. für je zwei einander fremde Somen x, y, für die f definiert ist,

$$f(x) + f(y) = f(x + y)$$

ist; und:

2°. für jedes Soma $a \in K_1$ die obere Schranke aller $|f(x)|$-Werte, mit $x \subset a$ und $x \in K$, endlich ist.

Ist f definiert für das Soma x, so sei *ihre positive Variation* $G(x)$ die (endliche) obere Schranke aller $f(e)$-Werte mit $e \subset x$ und $e \in K$; *ihre negative Variation* $g(x)$ sei die (endliche) untere Schranke dieser $f(e)$-Werte; *ihre Totalvariation* $T(x)$ sei definiert durch die Summe $G(x) + |g(x)|$.

Satz 11. $G(x)$, $g(x)$ und $T(x)$ sind beschränkt additive Somenfunktionen; dabei ist

$$G(x) + g(x) = f(x).$$

Satz 12. Die Somen, für die f, G, g und T definiert sind, bilden eine den Axiomen 1° — 4°, 6° und 7° genügende Teilstruktur $S(K_2)$ von $S(K)$.

Definition. Für jedes Soma x, zu welchem es ein Soma $a \in K_2$ mit $x \subset a$ gibt, definieren wir *die äussere T-Funktion* $T_a(x)$ gleich der unteren Schranke aller $T(a)$-Werte mit $x \subset a$ und $a \in K_2$. Die Klasse dieser Somen sei K_3.

Satz 13. Die Somen der Klasse K_3 bilden eine den Axiomen 1° — 4°, 6°, 7° genügende Teilstruktur $S(K_3)$ von S.

Satz 14. Für je zwei Somen x, y der Klasse K_3 ist

$$T_a(x + y) \leqq T_a(x) + T_a(y).$$

Satz 15. Für die zu K_2 gehörenden Somen (x) von K_3 fallen $T_a(x)$ und $T(x)$ zusammen.

Definition. Ein Soma a heisse *$T_a(x)$-messbar oder messbar in bezug auf $T_a(x)$*, wenn für jedes Soma $w \in K_3$

$$T_a(w) = T_a(wa) + T_a(w - wa)$$

ist.

In analoger Weise definiert man *die $G_a(x)$- und die $|g|_a(x)$-Messbarkeit* von Somen.

Satz 16. Fügt man $T_a(x)$ für die nicht zur Klasse K_3 gehörenden Somen den Wert $+\infty$ zu, so genügt die sodann für jedes Soma definierte äussere T-Funktion den Bedingungen: I*b*. es gibt ein Soma w mit $T_a(w)$ endlich und $\neq 0$; III. aus $a \subset b$ folgt immer $T_a(a) \leqq T_a(b)$; *V*. für jedes Soma a mit $T_a(a)$ endlich ist $T_a(a)$ gleich der unteren Schranke der Werte $T_a(\bar{x})$ für alle $T_a(x)$-messbaren Somen $(\bar{x})$ mit $a \subset \bar{x}$.

Beweis. Es genügt V. zu beweisen.

Aus $a \in K_2$, $w \in K_2$ folgt, mit den Sätzen 11, 12 und 15,

$$T_a(w) = T_a(wa) + T_a(w - w_a).$$

Zu $w \in K_3$ und willkürlich positivem ε gibt es ein $\overline{w} \in K_2$ mit $w \subset \overline{w}$ und

$$T(\overline{w}) < T_a(w) + \varepsilon.$$

Nun ist $wa \subset \overline{w}a$ und $w - wa \subset \overline{w} - \overline{w}a$; folglich

$$T_a(w) > T(\overline{w}) - \varepsilon = T_a(a\overline{w}) + T_a(\overline{w} - a\overline{w}) - \varepsilon \geqq T_a(aw) + T_a(w - aw) - \varepsilon,$$

also auch

$$T_a(w) \geqq T_a(aw) + T_a(w - aw).$$

Mit den Sätzen 13 und 14 liefert dies

$$T_a(w) = T_a(aw) + T_a(w - aw).$$

Jedes Soma $a \in K_2$ ist somit $T_a(x)$-messbar. Daraus folgt V.

Aus Satz 16 folgt, dass alle in den Paragraphen 2—6 unserer Arbeit: *Mass- und Integrationstheorie in Strukturen* [2]) erhaltenen Eigenschaften der Somenfunktion $m^\circ(x)$ sich ohnemehr auf die (erweiterte) Funktion $T_a(x)$ übertragen lassen.

Definition. Zu jedem Soma a, mit $T_a(a)$ endlich, wähle man ein $T_a(x)$-messbares Soma b mit $a \subset b$; man lege *der zu* $T_a(x)$ *adjungierten Massfunktion* $T_i(x)$ den Wert

$$T_a(b) - T_a(b - a)$$

bei. Für jedes Soma a mit $T_a(a) = \infty$ sei auch $T_i(a) = \infty$.

Durch obige Definition ist $T_i(a)$ eindeutig bestimmt.

Die in den Paragraphen 2—6 der eben zitierten Arbeit erhaltenen Eigenschaften der Somenfunktion $m_0(x)$ gelten nun ohnemehr für die Funktion $T_i(x)$. Daraus folgt u.a.:

Satz 17. $T_i(x)$ hat die Eigenschaften: I*b*. es gibt ein Soma w mit $T_i(w)$ endlich und $\neq 0$; III. aus $a \subset b$ folgt immer $T_i(a) \leqq T_i(b)$; IV. zu jedem Soma a mit $T_i(a)$ endlich gibt es ein $T_i(x)$-messbares Soma b mit $a \subset b$ und $T_i(b)$ ebenfalls endlich; V′. für jedes Soma a mit $T_i(a)$ endlich ist $T_i(a)$ gleich der oberen Schranke der Werte $T_i(\bar{x})$ für alle $T_i(x)$-messbaren Somen $(\bar{x})$ mit $\bar{x} \subset a$.

Die T_a- und die T_i-messbaren Somen fallen zusammen. Notwendig und hinreichend zur T_a- und T_i-Messbarkeit eines Somas a mit $T_a(a)$ oder $T_i(a)$ endlich ist die Relation $T_a(a) = T_i(a)$.

Die T_a- und T_i-messbaren Somen bilden eine den Axiomen 1° — 4°, 6°,

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[2]) Siehe Acta math. **73** (1941), S. 131—173.

7° genügende Struktur $S(K_5)$, welche $S(K)$ und $S(K_2)$ als Teilstrukturen enthält [3]); ebenso enthält $S(K_5)$ eine die gleichen Axiome erfüllende Teilstruktur $S(K_4)$, die alle und nur alle diejenigen T_a-messbaren Somen umfasst, welche in einem Soma der Klasse K_2 enthalten sind. —

Ist die Funktion $G(x)$ nicht immer Null, so hat $G_a(x)$ dieselben Eigenschaften wie $T_a(x)$; das gleiche gilt für $|g(x)|$ und $|g|_a(x)$; für die nicht zur Klasse K_3 gehörenden Somen werde dabei einer jeden dieser Funktionen der Wert $+\infty$ beigelegt. Ist $G(x) = 0$ für jedes Soma $x \in K_2$ so ist $G_a(a) = 0$ für jedes $a \in K_3$, und jedes Soma von S ist $G_a(x)$-messbar; eine analoge Bemerkung gilt für $|g(x)|$.

Satz 18. Jedes $T_a(x)$-messbare Soma ist auch $G_a(x)$- und $|g|_a(x)$-messbar.

Definition. Für jedes Soma $a \in K_5$ und mit $T_a(a)$ endlich (also $\in K_4$) sei *das f-Mass* $m_f(a)$ definiert durch:

$$m_f(a) = G_a(a) - |g|_a(a).$$

Dann ist für jedes solche Soma a

$$m_G(a) = G_a(a), \quad m_g(a) = -|g|_a(a),$$

wodurch

$$m_f(a) = m_G(a) + m_g(a)$$

ist; ausserdem gilt

$$m_T(a) = m_G(a) - m_g(a).$$

Satz 19. Für $a \in K_2$ ist

$$m_f(a) = f(a), \; m_T(a) = T(a), \; m_G(a) = G(a) \text{ und } m_g(a) = g(a).$$

Definition. Für jedes Soma $a \in K_5$ mit $T_a(a)$ unendlich sei

$m_G(a) =$ obere Schranke aller $m_G(a \,.\, w)$-Werte mit $w \in K_2$,

$m_g(a) =$ untere Schranke aller $m_g(a \,.\, w)$-Werte mit $w \in K_2$,

$$m_f(a) = m_G(a) + m_g(a),$$

und

$$m_T(a) = m_G(a) - m_g(a),$$

wobei zu beachten ist, dass $m_f(a)$ als unbestimmt betrachtet werden soll, wenn $m_G(a)$ und $m_g(a)$ beide unendlich sind.

Satz 20. Für $a \in K_5$, $b \in K_5$, $ab = 0$ wird immer

$$m_T(a + b) = m_T(a) + m_T(b)$$

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[3]) Zum Beweise der behauptung, dass $S(K)$ eine Teilstruktur von $S(K_5)$ ist, vergleiche man J. RIDDER, Fund. math. 24 (1935), S. 82 (Beweis von Satz XVI).

sein; es ist

$$m_f(a) + m_f(b) = m_f(a + b),$$

sobald bekannt ist, dass einer der beiden Glieder einen endlichen oder bestimmt unendlichen Wert hat.

§ 3. *Bildung einer Produktstruktur.*

Definition. $S^{(1)}$ und $S^{(2)}$ seien den Axiomen 1° — 4°, 6°, 7° genügende Strukturen. Unter *der Produktstruktur* $S^{(1)} \times S^{(2)}$ verstehen wir eine Struktur, deren Somen „Summen" endlich vieler „Produkte" $a_j^{(1)} \times a_j^{(2)}$ sind, mit $a_j^{(1)} \in S^{(1)}$, $a_j^{(2)} \in S^{(2)}$; dabei soll die „Summe" $\{a_j^{(1)} \times a_j^{(2)}\}$ $(j = 1, \ldots, m) \subset$ die „Summe" $\{b_k^{(1)} \times b_k^{(2)}\}$ $(k = 1, \ldots, n)$ sein, falls zu jedem „Produkt" $a_j^{(1)} \times a_j^{(2)}$ mit $a_j^{(1)} \neq 0$ und $a_j^{(2)} \neq 0$ endlich viele Somen $0 \neq a_{j;p}^{(1)} \in S^{(1)}$ $(p = 1, \ldots, P_j)$, $0 \neq a_{j;q}^{(2)} \in S^{(2)}$ $(q = 1, \ldots, Q_j)$ gehören, mit

$$a_{j;p}^{(1)} \cdot a_{j;\bar{p}}^{(1)} = 0 \text{ für } p \neq \bar{p}, \quad a_{j;q}^{(2)} \cdot a_{j;\bar{q}}^{(2)} = 0 \text{ für } q \neq \bar{q},$$

$$\sum_{p=1}^{P_j} a_{j;p}^{(1)} = a_j^{(1)}, \quad \sum_{q=1}^{Q_j} a_{j;q}^{(2)} = a_j^{(2)},$$

und der weiteren Eigenschaft, dass sich zu jedem Paar $a_{j;p}^{(1)}$, $a_{j;q}^{(2)}$ ein „Produkt" $b_k^{(1)} \times b_k^{(2)}$ angeben lässt mit

$$a_{j;p}^{(1)} \subset b_k^{(1)} \text{ und } a_{j;q}^{(2)} \subset b_k^{(2)}.$$

Satz 21. Die Produktstruktur $S^{(1)} \times S^{(2)}$ genügt den Axiomen 1° — 4°, 6°, 7°, und, falls Axiom 5° in $S^{(1)}$ und in $S^{(2)}$ gilt, auch Axiom 5°. Dabei ist das Produkt

$$\{a_j^{(1)} \times a_j^{(2)}\}_{(j=1,\ldots,m)} \cdot \{b_k^{(1)} \times b_k^{(2)}\}_{(k=1,\ldots,n)}$$

gleich

$$\{(a_j^{(1)} \cdot b_k^{(1)}) \times (a_j^{(2)} \cdot b_k^{(2)})\}_{(j=1,\ldots,m;\ k=1,\ldots,n)},$$

und die Summe

$$\{a_j^{(1)} \times a_j^{(2)}\}_{(j=1,\ldots,m)} + \{b_k^{(1)} \times b_k^{(2)}\}_{(k=1,\ldots,n)}$$

gleich

$$\{a_j^{(1)} \times a_j^{(2)},\ b_k^{(1)} \times b_k^{(2)}\}_{(j),(k)},$$

also gleich der „Summe" aller „Produkte" $a_j^{(1)} \times a_j^{(2)}$ und aller „Produkte" $b_k^{(1)} \times b_k^{(2)}$. Das leere Soma ist „Summe" von „Produkten" $a^{(1)} \times a^{(2)}$ mit entweder $a^{(1)}$ oder $a^{(2)}$ oder beide gleich dem leeren Soma in der zugehörigen Struktur; falls Axiom 5° in $S^{(1)}$ und in $S^{(2)}$ gilt, ist das grösste Soma der Produktstruktur gleich $\{1^{(1)} \times 1^{(2)}\}$; hierbei deuten $1^{(1)}$ und $1^{(2)}$ die grössten Somen von $S^{(1)}$ bzw. $S^{(2)}$ an.

Definition. *Elementares Soma* von $S^{(1)} \times S^{(2)}$ sei jedes Soma, das nur

ein einzelnes „Produkt" $a^{(1)} \times a^{(2)}$, mit $a^{(1)} \in S^{(1)}$, $a^{(2)} \in S^{(2)}$ enthält.

Satz 22*a*. Jede Differenz

$$\{a^{(1)} \times a^{(2)}\} - \{a^{(1)} \times a^{(2)}\} \cdot \{b^{(1)} \times b^{(2)}\}$$

lässt sich als Summe von zwei einander fremden elementaren Somen darstellen [4]).

Satz 22*b*. Jede Differenz

$$\{a^{(1)} \times a^{(2)}\} - \{a^{(1)} \times a^{(2)}\} \cdot \sum_{j=1}^{n} \{b_j^{(1)} \times b_j^{(2)}\}$$

ist Summe von höchstens 2^n, einander fremden, vom leeren Soma verschiedenen elementaren Somen [5]).

Satz 23. Jedes Soma $\{a_j^{(1)} \times a_j^{(2)}\}_{(j=1,\ldots,m)}$ ist gleich einem Soma $\{b_j^{(1)} \times b_j^{(2)}\}_{(j=1,\ldots,n)}$, das „Summe" von einander fremden, elementaren Somen ist [6]).

§ 4. $\mathfrak{S}$ sei eine den Axiomen 1° — 4°, 6°, 7° genügende Struktur, welche eine mit der im vorigen Par. definierten Struktur $S^{(1)} \times S^{(2)}$ isomorphe Teilstruktur $\mathfrak{S}^{(1,2)}$ enthält.

Ausgehend von Teilstrukturen $S(K_2^{(1)})$ von $S^{(1)}$ und $S(K_2^{(2)})$ von $S^{(2)}$, welche ebenfalls die Axiome 1° — 4°, 6°, 7° erfüllen, und von für die Somen dieser Strukturen definierten, beschränkt additiven Somenfunktionen $f^{(1)}(a)$ bzw. $f^{(2)}(a)$ [vgl. § 2; die Klassen $K_1^{(1)}$ und $K_1^{(2)}$ sollen dabei mit $K_2^{(1)}$ bzw. $K_2^{(2)}$ zusammenfallend gedacht werden; die Existenz von Klassen $K^{(1)}$ und $K^{(2)}$ braucht nicht vorausgesetzt zu werden], kann man, wie in § 2, $S(K_2^{(1)})$ und $S(K_2^{(2)})$ erweitern zu die gleichen Axiome erfüllenden Strukturen $S(K_4^{(1)})$, $S(K_5^{(1)})$ bzw. $S(K_4^{(2)})$, $S(K_5^{(2)})$, für deren Somen die Sätze 18—20 gelten. Ein eventuelles grösstes Soma von $S^{(1)}$ oder $S^{(2)}$ ist ein Soma von $S(K_5^{(1)})$ bzw. $S(K_5^{(2)})$.

$\mathfrak{S}(\mathfrak{K}_2)$ sei die den Axiomen 1° — 4°, 6°, 7° genügende Teilstruktur von $\mathfrak{S}^{(1,2)}$, welche bei der Isomorphie von $\mathfrak{S}^{(1,2)}$ und $S^{(1)} \times S^{(2)}$ sich auf die Teilstruktur $S(K_2^{(1)}) \times S(K_2^{(2)})$ von $S^{(1)} \times S^{(2)}$ abbildet.

Wenn ein Soma $\mathfrak{a} \in \mathfrak{K}_2$ sich bei der Isomorphie auf ein Soma $\bar{\mathfrak{a}}$ von $S(K_2^{(1)}) \times S(K_2^{(2)})$ abbildet [also $\bar{\mathfrak{a}} \in K_2^{(1)} \times K_2^{(2)}$], welche gleich ist einer „Summe"

$$\left.\begin{array}{r} \{a_j^{(1)} \times a_j^{(2)}\}_{(j=1,\ldots,n)}, \text{ mit } a_j^{(1)} \in K_2^{(1)},\ a_j^{(2)} \in K_2^{(2)} \text{ und} \\ a_j^{(1)} \times a_j^{(2)},\ a_k^{(1)} \times a_k^{(2)} \text{ einander fremd } (j \neq k), \end{array}\right\} \quad \ldots \quad (1)$$

sei

$$\mathfrak{F}(\mathfrak{a}) = \sum_{j=1}^{n} f^{(1)}(a_j^{(1)}) \times f^{(2)}(a_j^{(2)}).$$

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[4]) Vergleiche loc. cit. 3), S. 87 (Satz XIX).
[5]) Vergleiche loc. cit. 3), S. 88 (Satz XIX*a*).
[6]) Vergleiche loc. cit. 3), S. 88 (Satz XX).

Man sieht unschwer ein, dass der Wert von $\mathfrak{F}(\mathfrak{a})$ unabhängig von der unter den angegebenen Bedingungen gewählten Darstellung (1) ist; $\mathfrak{F}(\mathfrak{a})$ *ist eine für die Somen von* $\mathfrak{K}_2$ *beschränkt additive Funktion* (im Sinne von § 2).

Es ist nun möglich zu $\mathfrak{F}(\mathfrak{a})$ gehörende positive und negative Variationen, $\mathfrak{G}(\mathfrak{a})$ bzw. $\mathfrak{g}(\mathfrak{a})$, und eine Totalvariation $\mathfrak{T}(\mathfrak{a})$ mit den in § 2 behandelten Eigenschaften einzuführen. Wie dort führt die Totalvariation zur äusseren $\mathfrak{T}$-Funktion, $\mathfrak{T}_a(\mathfrak{a})$, für die Somen einer ($\mathfrak{K}_2$ umfassenden) Teilklasse $\mathfrak{K}_3$ der Klasse aller Somen von $\mathfrak{S}$; nach einer Erweiterung von $\mathfrak{T}_a(\mathfrak{a})$ zu allen Somen von $\mathfrak{S}$ (vgl. Satz 16) gelangt man zur inneren $\mathfrak{T}$-Funktion, $\mathfrak{T}_i(\mathfrak{a})$, und zu der (die Axiome 1° — 4°, 6°, 7° erfüllende) Struktur $\mathfrak{S}(\mathfrak{K}_5)$ der $\mathfrak{T}_a$- und $\mathfrak{T}_i$-messbaren Somen [$\mathfrak{K}_2 \subset \mathfrak{K}_5$].

Satz 24. $\mathfrak{S}(\mathfrak{K}_5)$ enthält eine zur Produktstruktur $S(K_5^{(1)}) \times S(K_5^{(2)})$ isomorphe Teilstruktur [7]).

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[7]) Vgl. loc. cit. 3), S. 89 (Satz XXII), 91 (Satz XXIII). — Zur Ableitung von weiteren Eigenschaften vergleiche J. RIDDER, Nieuw Archief (2) **19** : 1 (1936), S. 31—33.

Mathematics. — *Zur Mass- und Integrationstheorie in Strukturen.* II. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

II. RIEMANN–STIELTJES*sche Integrale.*

§ 5. $S^{(1)}$ sei die von den Teilmengen der Menge aller reellen Zahlen gebildete Struktur, bei der $a \subset b$ bedeuten soll: a ist Teilmenge von b; $S^{(2)}$ sei eine weitere, die Axiome $1° - 4°$, $6°$, $7°$ erfüllende Struktur, in welcher wir Teilklassen von Somen, $[K^{(2)}, K_1^{(2)}]$, $K_2^{(2)} - K_5^{(2)}$, von der in § 2 besprochenen Art nebst beschränkt additiven Somenfunktionen $f^{(2)}(a)$, $G^{(2)}(a)$, $g^{(2)}(a)$, $T^{(2)}(a)$ und zugehörigen Massfunktionen $m_{f(2)}(a)$ u.s.w. betrachten wollen (vgl. § 4). In $S^{(1)}$ nehmen wir als $S(K^{(1)})$ die kleinste, die Axiome $1° - 4°$, $6°$, $7°$ erfüllende Teilstruktur, welche die Menge aller reellen Zahlen und diejenigen ihrer Teilmengen umfasst, deren Durchschnitte mit den offenen Intervallen I $(a < x < b)$ immer als Summe von endlich vielen offenen Intervallen aufgefasst werden können, und für die $K_1^{(1)}$ die Teilklasse von $K^{(1)}$ ist, deren jedes Soma aus endlich vielen offenen Intervallen aufgebaut ist (siehe § 2). $S(K^{(1)})$ enthält eine Struktur $S(K_2^{(1)})$ von der in § 2 angegebenen Art. Definiert man schliesslich die Funktion $f^{(1)}(a)$ für die Somen $a \in K_2^{(1)}$, ausgehend von dem elementargeometrischen Mass der offenen Intervalle, so lässt sich $S(K_2^{(1)})$ erweitern zu einer Struktur $S(K_5^{(1)})$, wobei man für die Somen dieser Struktur eine (nicht-negative) Massfunktion $m_{f(1)}(a)$ mit den in § 2 behandelten Eigenschaften erhält.

Wie im vorigen Paragraphen, gibt es eine Produktstruktur $S^{(1)} \times S^{(2)}$, welche wir hier mit $\mathfrak{S}$ zusammenfallend annehmen; aus den $K_j^{(i)}$-Klassen gehen Klassen $\mathfrak{K}_2$, $\mathfrak{K}_3$, $\mathfrak{K}_5$ von Somen von $S^{(1)} \times S^{(2)}$ nebst zu diesen gehörenden Somenfunktionen $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{g}$, $\mathfrak{T}$ und Massfunktionen $m_{\mathfrak{F}}$, u.s.w. hervor [1]).

Satz 25. Wenn ein zur eben eingeführten Klasse $\mathfrak{K}_2$ gehörendes Soma $\mathfrak{a}$ sich darstellen lässt:

$$a_1^{(1)} \times b_1^{(2)} + a_2^{(1)} \times b_2^{(2)} + \ldots + a_n^{(1)} \times b_n^{(2)},$$

mit $a_j^{(1)} \in K_2^{(1)}$, $b_j^{(2)} \in K_2^{(2)}$ $(j = 1, \ldots, n)$, **und** $a_j^{(1)} \times b_j^{(2)}$ **und** $a_k^{(1)} \times b_k^{(2)}$ **einander fremd** $(j \neq k)$, **so wird**

$$\mathfrak{G}(\mathfrak{a}) = \sum_{j=1}^{n} f^{(1)}(a_j^{(1)}) \times G^{(2)}(b_j^{(2)}); \quad \mathfrak{g}(\mathfrak{a}) = \sum_{j=1}^{n} f^{(1)}(a_j^{(1)}) \times g^{(2)}(b_j^{(2)})$$

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1) $\mathfrak{S}(\mathfrak{K}_5)$ und $S(K_5^{(1)}) \times S(K_5^{(2)})$ fallen hier zusammen.

und

$$\mathfrak{T}(\mathfrak{a}) = \sum_{j=1}^{n} f^{(1)}(a_j^{(1)}) \times T^{(2)}(b_j^{(2)})$$

sein [2]).

Definition. Die Struktur $S^{(2)}$ genüge den Axiomen $1^\circ - 4^\circ$, 6°, 7°; sie enthalte das Soma $a^{(2)} \neq 0$. Dann wird *eine auf* $a^{(2)}$ *definierte abstrakte Ortsfunktion* F durch zwei Somenfunktionen $\alpha(x^{(2)})$ und $\beta(x^{(2)})$ gegeben, die beide auf jedem vom leeren Soma verschiedenen Soma $x^{(2)} \subset a^{(2)}$ definiert sind und folgende Eigenschaften haben:

1. es gibt zwei endliche oder unendliche Zahlen λ_1, λ_2, so dass für jedes Soma $x^{(2)} \subset a^{(2)}$ und $\neq 0$

$$\lambda_1 \leqq \alpha(x^{(2)}) \leqq \beta(x^{(2)}) \leqq \lambda_2$$

ist;

2. $\alpha(x^{(2)})$ ist monoton abnehmend, $\beta(x^{(2)})$ monoton zunehmend;

3. ist $\lambda_1 < \lambda_2$, so gibt es mindestens eine im offenen Intervall (λ_1, λ_2) überall dicht liegende Folge von Zahlen $\{y_k\}$, denen die Somen $\{x_k^{(2)}\}$, mit jedem $x_k^{(2)} \subset a^{(2)}$, eindeutig zugeordnet sind, so dass erstens $\alpha(x_k^{(2)}) \geqq y_k$ und dass zweitens $\beta(a^{(2)} - x_k^{(2)}) \leqq y_k$ ist.

Definition A. Die in der vorigen Definition betrachtete Ortsfunktion F sei *nicht-negativ*; das soll heissen:

$$\alpha(a^{(2)}) \geqq 0.$$

Sie sei ausserdem *auf* $a^{(2)}$ *beschränkt*, m.a.W.: $\beta(a^{(2)})$ und $\alpha(a^{(2)})$ seien endlich. Schliesslich sei $a^{(2)} \in K_4^{(2)}$.

Dann betrachten wir als *Pseudo-Soma* $[a^{(2)}; F]$ eine Kombination von zwei Klassen L und R von zu $\mathfrak{K}_2$ gehörenden Somen, bei der:

1^0. L alle und nur alle diejenigen Somen $(\mathfrak{a})$ von $\mathfrak{K}_2$ umfasst, deren jedes „Summe" ist von endlich vielen „Produkten" $a_j^{(1)} \times a_j^{(2)}$ mit: $a_j^{(2)} \subset a^{(2)}$, $a_j^{(2)} \in K_2^{(2)}$; zu jedem $a_j^{(2)}$ ein Soma $b_j^{(2)} \in S^{(2)}$, $a_j^{(2)} \subset b_j^{(2)} \subset a^{(2)}$, so dass das Intervall $0 \leqq x < \alpha(b_j^{(2)})$ [3]) das zugehörige Soma $a_j^{(1)} \in K_2^{(1)}$ enthält;

2^0. R alle und nur alle diejenigen Somen $(\mathfrak{a})$ von $\mathfrak{K}_2$ umfasst, deren jedes „Summe" von endlich vielen „Produkten" $\bar{a}_j^{(1)} \times \bar{a}_j^{(2)}$ ist, und folgende Eigenschaft hat: $a^{(2)}$ lässt sich in endlich viele, einander fremde Somen $(b_k^{(2)})$, $b_k^{(2)} \in S^{(2)}$, zerlegen, so dass bei jedem $b_k^{(2)}$ diejenigen Somen $\bar{a}_j^{(2)}$ mit

$$b_k^{(2)} \subset \bar{a}_j^{(2)}$$

eine Menge $\{\bar{a}_j^{(2)}\}_k$ bilden, für welche die zugehörige Somenmenge $\{\bar{a}_j^{(1)}\}_k$ eine Summe hat, die das Intervall $0 \leqq x \leqq \beta(\bar{b}_k^{(2)})$ enthält.

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[2]) Vgl. Mitt. I., loc. cit. 3), S. 95, 96 (Satz XXIV nebst Beweis).

[3]) Ist $\alpha(b_j^{(2)}) = 0$, so lese man: $0 \leqq x \leqq \alpha(b_j^{(2)})$.

Die unteren Schranken der $\mathfrak{T}$- und $\mathfrak{G}$- und die obere Schranke der $\mathfrak{g}$-Masse der Somen von R seien *das äussere $\mathfrak{T}$-, bzw. $\mathfrak{G}$-, bzw. $\mathfrak{g}$-Mass von* $[a^{(2)}; F]$, zugleich *das obere* RIEMANN–STIELTJES*sche* $T^{(2)}$-, *bzw.* $G^{(2)}$-, *bzw.* $g^{(2)}$-*Integral von F über* $a^{(2)}$; die oeberen Schanken der $\mathfrak{T}$- und $\mathfrak{G}$- und die untere Schranke der $\mathfrak{g}$-Masse der Somen von L seien *das innere $\mathfrak{T}$-, bzw. $\mathfrak{G}$-, bzw. $\mathfrak{g}$-Mass von* $[a^{(2)}; F]$, zugleich *das untere* RIEMANN-STIELTJES*sche* $T^{(2)}$-, *bzw.* $G^{(2)}$-, *bzw.* $g^{(2)}$-*Integral von F über* $a^{(2)}$. Sind äusseres und inneres $\mathfrak{T}$-, und dadurch auch äusseres und inneres $\mathfrak{G}$-Mass, und ebenso äusseres und inneres $\mathfrak{g}$-Mass, einander gleich, so seien diese gemeinsamen Werte bzw. *das $\mathfrak{T}$-, $\mathfrak{G}$- und $\mathfrak{g}$-Mass von* $[a^{(2)}; F]$ [4]; F heisse dann $T^{(2)}$-, $G^{(2)}$- *und* $g^{(2)}$-*integrierbar* (RS) *über* $a^{(2)}$, und es sei

$$\int_{a^{(2)}} (RS)\, F\, d\, m_{T^{(2)}}\,(x^{(2)}) = \text{das } \mathfrak{T}\text{-Mass von } [a^{(2)}; F],$$

$$\int_{a^{(2)}} (RS)\, F\, d\, m_{G^{(2)}}\,(x^{(2)}) = \text{das } \mathfrak{G}\text{-Mass von } [a^{(2)}; F],$$

$$\int_{a^{(2)}} (RS)\, F\, d\, m_{g^{(2)}}\,(x^{(2)}) = \text{das } \mathfrak{g}\text{-Mass von } [a^{(2)}; F];$$

jede derartige $T^{(2)}$-integrierbare Funktion F betrachten wir auch als $f^{(2)}$-*integrierbar* (RS) mit

$$\int_{a^{(2)}} (RS)\, F\, d\, m_{f^{(2)}}\,(x^{(2)}) = \int_{a^{(2)}} (RS)\, F\, d\, m_{G^{(2)}}\,(x^{(2)}) + \int_{a^{(2)}} (RS)\, F\, d\, m_{g^{(2)}}\,(x^{(2)}).$$

Definition A*. Wenn F *auf* $a^{(2)} \in K_1^{(2)}$ *beschränkt* ist, lassen sich zwei Ortsfunktionen F^+ (zugehörige Somenfunktionen α^+, β^+) und F^- (zugehörige Somenf. α^-, β^-) in folgender Weise einführen:

auf jedem Soma $p^{(2)} \subset a^{(2)}$ mit $\alpha\,(p^{(2)}) \geqq 0$ sei $\alpha^+\,(p^{(2)}) = \alpha\,(p^{(2)})$, $\beta^+(p^{(2)}) = \beta\,(p^{(2)})$, und $\alpha^-\,(p^{(2)}) = 0$, $\beta^-\,(p^{(2)}) = 0$,

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[4]) Läßt $S^{(1)} \times S^{(2)}$ sich in eine Struktur $\mathfrak{S}^*$ einbetten, in welcher dieselben Strukturaxiome wie in $S^{(1)} \times S^{(2)}$ gelten, Axiom 3^0 sogar in der umfassenderen Form 3^0: „jede Klasse von Somen hat eine Summe" (dies hat zur Folge, dass auch Axiom 5^0 erfüllt wird), und ist dabei für die zu einem Pseudo-Soma $[a^{(2)}; F]$ gehörenden Klassen L und R die Summe der Somen von L immer gleich dem Produkte der Somen von R: $\sum_{(L)} \mathfrak{a} = \prod_{(R)} \bar{\mathfrak{a}}$, so ist dieses letzte Soma als das zu F über $a^{(2)}$ gehörende *Ordinatensoma* zu betrachten. *Für eine $T^{(2)}$-integrierbare Funktion F ist dann das Ordinatensoma $\mathfrak{T}$-meßbar, und ihr $\mathfrak{T}$-Mass ist gleich dem $T^{(2)}$-Integral von F über $a^{(2)}$*; dabei sei das äussere und innere $\mathfrak{T}$-Mass in $\mathfrak{S}^*$ in derselben Weise aus $\mathfrak{T}(\mathfrak{a})$ abgeleitet zu denken wie (in den §§ 5 und 6) in $S^{(1)} \times S^{(2)}$; in $S^{(1)} \times S^{(2)}$ bleiben alle Messbarkeitsrelationen ungeändert erhalten. — Für die Eigenschaften einer Struktur mit den Axiomen 1^0, 2^0, 3^0, 4^0—7^0 siehe unsere Arbeit: *Über topologische Eigenschaften von Strukturen*, Verhand. Ned. Akad. v. Wetensch., Amsterdam, Sect. 1, **18**, Nr. 4, S. 1—43 (1944).

auf jedem Soma $q^{(2)} \subset a^{(2)}$ mit $\alpha(q^{(2)}) < 0$, $\beta(q^{(2)}) > 0$ sei $\alpha^+(q^{(2)}) = 0$, $\beta^+(q^{(2)}) = \beta(q^{(2)})$, und $\alpha^-(q^{(2)}) = 0$, $\beta^-(q^{(2)}) = -\alpha(q^{(2)})$,
auf jedem Soma $r^{(2)} \subset a^{(2)}$ mit $\beta(r^{(2)}) \leqq 0$ sei $\alpha^+(r^{(2)}) = 0$, $\beta^+(r^{(2)}) = 0$, und $\alpha^-(r^{(2)}) = -\beta(r^{(2)})$, $\beta^-(r^{(2)}) = -\alpha(r^{(2)})$.

Sind nun F^+ und F^- über $a^{(2)}$ $T^{(2)}$-integrierbar (RS), so betrachten wir F als $T^{(2)}$- *und* als $f_i^{(2)}$-*integrierbar* (RS) *über* $a^{(2)}$, und es sei:

$$\int_{a^{(2)}} (RS)\, F\, dm_{f^{(2)}}(x^{(2)}) = \int_{a^{(2)}} (RS)\, F^+\, dm_{f^{(2)}} - \int_{a^{(2)}} (RS)\, F^-\, dm_{f^{(2)}},$$

und

$$\int_{a^{(2)}} (RS)\, F\, dm_{T^{(2)}}(x^{(2)}) = \int_{a^{(2)}} (RS)\, F^+\, dm_{T^{(2)}} - \int_{a^{(2)}} (RS)\, F^-\, dm_{T^{(2)}}.$$

(*Geometrische* Definition des RIEMANN–STIELTJESschen Integrals.)

Satz 26. Zur Existenz des in der vorigen Definition eingeführten Integrals $\int_{a^{(2)}} (RS)\, F\, dm_{f^{(2)}}$ ist notwendig und hinreichend, dass es bei willkürlich positiven Zahlen ε, δ eine Überdeckung von $a^{(2)}$ durch endlich viele, einander fremde Somen $(x_j^{(2)}) \in K_2^{(2)}$ gebe, bei der von denjenigen Somen unter den $(x_j^{(2)})$ mit $\beta(x_j^{(2)} \cdot a^{(2)}) - \alpha(x_j^{(2)} \cdot a^{(2)}) > \varepsilon$ die Summe ihrer $T^{(2)}$-Masse $< \delta$ sei.

§ 6. **Definition.** Ein Soma $\sum_{j \in K} a_j$ wird *Summe der Somenklasse* $\{a_j\}$ mit $j \in K$ genannt, falls: α) jedes $a_j \subset \sum_{j \in K} a_j$; β) aus $a_j \subset b\,(j \in K)$ folgt $\sum_{j \in K} a_j \subset b$.

Satz 27. Hat die abzählbar unendliche Klasse: $a_1, a_2, \ldots, a_j, \ldots$ eine Summe, so ist diese eindeutig bestimmt.

Axiom $\bar{3}°$. Zu jedem Somenpaar gibt es eine Summe; für jede abzählbar unendliche Klasse von Somen: $a_1, a_2, \ldots, a_j, \ldots$ gibt es eine Summe $\sum_{j=1}^{\infty} a_j$.

Definition. Ein Soma $\prod_{j \in K} a_j$ wird *Produkt der Somenklasse* $\{a_j\}$ mit $j \in K$ genannt, falls: α) für jedes $j \in K$ $\prod_{j \in K} a_j \subset a_j$; β) aus $b \subset a_j\,(j \in K)$ folgt $b \subset \prod_{j \in K} a_j$.

In diesem Paragraphen betrachten wir eine den Axiomen 1°, 2°, $\bar{3}°$, 4°, 6°, 7° genügende Struktur $S^{(2)}$. Dann gilt:

Satz 28. Jede abzählbar unendliche Klasse von Somen: $a_1, a_2, \ldots$ hat ein eindeutig bestimmtes Produkt.

Auch hier gilt das Dualitätsprinzip von § 1, Ende, falls ausserdem Axiom 5° erfüllt wird, und bei Fortlassung von γ) auch wieder ohne dieses Axiom.

Auf dem Soma $a^{(2)}$ sei eine abstrakte Ortsfunktion F definiert. Dann kann man, ähnlich wie bei Punktfunktionen [5]), für reelles λ Teilsomen

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[5]) Vergleiche C. CARATHÉODORY, *Vorles. ü. reelle Funkt.*, 2e Aufl. (1927), S. 372 u.f. Angenommen wird, dass das leere Soma immer die Ungleichungen $\alpha \geqq \lambda$ und $\beta \leqq \lambda$ erfüllt.

$x^{(2)}(F \leqq \lambda)$ und $x^{(2)}(F \geqq \lambda)$ bestimmen; $x^{(2)}(F \geqq \lambda)$ ist das (immer existierende) grösste Teilsoma [6]) von $a^{(2)}$ mit $\alpha \geqq \lambda$, und ebenso $x^{(2)}(F \leqq \lambda)$ das (immer existierende) grösste Teilsoma [6]) von $a^{(2)}$ mit $\beta \leqq \lambda$ [7]).

Lemma [8]). In der (den Axiomen 1°, 2°, $\bar{3}$°, 4°, 6°, 7° genügenden) Struktur $S^{(2)}$ sollen F, $a^{(2)}$, $f^{(2)}$, $T^{(2)}$ die gleichen Eigenschaften wie in Definition A haben. Dann ist das obere R.–S. $T^{(2)}$-Integral von F über $a^{(2)}$ gleich

$$\lim_{\delta \to 0} \sum_{j=1}^{n} T_a^{(2)}[a^{(2)} - x^{(2)}(F \leqq y_{j-1})] \times \{y_j - y_{j-1}\},$$

während das untere R.–S. $T^{(2)}$-Integral von F über $a^{(2)}$ gleich

$$\lim_{\delta \to 0} \sum_{j=1}^{n} T_i^{(2)}[x^{(2)}(F \geqq y_j)] \times \{y_j - y_{j-1}\}$$

ist; hierbei ist δ das Maximum der Differenzen $(y_j - y_{j-1})$, gehörend zu einer Zerlegung $0 = y_0 < y_1 \ldots < y_n = M$ des Intervalles $[0, M]$, wobei die $\alpha^{(2)}$- und $\beta^{(2)}$-Werte für alle Teilsomen von $a^{(2)}$ kleiner als die (übrigens willkürliche) fest gewählte Konstante M sind.

Definition. F, $a^{(2)}$, $f^{(2)}$, $T^{(2)}$ sollen die gleichen Eigenschaften wie in Definition A haben. Dann heisse F *$T_a^{(2)}$-messber auf dem Soma $a^{(2)}$*, wenn für alle reellen y-Werte, eine höchstens abzählbar unendliche Menge E ausgenommen, die Somen $x^{(2)}(F \leqq y)$ und $x^{(2)}(F \geqq y)$ $T_a^{(2)}$-messbar sind, und wenn daneben für diese y-Werte

$$T_a^{(2)}\{a^{(2)} - x^{(2)}(F \leqq y)\} = T_a^{(2)}\{x^{(2)}(F \geqq y)\}$$

ist [9]).

Satz 29 [10]). Haben F, $a^{(2)}$, $f^{(2)}$, $T^{(2)}$ die gleichen Eigenschaften wie am Anfang der Definition A*, so ist es zur $T^{(2)}$- und $f^{(2)}$-Integrierbarkeit (RS) von F über $a^{(2)}$ notwendig und hinreichend, dass F auf $a^{(2)}$ $T_a^{(2)}$-messbar sei.

Sind für die Teilsomen $x^{(2)}$ von $a^{(2)}$ die zu F gehörigen Somenfunktionen α, β beschränkt, mit $m < \alpha \leqq \beta < M$, und wählt man die y_j-Werte $(j = 1, 2, \ldots, n-1)$ der Zerlegung $m = y_0 < y_1 < y_2 \ldots < y_n = M$ immer derartig, dass die zugehörigen Somen $x^{(2)}(F \geqq y_j)$ $T_a^{(2)}$-messbar sind, so ist

$$\int_{a^{(2)}} (RS)\, F\, dm_{T^{(2)}}(x^{(2)}) = \lim_{\delta \to 0} \sum_{j=0}^{n-1} y_j \cdot m_T[x^{(2)}(F \geqq y_j) - x^{(2)}(F \geqq y_{j+1})]$$

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[6]) Das soll heissen: alle Somen mit derselben Eigenschaft werden von diesem Soma umfasst.

[7]) Für das komplementäre Soma $a^{(2)} - x^{(2)}(F \geqq \lambda)$ ist $\beta \leqq \lambda$; ebenso ist für $a^{(2)} - x^{(2)}(F \leqq \lambda)$ $\alpha \geqq \lambda$.

[8]) Vergleiche Mitt. I., loc. cit. 3), S. 99 (Lemma *B*) und I., loc. cit. 2), S. 171, 172 (Lemma).

[9]) *Aus der $T_a^{(2)}$-Messbarkeit von $x^{(2)}(F \geqq y)$ für alle y, abzählbar viele Werte ausgenommen, lassen sich die übrigen, in dieser Definition enthaltenen Bedingungen ableiten.*

[10]) Zum Beweise vergleiche man I., loc. cit. 3), S. 101—103; auch I., loc. cit. 2), S. 172.

und

$$\int_{a^{(2)}}^{*} (RS)\, F\, d\, m_{f^{(2)}}(x^{(2)}) = \lim_{\delta \to 0} \sum_{j=0}^{n-1} y_j \cdot m_f [x^{(2)}(F \geqq y_j) - x^{(2)}(F \geqq y_{j+1})];$$

hierbei ist δ das Maximum der Differenzen $(y_j - y_{j-1})$ der Zerlegung von $[m, M]$.

Das bekannte Verfahren von DE LA VALLÉE POUSSIN führt zu (absolut konvergenten) uneigentlichen RIEMANN–STIELTJESschen $f^{(2)}$- und $T^{(2)}$-Integralen über Somen.

Die Integrale besitzen die gewöhnlichen elementaren Integraleigenschaften.

III. *Total-additives Mass.*

§ 7. In den drei letzten Kapiteln betrachten wir nur Strukturen, die den Axiomen 1°, 2°, $\bar{3}$°, 4°, 6°, 7° genügen.

Die in § 2 enthaltene Theorie eines *beschränkt* additiven Masses lässt sich in einer derartigen Struktur in leicht ersichtlicher Weise in eine Theorie eines *total*-additiven Masses abändern. Wir geben hier kurz die dazu notwendigen Änderungen in den Definitionen und Sätzen an.

K genüge in der Struktur S den gleichen Bedingungen wie in § 2; nur soll unter II. auch die Summe von abzählbar vielen Somen zugelassen werden.

Dann genügt auch $S(K)$ den Axiomen 1°, 2°, $\bar{3}$°, 4°, 6°, 7° und, falls S Axiom 5° erfüllt, daneben Axiom 5°.

Die Somenfunktion $f(x)$ soll *total-additiv* sein, d.h. in der Definition soll unter 1° die Additivität für abzählbar viele, paarweise fremde Somen gefordert werden, für deren jedes und für deren Summe f definiert ist [11]).

$G(x)$, $g(x)$, $T(x)$ werden nun ebenfalls total-additiv sein. Die Definitionen von $T_a(x)$ und $T_i(x)$ behalten den gleichen Wortlaut. *$S(K_2)$ und $S(K_3)$ sind den Axiomen 1°, 2°, 3°, 4°, 6°, 7° genügende, in bezug auf K_1 σ-beschränkte Strukturen;* mit „σ-beschränkt in bezug auf K_1" meinen wir, dass die Summe von je *abzählbar* vielen zu einer solchen Struktur gehörenden Somen auch zur Struktur gehört immer dann, wenn es ein Soma $a \in K_1$ gibt, welches die Summe als Teilsoma umfasst.

Satz 14'. Für abzählbar viele Somen (x_j), welche ebenso wie ihre Summe zur Klasse K_3 gehören, ist immer

$$T_a(x_1 + x_2 + \ldots) \leqq \sum_{(j)} T_a(x_j). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Der Wortlaut des Satzes 15 bleibt ungeändert; auch die der Definitionen der T_a-, G_a- und $|g|_a$-Messbarkeit von Somen und der Definition der adjungierten Funktion $T_i(x)$.

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[11]) Die Bedingung 2° folgt nun aus 1°, und kann somit fortgelassen werden. Vergleiche S. SAKS, *Théorie de l'intégrale*, Warszawa 1933, p. 248, 249.

Satz 16′. Fügt man $T_a(x)$ für die nicht zur Klasse K_3 gehörenden Somen den Wert $+\infty$ zu, so genügt die sodann für jedes Soma definierte äussere T-Funktion den Bedingungen: I^b. es gibt ein Soma w mit $T_a(w)$ endlich und $\neq 0$; III. aus $a \subset b$ folgt immer $T_a(a) \leqq T_a(b)$; V. für jedes Soma a mit $T_a(a)$ endlich ist $T_a(a)$ gleich der unteren Schranke der Werte $T_a(\bar{x})$ für alle $T_a(x)$-messbaren Somen $(\bar{x})$ mit $a \subset \bar{x}$; VI. die Summe von abzählbar vielen, $T_a(x)$-messbaren Somen ist $T_a(x)$-messbar; VII. für jede Folge von Somen $b_1 \subset b_2 \ldots \subset b_n \ldots$ mit $b = \sum_{j=1}^{\infty} b_j$ und $T_a(b)$ endlich ist

$$\lim_{j \to \infty} T_a(b_j) = T_a(b).$$

Beweis. Es genügt VI. und VII. zu beweisen.

Die I., loc. cit. 2), §§ 2—6 behandelten Eigenschaften der Somenfunktion $m_0(x)$ gelten ohnemehr für die Funktion $T_i(x)$. Ist insbes. $T_i(a+b)$ endlich und $ab = 0$, so gilt

$$T_i(a+b) \cong T_i(a) + T_i(b).$$

Daraus folgt für abzählbar unendlich viele paarweise fremde Somen $a_1, a_2, \ldots$ mit $T_i(a_1 + a_2 + \ldots)$ endlich:

$$T_i(a_1 + a_2 + \ldots) \cong \sum_{j=1}^{\infty} T_i(a_j).$$

Sind die einzelnen Somen $a_1, a_2, \ldots$ T_a-messbar, so ergibt sich aus dieser Ungleichung in Verbindung mit (2):

$$T_a(a_1 + a_2 + \ldots) \cong \sum_{(j)} m_T(a_j) \cong T_i(a_1 + a_2 + \ldots),$$

also, da immer $T_i \leqq T_a$ ist [12]),

$$m_T(a_1 + a_2 + \ldots) = T_a(\sum_{(j)} a_j) = T_i(\sum_{(j)} a_j) = \sum_{(j)} m_T(a_j). \quad . \quad . \quad (3)$$

Fordern wir nicht mehr die Endlichkeit von $T_i(a_1 + a_2 + \ldots)$, so folgt doch aus dem Vorigen, dass bei willkürlich gewähltem T_a-mesbarem Soma w, mit $T_a(w)$ endlich, auch $w(a_1 + a_2 + \ldots) = wa_1 + wa_2 + \ldots$ und $w - w(a_1 + a_2 + \ldots)$ T_a-messbar sind. Es ist somit

$$T_a(w) = T_a[w(a_1 + a_2 + \ldots)] + T_a[w - w(a_1 + a_2 + \ldots)]. \quad . \quad (4)$$

Daraus folgt jedoch [13]) die T_a-Messbarkeit der Summe $a_1 + a_2 + \ldots$ Nun leitet man auch leicht die Relation (4) für den Fall ab, in welchem die a_j nur T_a-messbar vorausgesetzt werden; das liefert dann Eigenschaft VI.

Mit (3) folgt VII. in den Fällen, in welchen jedes Soma b_j T_a-messbar ist.

Jedes Produkt von abzählbar vielen, T_a-messbaren Somen ist auch T_a-messbar [14]). Mit V. folgt daraus für jedes Soma w mit $T_a(w)$ endlich

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[12]) Siehe I., loc. cit. 2), S. 151 (Satz 35).
[13]) Siehe I., loc. cit. 2), S. 153 (Satz 39).
[14]) Siehe I., loc. cit. 2), S. 161, 162 (Satz 49).

die Existenz eines T_a-messbaren Somas $w \supset w$ mit $m_T(\overline{w}) = T_a(w)$. Dies ermöglicht den allgemeinen Fall von VII. in bekannter Weise auf den eben behandelten besonderen Fall zurückzuführen. —

Die I., loc. cit. 2), §§ 9—11 erhaltenen Eigenschaften der Somenfunktionen $m^{(1)}$, m_0 gelten nun ohnemehr für die Funktionen T_a, T_i [15]).

Die T_a- und T_i-messbaren Somen bilden eine den Axiomen 1°, 2°, $\overline{3}$°, 4°, 6°, 7° genügende Struktur $S(K_5)$, welche $S(K)$ und $S(K_2)$ als Teilstrukturen enthält. $S(K_5)$ enthält ebenfalls eine die Axiome 1°, 2°, 3°, 4°, 6°, 7° erfüllende Teilstruktur $S(K_4)$, die alle und nur alle diejenigen T_a-messbaren Somen enthält, die in einem Soma der Klasse K_2 enthalten sind; $S(K_4)$ ist in bezug auf K_1 σ-beschränkt.

Der den Satz 18 vorangehende Absatz behält den gleichen Wortlaut. Satz 18 bleibt ungeändert; ebenso Satz 19, wenn das f-Mass wie in § 2 definiert bleibt. Lässt man auch die den Satz 20 vorangehende Definition ungeändert, so lässt sich dieser Satz erweitern zu

Satz 20′. Für eine abzählbar unendliche Klasse $\{a_j\}$ von paarweise fremden, zu K_5 gehörenden Somen ist immer

$$m_T\left(\sum_{(j)} a_j\right) = \sum_{(j)} m_T(a_j);$$

ausserdem ist für diese Somen

$$m_f\left(\sum_{(j)} a_j\right) = \sum_{(j)} m_f(a_j)$$

immer dann, wenn das linke Glied einen endlichen oder bestimmt unendlichen Wert hat.

IV. Lebesgue–Stieltjes*sche Integrale.*

§ 8. Geringfügige Abänderungen in den Überlegungen der §§ 5 und 6 genügen um zum Lebesgue–Stieltjesschen Integral in einer die Axiome 1°, 2°, $\overline{3}$°, 4°, 6°, 7° erfüllende Struktur $S^{(2)}$ zu gelangen.

$S(K_2^{(2)})$ sei eine die Axiome 1°, 2°, 3°, 4°, 6°, 7° erfüllende und in bezug auf $K_2^{(2)}$ selbst σ-beschränkte Teilstruktur von $S^{(2)}$; $f^{(2)}$ sei eine für die Somen von $K_2^{(2)}$ definierte, total-additive Somenfunktion, d.h. aus $a_j \in K_2^{(2)}$ $(j = 1, 2, \ldots)$, $a_{j_1} \cdot a_{j_2} = 0$ $(j_1 \neq j_2)$, $\sum_{(j)} a_j \in K_2^{(2)}$ soll folgen: $f^{(2)}\left(\sum_{(j)} a_j\right) = \sum_{(j)} f^{(2)}(a_j)$. Dann lassen sich Somenklassen $K_3^{(2)}$, $K_4^{(2)}$, $K_5^{(2)}$ und Massfunktionen $m_{f^{(2)}}$, $m_{T^{(2)}}$ einführen, welche die im vorigen Paragraphen behandelten Eigenschaften haben. $S^{(1)}$ sei die von den Teilmengen der Menge aller reellen Zahlen gebildete Struktur; zu ihr seien Somenklassen und Massfunktionen in gleicher Weise gewählt wie in § 5.

Die Betrachtungen von §§ 3 und 4 führen nun, ebenso wie in dem in § 5 betrachteten Fall, zu Somenklassen $\mathfrak{K}_2$, $\mathfrak{K}_3$, $\mathfrak{K}_5$ in $S^{(1)} \times S^{(2)}$ und zu diesen

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[15]) Axiom VIII der zitierten Arbeit braucht hier nicht zu gelten, somit auch nicht die wenigen, sich auf diesem Axiom stützenden Masseigenschaften.

gehörenden, *beschränkt*-additiven Somenfunktionen $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{g}$, $\mathfrak{T}$ und *beschränkt*-additiven Massfunktionen $m_{\mathfrak{F}}$ u.s.w.

Satz 25 behält seine Gültigkeit.

Lässt man den Wortlaut der in §§ 5 und 6 weiter folgenden Definitionen ungeändert (nur lese man „L.–S." statt „R.–S.)", so bleiben auch alle weiteren Sätze dieser Paragraphen gültig.

Wir bemerken, dass zufolge der Eigenschaft von $S(K_4^{(2)})$ in bezug auf $K_2^{(2)}$ σ-beschränkt zu sein *die* in der Definition der $T^{(2)}$-Messbarkeit einer abstrakten Ortsfunktion F, welche auf einem Soma $a^{(2)} \in K_4^{(2)}$ beschränkt ist, *zugelassene Ausnahmemenge immer leer sein wird.* Dadurch ist es möglich diese Definition die in der Theorie der gewöhnlichen LEBESGUE-schen und der LEBESGUE–STIELTJESschen Integrale übliche Form zu geben.

V. *Erweiterung eines beschränkt-additiven Masses zu einem total-additiven Masse.*

§ 9. **Satz 30.** *S* genüge den Axiomen 1°, 2°, $\overline{3}$°, 4°, 6°, 7°. $S(K_2)$ sei, wie in § 2, eine die Axiome 1°, 2°, 3°, 4°, 6°, 7° erfüllende Teilstruktur [16]), für deren Somen eine beschränkt additive Somenfunktion $f(x)$ definiert sei. Ihre Totalvariation $T(x)$ sei *in* $S(K_2)$ *total-additiv* *, was heissen soll: ist $a = \sum_{j=1}^{\infty} a_j$, $a_j \in K_2$, $a_{j_1} \cdot a_{j_2} = 0$ $(j_1 \neq j_2)$ und *gehört auch a zur Klasse* K_2, so ist

$$T(a) = \sum_{j=1}^{\infty} T(a_j). \quad [17])$$

Dann gibt es eine kleinste, in bezug auf K_1 σ-beschränkte und $S(K_2)$ umfassende Teilstruktur $S(\bar{K}_2)$ von S, welche die Axiome 1°, 2°, 3°, 4°, 6°, 7° erfüllt; auf dieser lassen sich T, G, g und f zu im Sinne von § 7 total-additiven Somenfunktionen T, $\bar{G}$, $\bar{g}$ und $\bar{f}$ erweitern; dabei ist für $x \in S(\bar{K}_2)$

$$\bar{f}(x) = \bar{G}(x) + \bar{g}(x), \quad \bar{T}(x) = \bar{G}(x) - \bar{g}(x),$$

und sind $\bar{G}$ und $\bar{g}$ in bezug auf die Klasse $\bar{K}_2$ positive und negative Variation von $\bar{f}$. Durch diese Eigenschaften werden die Erweiterungen $\bar{f}$, $\bar{G}$, $\bar{g}$ und $\bar{T}$ auch eindeutig festgelegt [18]). —

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[16]) Auch hier ist es möglich sich K_1 und K_2 zusammenfallend zu denken, wobei dann die Klasse K fehlen darf.

[17]) Dann ist ebenso

$$G(a) = \sum_{(j)} G(a_j), \; g(a) = \sum_{(j)} g(a_j) \text{ und } f(a) = \sum_{(j)} f(a_j).$$

[18]) Für den Beweis vergleiche man O. HAUPT u. G. AUMANN, *Differential- und Integralrechnung*, Band III, Berlin 1938, S. 16—20, wo ein gleichartiger Satz für Mengen eines abstrakten Raumes abgeleitet wird. Vgl. auch I., loc. cit. 2), S. 160, 161 (Satz 48) und A. KOLMOGOROFF, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Behlin 1933, S. 15 u. 16.

Die Methoden von § 7 sind darauf wieder anwendbar um, von $\bar{f}$ auf $S(\bar{K}_2)$ ausgehend, zu einem total-additiven Masse $m_{\bar{f}}$ zu gelangen.

§ 10. Das im vorigen Paragraphen angewandte Erweiterungsverfahren ist wesentlich an die Existenz der Struktur S gebunden, in welcher $S(K_2)$ eingebettet ist. Hat $S(K_2)$ zwar die in Satz 30 geforderten Eigenschaften, steht jedoch daneben die Existenz einer Struktur S nicht fest, so wird ein anderes Verfahren notwendig um zu einer total-additiven Somenfunktion zu gelangen. Dies gelang bisher nur in besonderen Strukturen unter Anwendung eines von MAC NEILLE herrührenden Verfahrens [19]).

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[19]) Siehe F. WECKEN, Math. Ztschr. **45** (1939), S. 377—404, insbes. S. 391—393; M. H. MAC NEILLE, Proc. Nat. Acad. sci. USA **24** (1938), S. 188—193, insbes. S. 191—193.

Mathematics. — *Over het bestaan der oplossing van* $\triangle^k u = 0$, *die met haar* $k-1$ *eerste normale afgeleiden gegeven waarden aanneemt in de punten van een gegeven gesloten kromme.* I. By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

In een tweetal vroegere mededeelingen heb ik in de eerste plaats [1]) bewezen, dat de oplossing der vergelijking $\triangle^k u = 0$ ondubbelzinnig bepaald is in het gebied binnen een gesloten kromme, die aan zekere voorwaarden voldoet, als men verlangt, dat u in het gebied binnen de kromme doorloopende partiëele afgeleiden heeft tot inclusief die van de $2k^{de}$ orde en als is voorgeschreven, dat

$$u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2} \cdots \frac{\partial^{k-1} u}{\partial n^{k-1}}$$

naderen tot in elk punt der kromme gegeven waarden, als het punt, waar men die grootheden bepaalt, op willekeurige wijze van binnen uit tot een punt der kromme nadert, n duidt hierbij de richting van de normaal der kromme aan, wij zullen die steeds positief rekenen in de richting naar buiten.

Verder [2]) heb ik van die oplossing, aannemende dat zij bestaat, eenige eigenschappen afgeleid.

Ik wil nu de existentie bewijzen van een oplossing, die aan de genoemde eischen voldoet.

Wij zullen daarbij aannemen, dat de gesloten kromme C, die het beschouwde gebied begrenst, in ieder van haar punten een ondubbelzinnig bepaalde raaklijn heeft en zoo is, dat wanneer we een coördinatenstelsel aanbrengen met die raaklijn, tot x-as en de normaal tot y-as, de y van een punt der kromme in de omgeving van den oorsprong een analytische functie is van x,

$$y = c\,x^2 + \ldots$$

Het gebied binnen de kromme zal dan door een analytische substitutie $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ conform kunnen worden afgebeeld op het halfvlak $\eta < 0$.

Van de gegeven waarden voor u, $\frac{\partial u}{\partial n}$ enz. zullen wij aannemen, dat zij analytische functies zijn van den parameter, die de plaats op de kromme vast legt.

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[1]) Over de bepaaldheid der oplossingen van $\triangle^k u = 0$. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 222 (1945).

[2]) Eigenschappen der oplossingen van $\triangle^k u = 0$. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 229 (1945).

Het is nauwelijks twijfelachtig, of de hier geformuleerde voorwaarden, belangrijk kunnen worden verzwakt. Wij laten dat echter voor nader onderzoek over.

Wij zullen bewijzen, dat er onder de genoemde voorwaarden, steeds een analytische functie te vinden is, die aan de gestelde eischen voldoet.

Wat de vergelijking $\triangle^2 u = 0$ betreft (onder $\triangle$ verstaan wij steeds den operator van LAPLACE, onder $\triangle^2 u$ dus $\triangle\triangle u$ enz.) zijn er verschillende existentiebewijzen gegeven, waarbij drie typen zijn te onderscheiden 1^0 met behulp van achtereenvolgende benaderingen (zie b.v. A. KORN, Sur l'équilibre des plaques élastiques encastrées. Ann. de l'Ecole Normale 25 (1908)). 2^0 Met behulp van integraalvergelijkingen (zie b.v. G. LAURICELLA. Sur l'intégration de l'équation relative à l'équilibre des plaques encastrées. Acta Math. 32 (1909) en voor het analoge probleem in drie afmetingen K. SCHRÖDER, Zur Theorie der Randwertaufgaben der Differentialgleichung $\triangle\triangle u = 0$. Math. Zeitschrift 48 (1943), waar men vrij uitvoerige verdere litteratuur-opgaven vindt). 3^0 Met behulp van de directe methoden der variatierekening (zie b.v. K. FRIEDRICHS, Die Randwert- und Eigenwertprobleme der Theorie der elastischen Platten. Math. Ann. 98 (1928)).

In een vroegere meededeeling (Sur l'existence et la construction des solutions de certaines équations aux dérivées partielles du quatriéme ordre, Nederl. Akad. v. Wetensch. Proc. vol. XLV (1942)) heb ik zeer kort een bewijsmethode geschetst, die daarop neerkomt, dat de oplossing wordt teruggebracht tot het bepalen van een tweetal binnen C harmonische functies, wier waarden aan den rand door integraalvergelijkingen bepaald zijn [3]). Daar deze methode mij het meest geschikt voorkomt voor de uitbreiding op het onderhavige probleem, begin ik thans met een iets uitvoeriger uiteenzetting daarvan.

§ 1. Wij willen bewijzen, dat we een functie u kunnen bepalen, die binnen C voldoet aan $\triangle^2 u = 0$, terwijl aan C $u = f_1$, $\frac{\partial u}{\partial n} = f_2$ gegeven analytische functies van de plaats op de kromme zijn. Wij bedoelen daarmee, dat u en $\frac{\partial u}{\partial n}$ tot de gegeven waarden naderen, als het punt, waar we die functies berekenen, op willekeurige wijze naar een punt van C gaat. Voor punten in de nabijheid van C verstaan we onder $\frac{\partial u}{\partial n}$ de afgeleide in de richting van de normaal der door dat punt gaande kromme $\eta = c$, bij de in de inleiding genoemde afbeelding.

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[3]) Vergelijk in dit verband ook H. BREMEKAMP, Construction of the solution of $\triangle\triangle u = 0$, satisfying given boundary conditions in the case that the boundary is an Ellipse. Nieuw Archief voor Wiskunde, Bnd XXII.

Wij maken daarbij gebruik van de eigenschap, dat

$$u = r^2 u_1 + u_0, \quad \ldots \ldots \ldots \quad (1)$$

waarbij r den afstand aanwijst tot een willekeurig punt O, dat wij binnen C zullen kiezen, en waarbij binnen C $\Delta u_1 = \Delta u_0 = 0$, voldoet aan $\Delta^2 u = 0$.

Wij trachten nu de harmonische functies u_0 en u_1 zoo te bepalen, dat aan C

$$r^2 u_1 + u_0 = f_1, \quad (2)$$

$$\frac{\partial}{\partial n}(r^2 u_1 + u_0) = f_2. \quad (3)$$

Met behulp hiervan zoeken wij de waarden u_0 en u_1 aan den rand, $u_{0,c}$ en $u_{1,c}$, waardoor we volgens het theorema van DIRICHLET u_0 en u_1 kunnen vinden. Wij hebben

$$u_0(P) = -\frac{1}{2\pi}\int_C u_{0,c}(Q)\,\frac{\partial G(P,Q)}{\partial n}\,ds, \quad (4)$$

en

$$u_1(P) = -\frac{1}{2\pi}\int_C u_{1,c}(Q)\,\frac{\partial G(P,Q)}{\partial n}\,ds, \quad (5)$$

waarbij P een willekeurig punt binnen C is, Q de kromme C doorloopt en $G(P,Q)$ de gewone functie van GREEN voor de kromme C is.

Wij willen nu deze uitdrukkingen in (3) en (2) substitueeren. Daarbij stellen we vast, dat we onder de integralen, die ontstaan, als we in (4) en (5) (en in dergelijke integralen) P door een punt P_1, der kromme C vervangen, verstaan de limiet, waartoe de betrokken integraal nadert, als P van binnen uit naar P_1 gaat. Deze nadere omschrijving van de beteekenis der integraal is noodig, want als P en Q beide in P_1 vallen is $G(P,Q)$ niet gedefiniëerd en als eerst P en vervolgens Q naar P_1 naderen, wordt $\frac{\partial G(P,Q)}{\partial n}$ oneindig van de eerste orde. Wanneer we dus P zonder meer door P_1 vervangen, hebben de integralen geen beteekenis. Die beteekenis wordt nu door de bovenstaande definitie vastgelegd. Dat de daarbij genoemde limieten bestaan, is uit de theorie der vergelijking van LAPLACE welbekend. In de vergelijkingen (2) en (3) is het eerste lid ook op te vatten als de limiet, waartoe de daar geschreven uitdrukking nadert, als P van binnen uit naar het punt P_1 der kromme gaat. Dat is dus in overeenstemming met de boven omschreven opvatting der integralen. Wij vinden dus

$$2r\frac{\partial r}{\partial n_{P_1}}u_{1,c}(P_1) - \frac{r^2}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C u_{1,c}(Q)\,\frac{\partial G(P,Q)}{\partial n_Q}\,ds -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C u_{0,c}(Q)\,\frac{\partial G(P,Q)}{\partial n_Q}\,ds = f_2.$$

Door hieruit met behulp van (2) $u_{0,c}(Q)$ te elimineeren, vinden we

$$2r\frac{\partial r}{\partial n_{P_1}}u_{1,c}(P_1)-\frac{r^2}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C u_{1,c}(Q)\frac{\partial G(P,Q)}{\partial n_Q}ds+$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C r_Q^2 u_{1,c}(Q)\frac{\partial G(P,Q)}{\partial n_Q}ds-\frac{1}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C f_1(Q)\frac{\partial G(P,Q)}{\partial n_Q}ds=f_2.$$

Hierin stelt

$$-\frac{1}{2\pi}\int_C f_1(Q)\frac{\partial G(P,Q)}{\partial n_Q}ds$$

de harmonische functie $F_1(P)$ voor, die aan C de waarden $f_1(Q)$ aanneemt. Deze is bekend. We merken nog eens op, dat ook als P binnen de kromme ligt, de richting n_P volgens het voorgaande bekend is. Na de differentiatie is overeenkomstig het zooeven gezegde de limietovergang $P\to P_1$ uit te voeren. Wij kunnen nu deze vergelijking brengen in den vorm

$$2r_{P_1}\frac{\partial r_{P_1}}{\partial n_{P_1}}u_{1,c}(P_1)+\frac{1}{2\pi}\int_C (r_Q^2-r_P^2)u_{1,c}(Q)\frac{\partial^2 G(P,Q)}{\partial n_P\,\partial n_Q}ds=f_2-\frac{\partial F_1}{\partial n_{P_1}}. \qquad (6)$$

Ook hier ontstaat, als we P zonder meer door P_1 vervangen, een divergente integraal, de functie

$$(r_Q^2-r_P^2)\frac{\partial^2 G(P,Q)}{\partial n_P\,\partial n_Q}$$

wordt enkelvoudig oneindig, als P en Q beide naar P_1 gaan. Wij moeten echter weer onder de integraal de bovenomschreven limiet verstaan. Wij zullen nu bewijzen, dat wij even goed de hoofdwaarde van

$$\int_C (r_Q^2-r_P^2)u_{1,c}(Q)\frac{\partial^2 G(P,Q)}{\partial n_P\,\partial n_Q}ds$$

in den zin van CAUCHY kunnen nemen. Wij zetten dus van P_1 ter weerszijden gelijke bogen $P_1Q_1=P_1R_1=\varepsilon$ uit en zoeken de limiet voor $\varepsilon\to 0$ van de integraal langs den boog van Q_1 niet over P_1 naar R_1 en willen bewijzen

$$\lim_{P\to P_1}\int_C (r_Q^2-r_P^2)u_{1,c}(Q)\frac{\partial^2 G(P,Q)}{\partial n_P\,\partial n_Q}ds=$$

$$=\lim_{\varepsilon\to 0}\int_{Q_1}^{R_1}(r_Q^2-r_{P_1}^2)u_{1,c}(Q)\frac{\partial^2 G(P,Q)}{\partial n_{P_1}\,\partial n_Q}ds.$$

Wij brengen daarbij het coördinatenstelsel aan met de raaklijn in P_1, als x-as en de (naar buiten wijzende) normaal als y-as. De coördinaten

van Q noemen we (x, y). In verband met onze onderstellingen over de kromme en omdat we voor U_1 een analytische functie zoeken, kunnen wij dan stellen

$$(r_Q^2 - r_P^2)\, u_{1,c}(Q) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

waarbij de coëfficienten $a_0, a_1, \ldots$ van de ligging van P afhangen en $a_0 \to 0$, als $P \to P_1$. Wij hebben dus

$$\lim_{P \to P_1} \int_C (r_Q^2 - r_P^2)\, u_{1,c}(Q)\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds =$$

$$= \lim_{P \to P_1} \int_C (a_0 + a_1 x + \ldots)\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds.$$

Nu is

$$\int_C a_0\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds = a_0\, \frac{\partial}{\partial n_P} \int_C \frac{\partial G(P,Q)}{\partial n_Q}\, ds = 0,$$

voor ieder punt P binnen C, dus is ook de limiet als $P \to P_1$ nul.

Verder is, met gemakkelijk te begrijpen notatie,

$$\left.\begin{aligned} \lim_{P \to P_1} \int_C (a_1 x + a_2 x^2 + \ldots)\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds = \lim_{P \to P_1} \int_{Q_1}^{R_1} (a_1 x + a_2 x^2 + \ldots) \\ \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds + \lim_{P \to P_1} \int_{2\varepsilon} (a_1 x + a_2 x^2 + \ldots)\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, ds. \end{aligned}\right\} \quad (7)$$

De eerste term in het tweede lid is

$$\int_{Q_1}^{R_1} (a_1 x + a_2 x^2 + \ldots)\, \frac{\partial^2 G(P,Q)}{\partial n_{P_1}\, \partial n_Q}\, ds$$

en levert dus, als $\varepsilon \to 0$ de hoofdwaarde van

$$\int_C (a_0 + a_1 x \ldots)\, \frac{\partial^2 G(P_1,Q)}{\partial n_{P_1}\, \partial n_{Q_1}}\, ds,$$

den term met a_0 kunnen wij hier zonder bezwaar toevoegen, die is voor P in P_1 immers nul. Het komt er dus nog op aan, te bewijzen, dat de laatste term in (7) met ε onbepaald afneemt. Voeren we daarbij x als integratieveranderlijke in, dan komt dat daarop neer, dat wij bewijzen, dat uniform naar P_1

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon_1}^{\varepsilon_1} (b_1 x + b_2 x^2 + \ldots)\, \frac{\partial^2 G(P,Q)}{\partial n_P\, \partial n_Q}\, dx = 0$$

en het is duidelijk ,dat wij daarbij alleen te letten hebben op het deel van $G(P, Q)$ dat oneindig wordt, als P en Q naar P_1 gaan, dat is op een term die zich gedraagt als $ln\, PQ$. Beschouwen wij nu het geval, dat P langs de normaal naar P_1 gaat en noemen we $P_1P = -y_1$, dan is wegens onze onderstellingen over de kromme C

$$PQ^2 = (y_1 - y)^2 + x^2 = y_1^2 + c_1x^2 + \ldots, \tag{8}$$

waarin, als $|\, y_1 \,|$ beneden zekere grens ligt, c_1 positief is.

Verder is

$$\frac{\partial ln\, PQ}{\partial n_Q} = \frac{1}{PQ}\cos(PQ, n_Q) = \frac{1}{PQ}\{\cos QPP_1 \cos(n_Q, y) + \\ + \sin QPP_1 \sin(n_Q, y)\}$$

en daar

$$tg\,(n_Q, y) = 2cx + \ldots, \text{ dus } \sin(n_Q, y) = 2cx(1 + 4c^2x^2 + \ldots)^{-\frac{1}{2}},$$
$$\cos(n_Q, y) = (1 + 4c^2x^2 \ldots)^{-\frac{1}{2}},$$

$$\frac{\partial ln\, PQ}{\partial n_Q} = \frac{(y_1 - cx^2 \ldots)(1 - 2c^2x^2 \ldots) + 2cx^2(1 - 2c^2x^2 \ldots)}{PQ^2} = \\ = \frac{y_1(1 - 2c^2x^2 \ldots) + cx^2(1 - \ldots)}{(y_1^2 + c_1x^2 + \ldots)}$$

en, bedenkende dat differentiëeren naar n_P neerkomt op differentiëeren naar y

$$\frac{\partial^2 ln\, PQ}{\partial n_P\, \partial n_Q} = -\frac{y_1^2(1 - 2c^2x^2 \ldots) + y_1(2cx^2 \ldots) - c_1x^2(1 + \ldots)}{(y_1^2 + c_1x^2 + \ldots)^2} \quad . \quad . \tag{9}$$

en

$$(b_1x + b_2x^2 + \ldots)\frac{\partial^2 ln\, PQ}{\partial n_P\, \partial n_Q} = \\ - \frac{y_1^2(b_1x + \beta_2x^2 + \ldots) - y_1(2b_1cx^3 + \gamma_2x^4 \ldots) + (b_1\, c_1\, x^3 + \ldots)}{(y_1^2 + c_1x^2 + \ldots)^2}.$$

Wij hebben nu

$$\int_{-\varepsilon_1}^{\varepsilon_1} \frac{y_1^2\, b_1\, x}{(y_1^2 + c_1x^2)^2}\, dx = 0,$$

onafhankelijk van y_1 en van ε_1 en hetzelfde geldt voor alle termen met oneven machten van x. Verder

$$\left|\beta_2 \int_{-\varepsilon_1}^{\varepsilon_1} \frac{y_1^2\, x^2}{(y_1^2 + c_1x^2)^2}\, dx\right| < |\beta_2| \int_{-\varepsilon_1}^{\varepsilon_1} \frac{dx}{4|c_1|} = \frac{\varepsilon_1}{2}\left|\frac{\beta_2}{c}\right|,$$

en

$$\left|\gamma_2 \int_{-\varepsilon_1}^{\varepsilon_1} \frac{y_1 x^4}{(y_1^2 + c_1 x^2)^2}\, dx\right| < y_1 \gamma_2 \int_{-\varepsilon_1}^{\varepsilon_1} \frac{1}{c_1^2}\, dx = 2\varepsilon_1 \left|\frac{y_1 \gamma_2}{c_1^2}\right|,$$

waaruit blijkt, dat ook deze termen, uniform naar y_1, met ε_1 tot nul naderen en hetzelfde geldt voor alle termen, die hoogere machten van x bevatten.

Inderdaad komt dus de eerst gegeven definitie van de integraal in vergelijking (6) op hetzelfde neer als het vervangen van P door P_1 en het voorschrijven, dat de hoofdwaarde moet genomen worden. De hoofdwaarde van een oneigenlijke integraal over de kromme C, zooals in (6) voorkomt, zullen wij aanduiden door het teeken $\int'_C$. Brengen wij deze wijziging aan, dan ontstaat een integraalvergelijking, waaruit wij de functie $u_{1,c}$ kunnen bepalen. Integraalvergelijkingen, waarin van de er in voorkomende integraal de hoofdwaarde bedoeld wordt, komen reeds voor bij POINCARÉ in het Traité de Mécanique Céleste (T III p. 251, in het hoofdstuk over eb en vloed.). Zij zijn uitvoerig behandeld door G. BERTRAND (Ann. de l'Ecole Normale T 40. 1923). Om nu tot het bestaan der oplossing te kunnen besluiten, moeten wij ons nog overtuigen, dat de bijbehoorende homogene integraalvergelijking

$$2 r_P \frac{\partial r_{P_1}}{\partial n_{P_1}} u_{1,c}(P_1) + \frac{1}{2\pi} \int'_C (r_Q^2 - r_{P_1}^2)\, u_{1,c}(Q) \frac{\partial^2 G(P_1, Q)}{\partial n_P\, \partial n_Q}\, ds = 0. \quad (10)$$

geen oplossing heeft behalve $u_{1,c} = 0$. Was dat wel het geval, m.a.w. was -1 een eigenwaarde van den kern

$$\frac{1}{4\pi} \frac{r_Q^2 - r_{P_1}^2}{r_{P_1} \dfrac{\partial r_{P_1}}{\partial n_{P_1}}} \frac{\partial^2 G(P_1, Q)}{\partial n_{P_1}\, \partial n_Q},$$

dan zou (6) in het algemeen geen oplossing hebben, maar men zou bijzondere functies $f_2 - \dfrac{\partial F_1}{\partial n}$ kunnen construeeren, namelijk orthogonaal op de bijbehoorende eigenfuncties, waarbij de integraalvergelijking (6) oneindig veel oplossingen zou toelaten. Dan zou ook, zooals zoo dadelijk zal blijken, het vraagstuk, u zoo te bepalen, dat $\Delta^2 u = 0$ en aan C, $u = f_1$, $\dfrac{\partial u}{\partial n} = f_2$ oneindig veel oplossingen toelaten en wij hebben bewezen [4]), dat dat onmogelijk is.

Als op deze wijze $u_{1,c}$ bepaald is, geeft (1) $u_{0,c}$ waardoor volgens het theorema van DIRICHLET de functies u_1 en u_0 en dus ook u gevonden worden.

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[4]) Zie l.c. noot [1]).

Uit de onoplosbaarheid van de vergelijking (10) kunnen wij nog een gevolgtrekking maken. Schrijven wij die vergelijking in den vorm

$$2\,r_{P_1}\frac{\partial r_{P_1}}{\partial n_{P_1}}u_{1,c}(P_1)-\frac{r_{P_1}^2}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C' u_{1,c}(Q)\frac{\partial G(P_1,Q)}{\partial n_Q}\,ds=$$

$$=-\frac{1}{2\pi}\frac{\partial}{\partial n_{P_1}}\int_C' r_Q^2\,u_{1,c}(Q)\frac{\partial G(P_1,Q)}{\partial n_Q}\,ds$$

of

$$\frac{\partial}{\partial n_{P_1}}\{r_{P_1}^2\,u_{1,c}(P_1)\}=\frac{\partial\Psi(P_1)}{\partial n_{P_1}},$$

waarbij $\Psi(P)$ de harmonische functie voorstelt, die aan de kromme C de waarden $r^2u_{1,c}$ aanneemt, dan blijkt, dat wij het gevondene ook kunnen uitspreken in den vorm van de volgende stelling.

Er bestaat, behalve de constante 0, geen harmonische functie u, zoodanig dat aan de gesloten kromme C de functie r^2u overal dezelfde normale afgeleide heeft als de functie, die binnen C harmonisch is en aan C overal gelijk is aan r^2u.

§ 2. Wij kunnen nu de functie u, die binnen C voldoet aan $\triangle^2u=0$, terwijl aan C $u=f_1,\frac{\partial u}{\partial n}=f_2$, voorstellen door een integraal, wanneer we een op dat vraagstuk betrekking hebbende functie van GREEN $H_2(P,Q)$ invoeren. Dat geschiedt door de volgende definitie.

Voor willekeurige punten P en Q binnen C geldt:

1° $\triangle^2H_2(P,Q)=0$, als $P\neq Q$, (de differentiaties hebben betrekking op de coördinaten van P)

2° als $H_2(P,Q)=\varrho^2ln\varrho+h_2(P,Q)$, waarbij ϱ den afstand van P tot het willekeurige maar vaste punt Q binnen C voorstelt, heeft h_2 doorloopende partieële afgeleiden tot die van de vierde orde inclusief, als $P\to Q$.

3° $H_2(P,Q)$ en $\frac{\partial H_2(P,Q)}{\partial n}$ gaan naar nul, als P naar eenig punt van C gaat.

Het is duidelijk, dat wij om H_2 te bepalen, slechts h_2 hebben te bepalen, zoodanig dat binnen C $\triangle^2h_2=0$ en aan C $h_2=-\varrho^2\ln\varrho$.

$$\frac{\partial h_2}{\partial n}=-(2\ln\varrho+1)\frac{\partial\varrho}{\partial n},$$

hetgeen volgens het voorgaande mogelijk is.

Om nu de functie u, waarvan in het vorige sprake was, uit te drukken met behulp van H_2 leiden we vooreerst uit het theorema van GREEN

$$\int(U\triangle V-V\triangle U)\,d\sigma=\int\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)ds \qquad (11)$$

een hulpstelling af. Vervangen we V door $\triangle U'$, dan komt er

$$\int (U \triangle^2 U' - \triangle U \triangle U')\, d\sigma = \int \left(U \frac{\partial \triangle U'}{\partial n} - \frac{\partial U}{\partial n} \triangle U' \right) ds,$$

waaruit, door verwisseling van U en U',

$$\int (\triangle U \triangle U' - U' \triangle^2 U)\, d\sigma = \int \left(\triangle U \frac{\partial U'}{\partial n} - \frac{\partial \triangle U}{\partial n} U' \right) ds,$$

en nu door optelling

$$\int (U \triangle^2 U' - U' \triangle^2 U)\, d\sigma =$$

$$= \int \left(U \frac{\partial \triangle U'}{\partial n} - \frac{\partial u}{\partial n} \triangle U' + \triangle U \frac{\partial U'}{\partial n} - \frac{\partial \triangle U}{\partial n} U' \right) ds. \qquad (12)$$

Nemen wij hierin nu voor U de gevraagde functie u, voor U' de functie $H_2(P, Q)$ en voor het integratiegebied het gebied tusschen C en een cirkel om Q met een zoo kleinen straal δ, dat die geheel binnen C valt, dan wordt het linkerlid nul. De integraal rechts moet genomen worden langs C in positieven, langs den cirkel in negatieven zin. Voor de integraal langs C leveren de twee laatste termen van de integrand niets op wegens de definitie van H_2. Om de integraal over den cirkel te vinden, merken we vooreerst op, dat die onafhankelijk van δ is. Stellen we verder $H_2 = \varrho^2 ln \varrho + h_2$, dan zien we, dat de bijdrage van de termen, die van h afkomstig zijn, met δ tot nul nadert. Aan den cirkel is verder, als wij nu weer de normaal naar buiten trekken en daarbij den omloopszin van de integratie omkeeren,

$$\frac{\partial \varrho^2 ln\, \varrho}{\partial n} = \delta (2 ln\, \delta + 1),\ \triangle \varrho^2 ln\, \varrho = 4 ln\, \delta + 4,\ \frac{\partial}{\partial n} \triangle \varrho^2 ln\, \varrho = \frac{4}{\delta},$$

waaruit blijkt, dat ook daarbij alleen de eerste term van de integrand iets oplevert en wel $8\pi u(Q)$. Wij vinden dus

$$8\pi u(Q) = -\int_C \left(u \frac{\partial \triangle H_2}{\partial n} - \frac{\partial u}{\partial n} \triangle H_2 \right) ds. \qquad (13)$$

Uit de formule (12) kunnen wij ook op bekende wijze de wederkeerigheidseigenschap $H_2(P, Q) = H_2(Q, P)$ afleiden.

Mathematics. — *On the theory of linear integral equations.* I. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

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§ 1. *Introduction.*

Let R be a complete (not necessarily separable) HILBERT space. We shall use the following notations:

$f, g, \ldots\ldots$, the elements of R.
$\lambda, \mu, \ldots\ldots$, complex numbers.
$\bar{\lambda}, \bar{\mu}, \ldots\ldots$, the conjugate complex numbers of $\lambda, \mu, \ldots\ldots$
(f, g) , the inner product of f and g.
$\| f \|$, the non-negative numbers $(f, f)^{1/2}$.
$T, K, \ldots\ldots$, bounded, linear transformations in R, that is (for T), $\| Tf \| \leq M \| f \|$ for a certain $M \geq 0$ and $T(\lambda f + \mu g) = \lambda Tf + \mu Tg$ for arbitrary λ, μ, f, g.
$T^*, K^*, \ldots\ldots$, the adjoint transformations of $T, K, \ldots\ldots$, we have therefore (for T) $(Tf, g) = (f, T^* g)$ for arbitrary f, g.
H , a bounded, positive, self-adjoint transformation, that is, a bounded, linear transformation satisfying $(Hf, g) = (f, Hg)$ and $(Hf, f) \geq 0$ for arbitrary f, g.
$H^{1/2}$, the uniquely determined, bounded, positive, self-adjoint transformation, satisfying $(H^{1/2})^2 = H$.
$N(f)$, the non-negative number $(Hf, f)^{1/2} = \| H^{1/2} f \|$.
I , the identical transformation, $If = f$ for every f.
O , the nulltransformation, $Of = 0$ for every f.

We suppose that $H \neq O$. Then the set of all elements f, satisfying $Hf = 0$, is a subspace $[L]$, not identical with the whole space R. The orthogonal subspace will be denoted by $[M]$. As well-known, every element $f \in R$ can be written uniquely in the form $f = h + g$ with $h \in [L]$ and $g \in [M]$. By $g = Ef$ the *projection* E on $[M]$ is defined; the projection on $[L]$ is $I - E$, and we have $E \neq O$. From $H(I - E) f = 0$ for every $f \in R$ follows $Hf = HEf$, so that $H = HE$.

Two elements f and g will be called *H-orthogonal* when $(Hf, g) = 0$, and the system Q of elements is called *H-orthonormal* when, for $\varphi \in Q$, $\psi \in Q$, we have $(H\varphi, \psi) = 1$ for $\varphi = \psi$, and $= 0$ for $\varphi \neq \psi$. The elements $f_1, f_2, \ldots, f_n$ will be called *H-independent* when $H \sum_{i=1}^{n} \lambda_i f_i = 0$ implies $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Evidently, if $f_1, f_2, \ldots, f_n$ are *H*-inde-

pendent, they are linearly independent. It is also not difficult to prove that if the elements $\varphi_1, \ldots, \varphi_n$ form an H-orthonormal system, they are H-independent.

If $Tf = \lambda f$ for an element $f \neq 0$, this element is called a *characteristic element* of the transformation T, belonging to the *characteristic value* λ. The set of all characteristic elements, belonging to the same characteristic value λ, is a subspace of R, and the dimension of this subspace is called the *multiplicity* of the characteristic value λ.

The bounded, linear transformation K is said to be *completely continuous* when every bounded, infinite set of elements contains a sequence f_n such that the sequence Kf_n converges. We shall assume the following theorems about transformations of this kind to be known:

Theorem 1. *If K is completely continuous, the same is true of K^*.*

Theorem 2. *If K is completely continuous, every characteristic value $\lambda \neq 0$ of K has finite multiplicity. The number of different characteristic values λ_n is finite or enumerable and in this last case $\lim_{n \to \infty} \lambda_n = 0$.*

Theorem 3. *If K is completely continuous, and $\lambda \neq 0$ is a characteristic value of K, having a certain multiplicity, then λ is a characteristic value of K^* with the same multiplicity. In this case the equation $Kf - \lambda f = g$ has, for a given element g, a solution f for those and only those elements g that are orthogonal to all characteristic elements of K^*, belonging to the characteristic value λ. In the same way the equation $K^*f - \lambda f = g$ has, for a given element g, a solution f for those and only those elements g that are orthogonal to all characteristic elements of K, belonging to the characteristic value λ.*

If $\lambda \neq 0$ is no characteristic value of K, both the equations $Kf - \lambda f = g$ and $K^*f - \lambda f = g$ have uniquely determined solutions for every element g. In this case the complex number λ will be called a regular value of K.

§ 2. *Bounded, symmetrisable transformations.*

The bounded, linear transformation K is called *symmetrisable* (to the left, and relative to the transformation H), if the transformation HK is self-adjoint, that is, if $(HKf, g) = (f, HKg)$ for arbitrary f, g.

Theorem 4. *If K is symmetrisable, the same is true of $T = EK$. Further $Hf = 0$ implies $Tf = 0$.*

Proof. From $H = HE$ follows $HT = HEK = HK$; if therefore HK is self-adjoint, the same is true of HT.

Further $(HTg, f) = (g, HTf)$ or $(Tg, Hf) = (g, HTf)$ for arbitrary f, g; the relation $Hf = 0$ implies therefore $(g, HTf) = 0$ for every $g \in R$, hence $HTf = 0$. Then however $Tf \in [L]$, so that, since also $Tf = EKf \in [M]$, we have $Tf = 0$.

Theorem 5. *Let the symmetrisable transformation K be such that $Hf = 0$ implies $Kf = 0$. Then the characteristic values of K are real and*

characteristic elements, belonging to different characteristic values, are H-orthogonal.

Proof. Let $f \neq 0$ and $Kf = \lambda f$. If $(Hf, f) = 0$ we see, since $(Hf, f) = \| H^{1/2} f \|^2$, that $H^{1/2} f = 0$, so that $Hf = 0$ or, by hypothesis, $\lambda f = Kf = 0$, from which follows, on account of $f \neq 0$, that $\lambda = 0$. If $(Hf, f) \neq 0$ we find $\lambda(Hf, f) = (H\lambda f, f) = (HKf, f) = (f, HKf) = (f, H\lambda f) = \bar{\lambda}(Hf, f)$ or $\lambda = \bar{\lambda}$, which shows that λ is real.

Let now $\lambda \neq \mu$, $f \neq 0$, $g \neq 0$, $Kf = \lambda f$ and $Kg = \mu g$. Then $\lambda(Hf, g) = (HKf, g) = (f, HKg) = \bar{\mu}(f, Hg) = \mu(Hf, g)$ or $(\lambda - \mu)(Hf, g) = 0$, from which follows, since $\lambda - \mu \neq 0$, that $(Hf, g) = 0$.

Remark. Since, even if $Hf = 0$ does not imply $Kf = 0$ for the symmetrisable transformation K, it does imply $Tf = EKf = 0$ by Theorem 4, Theorem 5 is in any case valid for the symmetrisable transformation $T = EK$.

In what follows now, we shall assume that K, and therefore also $T = EK$, is symmetrisable, while, moreover, $T = EK$ is completely continuous. Then the following theorems hold [1]):

Theorem 6. *If $HK \neq O$, the transformation $T = EK$ has a characteristic value $\lambda \neq 0$, that is, there exists an element $\varphi \neq 0$ such that $T\varphi = \lambda\varphi$. In the particular case that $Hf = 0$ implies $Kf = 0$, the transformation K itself has also the characteristic value λ with characteristic element $\psi = \varphi + \lambda^{-1}(I - E)K\varphi$, hence $K\psi = \lambda\psi$.*

Theorem 7. *In the case that $Hf = 0$ implies $Kf = 0$, the relations $\varphi = E\psi$, $\psi = \varphi + \lambda^{-1}(I - E)K\varphi$ define a one-to-one correspondence between all characteristic elements ψ of K, belonging to the characteristic values $\neq 0$, and all characteristic elements φ of $T = EK$, belonging to the characteristic values $\neq 0$. Corresponding elements have the same characteristic value.*

As a consequence of Theorem 2, it is possible to range the characteristic values $\neq 0$ of T into a sequence λ_n such that every characteristic value $\neq 0$ occurs in this sequence as many times as denoted by its multiplicity, while moreover $|\lambda_1| \geq |\lambda_2| \geq \ldots$. Choosing now in the unitary space (space of finite dimension) of all characteristic elements belonging to a certain characteristic value $\lambda \neq 0$ a maximal system of linearly independent elements, we see readily that these elements, say $\chi_1, \ldots, \chi_p$, are H-independent. Indeed, $H\Sigma a_i\chi_i = 0$ (a_i complex) implying $T\Sigma a_i\chi_i = 0$ or $\Sigma\lambda a_i\chi_i = 0$, we find in virtue of $\lambda \neq 0$ and the linear independence of $\chi_1, \ldots, \chi_p$ that $a_1 = \ldots = a_p = 0$. Applying now to the elements $\chi_1, \ldots, \chi_p$ a process, wholly similar to SCHMIDT's well-known orthogonalization process, we obtain an H-orthonormal system, consisting of p elements, such that the unitary space determined by this system is identical with the

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[1]) A. C. ZAANEN, Ueber vollstetige symmetrische und symmetrisierbare Operatoren. Nieuw Arch. v. Wisk. (2) **22**, 57—80 (1943).

unitary space of all characteristic elements belonging to the characteristic value λ. Doing this for all characteristic values $\neq 0$, we may range the elements of all these H-orthonormal systems into a sequence φ_n, such that for every value of n the element φ_n belongs to the characteristic value λ_n. Evidently the whole sequence φ_n is also H-orthonormal, since for $\lambda_m = \lambda_n$ the relation $(H\varphi_m, \varphi_n) = 0$ follows from our definition of the sequence φ_n and for $\lambda_m \neq \lambda_n$ this relation follows from Theorem 5. In the special case that $Hf = 0$ implies $Kf = 0$, the transformations K and $T = EK$ have, by Theorem 7, the same characteristic values $\neq 0$, and it may be verified readily that every characteristic value $\neq 0$ has, for K and T, the same multiplicity. Consequently, denoting by ψ_n the characteristic element of K, corresponding by Theorem 7 with the characteristic element φ_n of T, we obtain the H-orthonormal sequence ψ_n.

Then we have [2])

Theorem 8. *$|\lambda_n| = \max N(Kf)/N(f)$ for all f satisfying the conditions $N(f) \neq 0$ and $(Hf, \varphi_1) = \ldots = (Hf, \varphi_{n-1}) = 0$. For $f = \varphi_n$ the maximum is attained. Further $HKf = 0$ or, which comes to the same thing, $N(Kf) = 0$ if and only if $(Hf, \varphi_n) = 0$ for every value of n.*

In the particular case that $Hf = 0$ implies $Kf = 0$, the elements φ may be replaced by the corresponding elements ψ in both parts of the theorem.

Theorem 9. *If $a_n = (Hf, \varphi_n)$, then*

$$\lim_{k \to \infty} N(Kf - \sum_{n=1}^{k} \lambda_n a_n \varphi_n) = 0 \quad \textit{(Expansion Theorem)},$$

$$(HKf, f) = \Sigma \lambda_n |a_n|^2$$

for any element f.

In the particular case that $Hf = 0$ implies $Kf = 0$, the elements φ_n may be replaced by the corresponding elements ψ_n.

Theorem 10. *Let λ_{n_i} $(i = 1, 2, \ldots)$ be the subsequence of all positive characteristic values where $\lambda_{n_1} \geq \lambda_{n_2} \geq \ldots$. Then $\lambda_{n_i} = \max (HKf, f)/N^2(f)$ for all f satisfying the conditions $N(f) \neq 0$ and $(Hf, \varphi_{n_1}) = \ldots = (Hf, \varphi_{n_{i-1}}) = 0$. For $f = \varphi_{n_i}$ the maximum is attained.*

A similar statement holds for the subsequence of all negative characteristic values.

In the particular case that $Hf = 0$ implies $Kf = 0$, we may replace the elements φ by the corresponding elements ψ.

In the last theorem the characteristic value λ_{n_i} was characterized by a certain maximum property. It is a disadvantage however that for this characterization the elements $\varphi_{n_1}, \ldots, \varphi_{n_{i-1}}$ must be known. The question arises therefore whether this may be avoided. That this is indeed the case, is shown by the following theorem:

Theorem 11. *Let the elements $p_1, \ldots, p_{i-1}$ be arbitrarily given and let $\mu_i =$ upper bound $(HKf, f)/N^2(f)$ for all f satisfying the conditions*

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[2]) See [1]).

$N(f) \neq 0$ *and* $(Hf, p_1) = \ldots = (Hf, p_{i-1}) = 0$. *The number* μ_i *depends evidently on the elements* $p_1, \ldots, p_{i-1}$. *Letting now these elements run through the whole space* R, *we have* $\lambda_{n_i} = \min \mu_i$.

A similar statement holds for the negative characteristic values.

Proof. We shall prove first that it is possible to find an element $f = \sum_{k=1}^{i} c_k \varphi_{n_k}$ such that the conditions $N(f) = 1$ and $(Hf, p_1) = \ldots = (Hf, p_{i-1}) = 0$ are satisfied. These conditions are equivalent with

$$\sum_{k=1}^{i} |c_k|^2 = 1 \text{ and } \sum_{k=1}^{i} c_k (H\varphi_{n_k}, p_h) = 0 \quad (h = 1, \ldots, i-1),$$

and it follows immediately from a well-known theorem that the $i-1$ homogeneous linear equations

$$\sum_{k=1}^{i} c_k (H\varphi_{n_k}, p_h) = 0 \quad (h = 1, \ldots, i-1)$$

have indeed a solution $c_1, \ldots, c_i$ for which $\sum_{k=1}^{i} |c_k|^2 = 1$. Observing that

$$(HK\varphi_{n_k}, \varphi_{n_k}) = (HT\varphi_{n_k}, \varphi_{n_k}) = \lambda_{n_k}, \text{ and}$$

$$(HK\varphi_{n_k}, \varphi_{n_l}) = (HT\varphi_{n_k}, \varphi_{n_l}) = 0$$

for $k \neq l$, we find then for $f = \sum_{k=1}^{i} c_k \varphi_{n_k}$ the inequality

$$(HKf, f)/N^2(f) = (HKf, f) = \sum_{k,l=1}^{i} c_k \bar{c}_l (HK\varphi_{n_k}, \varphi_{n_l}) =$$

$$= \sum_{k=1}^{i} \lambda_{n_k} |c_k|^2 \geqq \lambda_{n_i} \sum_{k=1}^{i} |c_k|^2 = \lambda_{n_i};$$

it is clear therefore that $\mu_i \geq \lambda_{n_i}$. On the other hand we have, by Theorem 10, for

$$p_1 = \varphi_{n_1}, \ldots, p_{i-1} = \varphi_{n_{i-1}}, (Hf, p_1) = \ldots = (Hf, p_{i-1}) = 0,$$

the relation $\max (HKf, f)/N^2(f) = \lambda_{n_i}$. Hence $\lambda_{n_i} = \min \mu_i$ [3]).

§ 3. *Expression of the solutions of* $Tf - \lambda f = g$ *and* $Kf - \lambda f = g$ *in terms of the characteristic elements.*

We suppose again the transformation K to be symmetrisable and the transformation $T = EK$ to be completely continuous. Then, if $\lambda \neq 0$ is not one of the characteristic values $\lambda_n \neq 0$ of T, it is a regular value (see Theorem 3), on account of Theorem 3 the equation $Tf - \lambda f = g$ has therefore a uniquely determined solution for every element g. If on the other hand λ is identical with one of the characteristic values λ_n, the equation $Tf - \lambda f = g$ has a solution f for those and only those elements

[3]) Compare R. COURANT und D. HILBERT, Methoden der Math. Physik I, Ch. III, § 4, 3.

g that are orthogonal to all characteristic elements of T^*, belonging to the characteristic value $\bar{\lambda}$. Since however all numbers λ_n are real, we see that g must be orthogonal to all characteristic elements of T^*, belonging to the characteristic value λ.

Theorem 12. *For $\lambda \neq 0$, the equation $Tf - \lambda f = g$ has a solution f for those and only those elements g that are H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. (If λ is no characteristic value of T, this means that g may be any element.)*

In the case that $Hf = 0$ implies $Kf = 0$, the same statement holds for the transformation K.

Proof. We shall prove first that, if the p-dimensional unitary space of all characteristic elements of T, belonging to the characteristic value λ, is determined by the linearly independent elements $\chi_1, \ldots, \chi_p$, the p-dimensional unitary space of all characteristic elements of T^*, belonging to the characteristic value λ, is determined by $H\chi_1, \ldots, H\chi_p$. Indeed, the relation $T\chi = \lambda\chi$ implies, for every f,

$$(T^*H\chi, f) = (H\chi, Tf) = (\chi, HTf) = (HT\chi, f) = (\lambda H\chi, f);$$

hence $T^*H\chi = \lambda H\chi$. The elements $H\chi_1, \ldots H\chi_p$ are therefore characteristic elements of T^*. We have still to show that they are linearly independent. This follows from the fact that

$$\sum_{i=1}^{p} a_i H\chi_i = 0 \text{ or } H \sum_{i=1}^{p} a_i \chi_i = 0$$

implies

$$T \sum_{i=1}^{p} a_i \chi_i = 0 \text{ or } \sum_{i=1}^{p} \lambda a_i \chi_i = 0,$$

so that, since $\lambda \neq 0$, we have $a_1 = \ldots = a_p = 0$. Finally we observe that, if g is orthogonal to an element $H\chi$, this means that g and χ are H-orthogonal. The result is therefore that $Tf - \lambda f = g$ has a solution f for those and only those elements that are H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. Evidently this solution is only determined to within an arbitrary linear combination of these characteristic elements. This completes the proof of the first part.

Let now $Hf = 0$ imply $Kf = 0$. Then, since $H = HE$ or $H(I - E) = O$, we have also $K(I - E) = O$ or $K = KE$. We shall prove now that if one of the equations $Kf - \lambda f = g$ and $Tf - \lambda f = g$ has a solution, so has the other. Indeed, from $Kf - \lambda f = g$ follows, since $K = KE$,

$$TEf - \lambda Ef = EKEf - \lambda Ef = E(Kf - \lambda f) = Eg = g - (I - E)g,$$

hence $TEf - \lambda(Ef - \lambda^{-1}(I - E)g) = g$ or, on account of $T(I - E)g = 0$,

$$T(Ef - \lambda^{-1}(I - E)g) - \lambda(Ef - \lambda^{-1}(I - E)g) = g.$$

The element $f_1 = Ef - \lambda^{-1}(I - E)g$ satisfies therefore the relation $Tf_1 - \lambda f_1 = g$.

Conversely, from $Tf - \lambda f = g$ we infer

$$Kf - \lambda f = EKf + (I - E)Kf - \lambda f = g + (I - E)Kf,$$

hence $Kf - \lambda(f + \lambda^{-1}(I - E)Kf) = g$ or, on account of $K(I - E)Kf = 0$,

$$K(f + \lambda^{-1}(I - E)Kf) - \lambda(f + \lambda^{-1}(I - E)Kf) = 0.$$

The element $f_2 = f + \lambda^{-1}(I - E)Kf$ satisfies therefore the relation $Kf_2 - \lambda f_2 = g$.

Thus we find that the equation $Kf - \lambda f = g$ has a solution f for those and only those elements g that are H-orthogonal to all characteristic elements φ of T, belonging to the characteristic value λ. Observing finally that $(Hg, \varphi) = 0$ is equivalent with $(Hg, \psi) = 0$, where ψ is the characteristic element of K corresponding with φ, we obtain the desired result. Evidently the solution of $Kf - \lambda f = g$ is only determined to within an arbitrary linear combination of the characteristic elements of K, belonging to the characteristic value λ.

Theorem 13. *Let $\lambda \neq 0$, and let the element g be H-orthogonal to all characteristic elements of T, belonging to the characteristic value λ. (If λ is no characteristic value, the element g is therefore arbitrary.) Then every solution of $Tf - \lambda f = g$ satisfies the relation*

$$\lim_{k \to \infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \varphi_n\right) = 0,$$

where $a_n = (Hg, \varphi_n)$ for $\lambda_n \neq \lambda$, and where Σ' denotes that for those values of n for which $\lambda_n = \lambda$ the coefficient of φ_n has the value-(Hf, φ_n). For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Tf - \lambda f = g$.

In the case that $Hf = 0$ implies $Kf = 0$, every solution of $Kf - \lambda f = g$ satisfies the relation

$$\lim_{k \to \infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \psi_n\right) = 0,$$

where $a_n = (Hg, \psi_n)$ for $\lambda_n \neq \lambda$, and where Σ' denotes that for those values of n for which $\lambda_n = \lambda$ the coefficient of ψ_n has the value-(Hf, ψ_n). For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Kf - \lambda f = g$.

Proof. Let $Tf - \lambda f = g$. Writing $b_n = (Hf, \varphi_n)$, we have by Theorem 9

$$\lim_{k \to \infty} N\left(Tf - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = \lim_{k \to \infty} N\left(Kf - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = 0,$$

hence

$$\lim_{k \to \infty} N\left(\lambda f + g - \sum_{n=1}^{k} \lambda_n b_n \varphi_n\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From

$$(HTf, \varphi_n) = (f, HT\varphi_n) = (Hf, \lambda_n \varphi_n) = \lambda_n b_n$$

we derive

$$\lambda_n b_n = (HTf, \varphi_n) = (H(g + \lambda f), \varphi_n) = a_n + \lambda b_n,$$

so that for $\lambda_n \neq \lambda$ we find $b_n = -a_n/(\lambda - \lambda_n)$. It follows therefore from (1) that

$$\lim_{k\to\infty} N\left(f + \frac{g}{\lambda} - \sum_{n=1}^{k} \frac{\lambda_n}{\lambda} b_n \varphi_n\right) =$$

$$= \lim_{k\to\infty} N\left(f + \frac{g}{\lambda} + \sum_{n=1}^{k}{}' \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} a_n \varphi_n\right) = 0.$$

Since, if g is given, f is determined to within a linear combination of those elements φ_n for which $\lambda_n = \lambda$, there exists a solution f for every set of arbitrarily prescribed values of the coefficients $b_n = (Hf, \varphi_n)$ for these values of n. This completes the proof of the first part.

The proof of the second part runs in a similar way, substituting everywhere K for T and ψ_n for φ_n.

§ 4. *Self-adjoint transformations as a special case of symmetrisable transformations.*

Identifying the bounded, positive, self-adjoint transformation $H \neq O$ with the identical transformation I, we see that the subspace $[L]$ of all elements satisfying $Hf = If = 0$ contains only the nullelement, so that the orthogonal subspace $[M]$ coincides with the whole space R. The projection E on $[M]$ is therefore the transformation I. The notions of H-orthogonality and H-independence are now identical with the usual notions of orthogonality and linear independence, while $N(f) = \| f \|$. That the bounded linear transformation K is symmetrisable means now that $IK = K$ is self-adjoint, further we have $T = EK = K$.

A bounded symmetrisable transformation K such that $T = EK$ is completely continuous is therefore in this case simply a completely continuous self-adjoint transformation K. The condition that $Hf = If = 0$ implies $Kf = 0$, is always satisfied. The Theorems 4 and 7 lose their meaning; we leave it to the reader to pronounce the other theorems for this case.

§ 5. *Transformations of the form $K = AH$, where A is self-adjoint.*

Theorem 14. *If A is a bounded, self-adjoint transformation, the transformation $K = AH$ is symmetrisable. Further $Hf = 0$ implies $Kf = 0$.*

Proof. $(HKf, g) = (HAHf, g) = (f, HAHg) = (f, HKg)$; the transformation HK is therefore self-adjoint, in other words, K is symmetrisable. The proof of the second part is trivial.

Making now the assumption that one at least of the transformations A and H is completely continuous, the transformation $K = AH$ is symmetrisable and completely continuous. The same holds then for $T = EK$. All theorems in §§ 2—3 are therefore valid for the transformation K. It is possible, however, to prove somewhat more.

Theorem 15. (*Expansion Theorem.*) *If ψ_i is the H-orthonormal sequence of characteristic elements of $K = AH$, belonging to the sequence*

of characteristic values $\lambda_i \neq 0$, *and if* $a_i = (Hf, \psi_i)$, *then*

$$Kf = \Sigma \lambda_i a_i \psi_i + p,$$

where $Hp = 0$. *For* $n \geq 2$ *we have*

$$K^n f = \Sigma \lambda_i^n a_i \psi_i.$$

Proof [4]). We observe first that, the system ψ_i being H-orthonormal, the system $H^{1/2}\psi_i$ is orthonormal, since $(H\psi_i, \psi_k) = (H^{1/2}\psi_i, H^{1/2}\psi_k)$. Writing $a_i = (g, H^{1/2}\psi_i)$ for an arbitrary g, the sums $s_k = \sum_{i=1}^{k} a_i H^{1/2}\psi_i$ converge therefore to an element h. Taking $g = H^{1/2}f$, we find then $\Sigma a_i H^{1/2}\psi_i = h$, where $a_i = (H^{1/2}f, H^{1/2}\psi_i) = (Hf, \psi_i)$. From this we derive

$$AH^{1/2}h = AH^{1/2}\Sigma a_i H^{1/2}\psi_i = \Sigma a_i AH\psi_i = \Sigma \lambda_i a_i \psi_i.$$

The convergence of the series $\Sigma\lambda_i a_i \psi_i$ enables us now to make $k \to \infty$ in the relation

$$\lim_{k\to\infty} N\left(Kf - \sum_{i=1}^{k} \lambda_i a_i \psi_i\right) = 0,$$

proved in Theorem 9. Writing $Kf - \Sigma \lambda_i a_i \psi_i = p$, we obtain then $N(p) = 0$; hence, $N(p) = 0$ being equivalent with $Hp = 0$,

$$Kf = \Sigma \lambda_i a_i \psi_i + p,$$

where $Hp = 0$.

From this we deduce

$$K^2 f = \Sigma \lambda_i a_i K\psi_i + Kp = \Sigma \lambda_i^2 a_i \psi_i + Kp,$$

but, since $Hp = 0$, we have $Kp = AHp = 0$; hence

$$K^2 f = \Sigma \lambda_i^2 a_i \psi_i.$$

The relation

$$K^n f = \Sigma \lambda_i^n a_i \psi_i$$

for $n > 2$ follows easily by induction.

It may be asked whether the element p, occurring in Theorem 15, is not always identical with the nullelement. The answer to this question is given by

Theorem 16. *The element* p *in Theorem* 15 *is not necessarily identical with the nullelement.*

Proof. Let R be a complete, separable HILBERT space in which φ_n is a complete orthonormal system, and let $\mu_3, \mu_4, \ldots$ and $\nu_3, \nu_4, \ldots$ be sequences of positive numbers for which $\lim \mu_n = \lim \nu_n = 0$ and $\mu_3 > \mu_4 > \ldots$, $\nu_3 > \nu_4 > \ldots$. Defining the positive, self-adjoint transformation H and the self-adjoint transformation A by

$$H\varphi_1 = \varphi_1,\ H\varphi_2 = 0,\ H\varphi_i = \mu_i \varphi_i \qquad (i = 3, 4, \ldots),$$
$$A\varphi_1 = \varphi_2,\ A\varphi_2 = \varphi_1,\ A\varphi_i = \nu_i \varphi_i \qquad (i = 3, 4, \ldots),$$

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[4]) This proof is simpler than that in [1]).

it is not difficult to see that both H and A are completely continuous. We have

$$AH\varphi_1 = \varphi_2,\ AH\varphi_2 = 0,\ AH\varphi_i = \nu_i \mu_i \varphi_i \qquad (i = 3, 4, \ldots).$$

To find the characteristic elements of AH belonging to characteristic values $\neq 0$, we write $AHf = \lambda f$ for $f = \sum_{i=1}^{\infty} a_i \varphi_i$ and $\lambda \neq 0$. From this we derive

$$a_1 \varphi_2 + \sum_3 \nu_i \mu_i a_i \varphi_i = \sum_1 \lambda a_i \varphi_i;$$

hence $a_1 = a_2 = 0$ and $\nu_i \mu_i a_i = \lambda a_i (i = 3, 4, \ldots)$. Since $\nu_i \mu_i \neq \nu_k \mu_k$ for $i \neq k$ we have therefore $\lambda = \nu_k \mu_k$ for a certain value of $k (\geq 3)$ and $a_i = 0$ for $i \neq k$, which shows that the elements $a_k \varphi_k$ $(k \geq 3)$ are the only characteristic elements with characteristic values $\neq 0$. Making them H-normal, we obtain $a_k = \mu_k^{-1/2}$, so that, by Theorem 15,

$$AHf = \sum_3 \nu_i (Hf, \varphi_i) \varphi_i + p$$

for every f. Taking $f = \varphi_1$, we have $(Hf, \varphi_i) = (H\varphi_1, \varphi_i) = (\varphi_1, \varphi_i) = 0$ $(i \geq 3)$ and $AHf = AH\varphi_1 = \varphi_2$; hence $p = \varphi_2 \neq 0$.

Theorem 17. *Let $\lambda \neq 0$, and let the element g be H-orthogonal to all characteristic elements of $K = AH$ belonging to the characteristic value λ. (If λ is no characteristic value, the element g is therefore arbitrary.) Then every solution of $Kf - \lambda f = g$ satisfies a relation of the form*

$$f = -\frac{g}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i + q,$$

where $a_i = (Hg, \psi_i)$ for $\lambda_i \neq \lambda$, $Hq = 0$, and where Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of ψ_i has the value (Hf, ψ_i). For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of $Kf - \lambda f = g$.

Proof. Let $Kf - \lambda f = g$. By Theorem 15 we have

$$Kf = AHf = \Sigma \lambda_i b_i \psi_i + p,$$

where $b_i = (Hf, \psi_i)$ and $Hp = 0$. Since, as we have proved in Theorem 13, $b_i = -a_i/(\lambda - \lambda_i)$ for $\lambda_i \neq \lambda$, we obtain

$$\lambda f + g = -\Sigma' \frac{\lambda_i}{\lambda - \lambda_i} a_i \psi_i + p$$

or

$$f = -\frac{g}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i + q,$$

where we have written $q = p/\lambda$. The last statement of the theorem has been proved already in Theorem 13.

Theorem 18. *The self-adjoint transformation $\widetilde{K} = H^{1/2} A H^{1/2}$ possesses the same sequence λ_i of characteristic values $\neq 0$ as the transformation $K = AH$. If ψ_i is an H-orthonormal sequence of characteristic elements*

of K, corresponding with the characteristic values λ_i, then $H^{1/2}\psi_i$ is an orthonormal sequence of characteristic elements of $\widetilde{K}$, also corresponding with the characteristic values λ_i.

Proof. Let $K\psi = AH\psi = \lambda\psi$ where $\lambda\psi \neq 0$. Then, writing $H^{1/2}\psi = \chi$, we have

$$\widetilde{K}\chi = H^{1/2}AH\psi = \lambda H^{1/2}\psi = \lambda\chi,$$

where $\lambda\chi \neq 0$ since $\lambda \neq 0$ and $AH^{1/2}\chi = AH\psi = \lambda\psi \neq 0$. Conversely, if $\widetilde{K}\chi = \lambda\chi$ where $\lambda\chi \neq 0$, we find, writing $\psi = \lambda^{-1}AH^{1/2}\chi$, that

$$H^{1/2}\psi = \lambda^{-1}\widetilde{K}\chi = \chi;$$

hence

$$K\psi = AH^{1/2}H^{1/2}\psi = AH^{1/2}\chi = \lambda\psi,$$

where $\lambda\psi \neq 0$ since $H^{1/2}\lambda\psi = \lambda\chi \neq 0$. This shows that K and $\widetilde{K}$ have the same characteristic values $\neq 0$, and that with the H-orthonormal sequence ψ_i of characteristic elements of K corresponds the orthonormal sequence $H^{1/2}\psi_i$ of characteristic elements of $\widetilde{K}$.

Mathematics. — *On the theory of linear integral equations.* II. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the first paper bearing the same title, to which we shall refer with I. Let a_i, b_i $(i = 1, \ldots, m)$ be real and such that $a_i < b_i$ $(i = 1, \ldots, m)$. Then $\Delta = [a_1, b_1; a_2, b_2; \ldots; a_m, b_m]$ is an interval in the m-dimensional Euclidean space. The point $(x_1, x_2, \ldots, x_m)$ in this space will be denoted by x. Further we shall denote the function space of all functions $f(x)$, with complex values, such that $|f(x)|^2$ is summable (in the sense of LEBESGUE) over Δ, by $L_2^{(m)}(\Delta)$ or $L_2(\Delta)$ or shortly by L_2. As well-known, L_2 is a HILBERT space, if addition and multiplication with complex numbers are defined in the usual way, and the inner product of f and g as

$$(f, g) = \int_{\Delta} f(x)\,\overline{g(x)}\,dx,$$

so that

$$\|f\| = \left(\int_{\Delta} |f|^2\,dx\right)^{1/2}.$$

Convergence of the sequence $s_n(x)$ to $f(x)$ in the space L_2 means that $\lim_{n\to\infty} \|f - s_n\| = 0$; convergence of the series $\sum_{i=1}^{\infty} f_i(x)$ to $f(x)$ means that $\lim_{n\to\infty} \|f - \sum_{i=1}^{n} f_i\| = 0$; to avoid misunderstandings, we shall say that $\sum_{i=1}^{\infty} f_i(x)$ converges *in mean* to $f(x)$, and write

$$f(x) \backsim \sum_{i=1}^{\infty} f_i(x),$$

reserving the term convergence for ordinary point-wise convergence. The interval $[a_1, b_1; \ldots; a_m, b_m; a_1, b_1; \ldots; a_m, b_m]$ in $2m$-dimensional Euclidean space will be denoted by $\Delta \times \Delta$, and the function space of all functions $f(x, y)$ $(x, y \in \Delta)$, with complex values, for which $|f(x, y)|^2$ is summable over $\Delta \times \Delta$, by $L_2^{(2m)}(\Delta)$ or $L_2^{(2m)}$. This function space is also a HILBERT space with inner product

$$(f, g)_{2m} = \int_{\Delta\times\Delta} f(x, y)\,\overline{g(x, y)}\,dx\,dy,$$

so that

$$\|f\|_{2m} = \left(\int_{\triangle \times \triangle} |f(x, y)|^2 \, dx \, dy \right)^{1/2}.$$

Convergence in mean of $\sum_{i=1}^{\infty} f_i(x, y)$ to $f(x, y)$ means that

$$\lim_{n \to \infty} \| f(x, y) - \sum_{i=1}^{n} f_i(x, y) \|_{2m} = 0,$$

and we shall write

$$f(x, y) \backsim \sum_{i=1}^{\infty} f_i(x, y).$$

Let now the measurable function $K(x, y)$ be defined in $\triangle \times \triangle$, such that

$$|K(x, y)|_{2m} = \left(\int_{\triangle \times \triangle} |K(x, y)|^2 \, dx \, dy \right)^{1/2}$$

is finite, in other words, $K(x, y) \in L_2^{(2m)}$. Then it is well-known that the linear transformation K in the space L_2, defined by

$$Kf = \int_{\triangle} K(x, y) f(y) \, dy$$

is completely continuous. If, moreover, $\| K(x, y) \|_{2m} \neq 0$, the transformation K is not identical with the nulltransformation O. We shall consider now the *linear integral equation* $Kf - \lambda f = g$ or

$$\int_{\triangle} K(x, y) f(y) \, dy - \lambda f(x) = g(x) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f, g \in L_2$, the *kernel* $K(x, y) \in L_2^{(2m)}$, and λ is a complex number. If $K(x, y) = K(y, x)$ for almost all points $(x, y) \in \triangle \times \triangle$, the kernel $K(x, y)$ is called *Hermitean*. It follows easily by FUBINI's Theorem on successive integrations that then $(Kf, g) = (f, Kg)$ for arbitrary $f, g \in L_2$, so that in this case the transformation K is completely continuous and self-adjoint.

§ 2. *Integral equation with Hermitean kernel, belonging to* $L_2^{(2m)}$.

Let $K(x, y) \in L_2^{(2m)}$ be Hermitean. Then the theorems, proved in I (for the special case, considered in I, § 4), yield the following results [1]):

[1]) A part of the theorems in this paragraph was proved in a different way, following the method of E. SCHMIDT, by F. SMITHIES, The eigenvalues and singular values of integral equations, Proc. of the London Math. Soc. (2) 43 (1937), 255—279.

Theorem 1. *Every characteristic value $\lambda \neq 0$ of (1) has finite multiplicity. The total number of different characteristic values λ_i is finite or enumerable, and in this last case* $\lim_{i=\infty} \lambda_i = 0$.

Proof. Follows from I, Theorem 2.

Theorem 2. *The characteristic values are real, and characteristic functions, belonging to different characteristic values, are orthogonal.*

Proof. Follows from I, Theorem 5.

Theorem 3. *If $\lambda \neq 0$ is a characteristic value of (1), this equation has, for a given function $g(x) \in L_2$, a solution $f(x) \in L_2$ for those and only those functions $g(x)$ that are orthogonal to all characteristic functions, belonging to the characteristic value λ. If $\lambda \neq 0$ is no characteristic value, so that it is a regular value, the equation (1) has a uniquely determined solution for every $g(x) \in L_2$.*

Proof. Follows from the preceding theorem and I, Theorem 3.

Theorem 4. *If $\| K(x, y) \|_{2m} \neq 0$, the equation (1) has a characteristic value $\neq 0$.*

Proof. Follows from I, Theorem 6.

Let now $\lambda_i (|\lambda_1| \geq |\lambda_2| \geq \ldots)$ be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\varphi_i(x)$ a corresponding orthonormal sequence of characteristic functions. Then we have

Theorem 5. *If* $a_i = (f, \varphi_i) = \int_{\Delta} f(x)\overline{\varphi_i(x)}\, dx$, *then*

$$\int_{\Delta} K(x, y) f(y)\, dy \backsim \Sigma \lambda_i a_i \varphi_i(x) \quad \textit{(Expansion Theorem)},$$

$$\int_{\Delta \times \Delta} K(x, y)\overline{f(x)} f(y)\, dx\, dy = \Sigma \lambda_i |a_i|^2$$

for any $f(x) \in L_2$.

Proof. Follows from I, Theorem 9.

Theorem 6 (COURANT's Theorem). *Let λ_{n_i} $(i = 1, 2, \ldots)$ be the subsequence of all positive characteristic values where $\lambda_{n_1} \geqq \lambda_{n_2} \geqq \ldots$, let the functions $p_1(x), p_2(x), \ldots, p_{i-1}(x)$ be arbitrarily given, and let*

$$\mu_i = \textit{upper bound} \int_{\Delta \times \Delta} K(x, y)\overline{f(x)} f(y)\, dx\, dy \Big/ \int_{\Delta} |f|^2\, dx$$

for all $f(x) \in L_2$ satisfying $\int_{\Delta} |f|^2\, dx \neq 0$ *and*

$$\int_{\Delta} f\overline{p_1}\, dx = \ldots = \int_{\Delta} f\overline{p_{i-1}}\, dx = 0.$$

The number μ_i depends on $p_1(x), \ldots, p_{i-1}(x)$. Letting now these functions run through the whole space L_2, we have $\lambda_{n_i} = \min \mu_i$ [2]).

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 7. *Let $\lambda \neq 0$, and let $g(x) \in L_2$ be orthogonal to all characteristic functions of (1), belonging to the characteristic value λ. (If λ is no characteristic value, the function $g(x)$ is therefore arbitrary.) Then the solution of (1) is given by*

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \varphi_i(x),$$

where $a_i = \int_{\triangle} g(x)\overline{\varphi_i(x)}\,dx$ for $\lambda_i \neq \lambda$, and where Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of $\varphi_i(x)$ may have any arbitrary value.

Proof. Follows from I, Theorem 13.

It is not possible to obtain an expansion theorem for the kernel itself directly from the theorems in I. A little consideration will show, however, that the following theorem holds:

Theorem 8. *We have*

$$\int_{\triangle} |K(x,y)|^2\,dy = \Sigma \lambda_i^2 |\varphi_i(x)|^2 \text{ for almost every } x \in \triangle, \quad . \quad . \quad (2)$$

$$\int_{\triangle \times \triangle} |K(x,y)|^2\,dx\,dy = \Sigma \lambda_i^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$K(x,y) \backsim \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \text{ (Expansion Theorem)} \quad . \quad . \quad . \quad (4)$$

Proof. From $\| K(x,y) \|_{2m} < \infty$ follows that the function $k(x) = K(x,y)$ belongs to the space $L_2^{(m)}$ for almost every $y \in \triangle$. We shall show now that the relations $(k, \varphi_i) = \lambda_i \overline{\varphi_i(y)}$ and $(k, g) = 0$ for every $g(x) \in L_2$ orthogonal to all $\varphi_i(x)$, hold for almost every $y \in \triangle$, so that it will be possible to write $k = \Sigma (k, \varphi_i)\varphi_i = \Sigma \lambda_i \overline{\varphi_i(y)}\,.\,\varphi_i$ in the terminology of HILBERT space. Indeed,

$$(k, \varphi_i) = \int_{\triangle} K(x,y)\,\overline{\varphi_i(x)}\,dx = \overline{\int_{\triangle} K(y,x)\,\varphi_i(x)\,dx} = \lambda_i \overline{\varphi_i(y)}$$

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[2]) R. COURANT, Zur Theorie der linearen Integralgleichungen, Math. Annalen 89 (1923), 161—178, proved his theorem for a continuous kernel, approximating this kernel by a sequence of kernels having only a finite number of characteristic values. He remarks that his result remains valid in the case that $\int\int |K(x,y)|^2\,dx\,dy$ is finite, and $\int |K(x,y)|^2\,dy$ is bounded. As we see here, the first condition alone is sufficient (and, as may be proved, not even necessary).

for almost every $y \in \triangle$, and, if $(g, \varphi_i) = 0$ for all values of i, so that by Theorem 5 $\int_{\triangle} K(x, y)\, g(y)dy = 0$ almost everywhere, we have

$$(k, g) = \int_{\triangle} K(x, y)\, \overline{g(x)}\, dx = \overline{\int_{\triangle} K(y, x)\, g(x)\, dx} = 0$$

for almost every $y \in \triangle$.

The relation $k = \Sigma(k, \varphi_i)\varphi_i$ implies

$$\| k - \sum_{i=1}^{n} (k, \varphi_i)\varphi_i \|^2 = \sum_{i=n+1} |(k, \varphi_i)|^2;$$

hence

$$\int_{\triangle} | K(x, y) - \sum_{i=1}^{n} \lambda_i \varphi_i(x)\, \overline{\varphi_i(y)} |^2\, dx = \sum_{i=n+1} \lambda_i^2\, |\varphi_i(y)|^2 \quad . \quad . \quad (5)$$

for almost every $y \in \triangle$. Taking $n = 0$, we obtain (2), and, integrating (5) over y, we see that

$$\int_{\triangle \times \triangle} | K(x, y) - \sum_{i=1}^{n} \lambda_i \varphi_i(x)\, \overline{\varphi_i(y)} |^2\, dx\, dy = \sum_{i=n+1} \lambda_i^2.$$

For $n = 0$ we have (3), and, making $n \to \infty$, we find (4).

By direct computation it may be verified readily that the *iterated* kernels $K_n(x, y) = \int_{\triangle} K(x, z) K_{n-1}(z, y)dz$ $(n = 2, 3, \ldots)$ exist almost everywhere in $\triangle \times \triangle$, that they are also Hermitean and belong to the space $L_2^{(2m)}$. It is not difficult to prove that the sequence λ_i^n $(i = 1, 2, \ldots)$ is the sequence of all characteristic values $\neq 0$ of $K_n(x, y)$, and that $\varphi_i(x)$ $(i = 1, 2, \ldots)$ is a corresponding sequence of characteristic functions. The Theorems 1—8 hold therefore for the integral equation with kernel $K_n(x, y)$, replacing everywhere λ_i by λ_i^n.

The numbers $\int_{\triangle} K_n(x, x)dx$ are called the *traces* of $K(x, y)$. If n is even $(n = 2p)$, we have

$$\int_{\triangle} K_n(x, x)\, dx = \int_{\triangle \times \triangle} | K_p(x, z) |^2\, dx\, dz = \Sigma \lambda_i^{2p} = \Sigma \lambda_i^n.$$

We shall prove that this relation is also true for odd $n > 1$.

Theorem 9. $\int_{\triangle} K_n(x,x)dx = \Sigma \lambda_i^n$ *for* $n \geq 2$.

Proof. Let $n \geq 2$. From

$$\left| K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n |\varphi_i(x)|^2 \right| =$$

$$= \left| \int_{\triangle} K(x,z) \left[K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right] dz \right| \leq \left(\int_{\triangle} |K(x,z)|^2 dz \right)^{1/2} \cdot$$

$$\cdot \left(\int_{\triangle} \left| K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right|^2 dz \right)^{1/2},$$

holding for almost every $x \in \triangle$, follows on account of SCHWARZ's inequality

$$\left| \int_{\triangle} K_n(x,x)\,dx - \sum_{i=1}^{p} \lambda_i^n \right| = \left| \int_{\triangle} \left\{ K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n |\varphi_i(x)|^2 \right\} dx \right| \leq$$

$$\leq \int_{\triangle} \left| K_n(x,x) - \sum_{i=1}^{p} \lambda_i^n |\varphi_i(x)|^2 \right| dx \leq$$

$$\left(\int_{\triangle \times \triangle} |K(x,z)|^2\,dx\,dz \right)^{1/2} \cdot \left(\int_{\triangle \times \triangle} \left| K_{n-1}(z,x) - \sum_{i=1}^{p} \lambda_i^{n-1} \varphi_i(z) \overline{\varphi_i(x)} \right|^2 dx\,dz \right)^{1/2};$$

hence, since the last factor on the right tends to 0 as $p \to \infty$ by Theorem 8, (4),

$$\int_{\triangle} K_n(x,x)\,dx = \Sigma \lambda_i^n.$$

Remark. It is well-known that if $K(x,y)$ satisfies the conditions that $\int_{\triangle} |K(x,y)|^2\,dy$ is finite for every $x \in \triangle$ and

$$\lim_{x_2 \to x_1} \int_{\triangle} |K(x_2,y) - K(x_1,y)|^2\,dy = 0,$$

several of the results in this paragraph can be improved. The convergence in mean in the Theorems 5 and 7, and also the convergence almost everywhere in Theorem 8, (2), may be replaced by uniform convergence. Moreover, for $n \geq 2$, the series $\Sigma \lambda_i^n \varphi_i(x) \overline{\varphi_i(y)}$ converges uniformly in $\triangle \times \triangle$ to $K_n(x,y)$.

§ 3. *Positive Hermitean kernels.*

It was proved by J. MERCER that, given the positive Hermitean kernel $H(x, y)$, continuous in $\triangle \times \triangle$, we have, uniformly in $\triangle \times \triangle$,

$$H(x, y) = \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)},$$

where $\lambda_i (i = 1, 2, \ldots)$ is the sequence of all characteristic values $\neq 0$ of $H(x, y)$ and $\varphi_i(x)$ a corresponding orthonormal sequence of characteristic functions. As well-known, all λ_i are non-negative (follows easily from Theorem 5).

Theorem 10. *If in the space $L_2(\triangle)$ the bounded, positive, self-adjoint transformation H is given by*

$$Hf = \int_{\triangle} H(x, y) f(y)\, dy,$$

where $H(x, y)$ is a positive Hermitean kernel, continuous in $\triangle \times \triangle$, then there exists a positive Hermitean kernel $H_{1/2}(x, y)$ for which

$$\int_{\triangle} |H_{1/2}(x, y)|^2\, dy$$

is bounded, such that the uniquely determined, bounded, positive, self-adjoint transformation $H^{1/2}$ is given by

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy.$$

Proof. From

$$H(x, y) = \Sigma \lambda_i \varphi_i(x) \overline{\varphi_i(y)},$$

holding uniformly in $\triangle \times \triangle$, follows

$$\int_{\triangle} H(x, x)\, dx = \Sigma \lambda_i = \Sigma (\lambda_i^{1/2})^2;$$

the series $\Sigma \lambda_i^{1/2} \varphi_i(x) \overline{\varphi_i(y)}$ converges therefore in mean (convergence in the HILBERT space $L_2^{(2m)}$) to a function $H_{1/2}(x, y) \in L_2^{(2m)}$, and it is not difficult to see that the transformation A, defined by

$$Af = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy,$$

has $\lambda_i^{1/2} (i = 1, 2, \ldots)$ as the sequence of its characteristic values $\neq 0$, and

$\varphi_i(x)$ as a corresponding sequence of characteristic functions. This transformation is therefore identical with $H^{1/2}$, so that

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy.$$

By Theorem 8 we have

$$\int_{\triangle} |H_{1/2}(x, y)|^2\, dy = \Sigma \lambda_i |\varphi_i(x)|^2 = H(x, x)$$

for almost every $x \in \triangle$, and, since it is allowed to change the values of $H_{1/2}(x, y)$ in a set of measure 0, we may even suppose this relation to hold for every $x \in \triangle$. Observing finally that $H(x, x)$, as a continuous function, is bounded, we obtain the desired result.

Mathematics. — *A theorem concerning analytic continuation.* By J. DE GROOT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 26, 1946.)

1. Two one-valued analytic functions $f_1(z)$ and $f_2(z)$, originally defined in the respective regions G_1 and G_2, are identical in $G_1 + G_2$ if they take the same values in the points of an arbitrarily small region belonging both to G_1 and G_2; and, what is more, they are identical even if they take the same values in infinitely many points which have a limit-point belonging both to G_1 and G_2. In this case we say that the original $f_1(z)$, defined only in G_1, may be continued analytically in the region G_2.

Conversely we may ask ourselves the following question: in the complex Z-plane be given infinitely many points $z_1, z_2, z_3, \ldots$ converging to the point z'. Each point z_i $(i = 1, 2, \ldots)$ is given a certain complex "function-value" w_i. To which extent is it possible to find a one-valued analytic function $f(z)$, defined in a certain region in the Z-plane, which exactly in the points z_i takes the given values w_i; in other words, to which extent is it possible to continue the function defined in z_i to a certain region? In the following we shall — after giving the problem a more accurate form — give a necessary and sufficient condition for analytic continuation (theorem I). For part of the proof we need HANS FREUDENTHAL's mean value theorem of the theory of complex functions, which is generalized by us for higher difference-quotients (theorems II and III).

2. In the problem of continuity mentioned in 1. one must make a distinction between the two following principally different cases: 1°. the region where one wishes the function to be continued does not necessarily contain the limit-point z'; 2°. the region in question does contain z'. Case 1°. immediately leads to the well-known problem of composing an integral function which in the points z_i $(i = 1, 2, \ldots)$, that nowhere in the finite have a limit-point, takes the prescribed value w_i (comp. K. KNOPP, Funktionentheorie (Sammlung Göschen) II, 4th ed., p. 38 problem 3, p. 45 problem 4). The following function $f(z)$ (composed by J. VAN IJZEREN) satisfies this demand:

$$f(z) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{z_i}\right) e^{\frac{z}{z_i} + \frac{1}{2}\left(\frac{z}{z_i}\right)^2 + \ldots \frac{1}{k_i}\left(\frac{z}{z_i}\right)^{k_i}} \cdot \sum_{i=1}^{\infty} \left(\frac{w_i}{p_i \left(1 - \frac{z}{z_i}\right)} - g_i(z) \right).$$

Here Π is a WEIERSTRASZ-product, being 0 in the points z_i; p_i is the value of this infinite product if it is divided by $\left(1 - \frac{z}{z_i}\right)$, while $g_i(z)$ are suitably chosen integral rational functions securing convergence (in case a $z_j = 0$ the shape of $f(z)$ must be slightly altered). $f(z)$ is an obviously

integral function, which in the points z_i takes the values w_i. By submitting the complex plane and thereby the points z_i and the variable z to the following linear transformation

$$z_i = \frac{1}{z_i' - z'}, \qquad z = \frac{1}{\bar{z} - z'},$$

where the sequence of points z_i' converges to z', and by substituting these values in $f(z)$ one finds an analytic function $f(\bar{z})$ which is everywhere one-valuedly analytic except perhaps in the point z', and which in the given points z_i' takes the given values w_i. Thus $f(\bar{z})$ offers a solution to the problem put in case 1°.

We may further confine ourselves to case 2°., where analytic continuation must be possible in a region containing the limit-point z' of the point-sequence z_i. If such a function exists it is (according to 1.) uniquely defined (contrary to the problem put in 1°., where several continuation-functions $f(\bar{z})$ are possible). The required function must take the given values w_i in the points z_i. It is, however, asking too much when we demand that the region in which analytic continuation is possible contains besides z' also all points z_i, as in this case — because of the uniqueness of the function — the required continuation would be impossible by every change, however small, of but one of the values w_i. So we finally ask ourselves the following question: on which conditions is it possible to find an analytic function, defined in a region containing the limit-point z' of the sequence z_i (and so nearly all points z_i), which in almost all points z_i takes prescribed values w_i?

3. It must be possible to expand the required continued function $f(z)$ into a power-series in z':

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z')}{n!}(z - z')^n. \tag{1}$$

Thus necessarily all derivates of $f(z)$ in z' exist. The values w_i in z_i therefore must be chosen such that the limit of the n^{th} difference-quotient exists for every natural number n if the $(n+1)$ points $z_{i_0}, z_{i_1}, \ldots, z_{i_n}$ on which this difference-quotient is defined, tend to z'. Further we must demand that these limits do not tend too quickly to the infinite for $n \to \infty$ and that in such a way that the convergence-radius of (1) is > 0. At first sight one might expect these two conditions to be sufficient for the required analytic continuation. We shall, however, show by an example that this is not the case. We define $f(x)$ with real x by

$$f(x) = e^{-\frac{1}{x^2}} \qquad \text{for } x \neq 0 \;\; ; \;\; f(0) = 0.$$

Apparently all derivates in 0 exist and have the value 0. Now, if on the real axis we take a sequence of points converging to 0 and give these points the corresponding function-values $f(x)$, the above-mentioned con-

ditions are satisfied, i.e., when one runs through the sequence of points the n^{th} difference-quotient tends to the fixed limit 0 for every n, and these limits, i.e., the n^{th} derivates in 0, all are 0, so they all certainly are (uniformly) bounded. Yet analytic continuation to a neighbourhood of the origin is impossible as $e^{-\frac{1}{z^2}}$ is not analytic in the origin.

Now it is possible by sharpening the above-mentioned conditions to find a sufficient condition, namely by demanding as well that the n^{th} difference-quotients "tend uniformly in n to their limits". We prefer, however, to derive a sufficient condition (which will prove to be necessary as well), which moreover, as we may see immediately, is a direct result of the first-mentioned sufficient condition.

4. The n^{th} difference-quotient of $f(z)$ may, as is generally known, be defined like follows:

$$[z_1](f) = f(z_1) \qquad \text{and} \quad [z_1\, z_2](f) = \frac{[z_1](f) - [z_2](f)}{z_1 - z_2}$$

are the 0th and first difference-quotients of $f(z)$ in z_1, and z_1 and z_2 respectively. By induction one defines the n^{th} difference-quotient by

$$[z_1\, z_2 \ldots . z_n\, z_{n+1}](f) = n.\, \frac{[z_1\, z_3 \ldots . z_{n+1}](f) - [z_2\, z_3 \ldots . z_{n+1}](f)}{z_1 - z_2}.$$

For shortness we introduce the following notation:

$$[z_i\, z_{i+1} \ldots z_{i+n}](f) = \triangle_i^n \qquad (n = 1, 2, \ldots ; i = 1, 2, \ldots).$$

Our function $f(z)$ now has already been defined in the points z_j; for $f(z_j) = w_j$. Thus we may compose the difference-quotients $\triangle_j^n$ defined on a number of points z_j.

We now demand *that there may be found a sufficiently great index i* (which we may fix from then onwards), *a positive number r and a natural number n_0 such that for every natural number $n \geqq n_0$*

$$|\triangle_i^n| \leqq n!\, r^n \qquad (n = n_0, n_0 + 1, \ldots). \tag{2}$$

We shall now prove

Theorem I. *If the values $w_j = f(z_j)$ satisfy condition (2) it is possible to find a neighbourhood of the limit-point z' where f may be continued analytically; i.e., it is possible to find a uniquely defined analytic function $f(z)$ (mentioned in (3)), which in almost all points z_j takes the given values w_j. Conversely, if $f(z)$ is a one-valued analytic function in a neighbourhood of point z' and if $z_1, z_2, \ldots$ is a sequence of points converging to z', then condition (2) holds true for every sequence of this kind. Therefore condition (2) is a necessary and sufficient condition for the required analytic continuation.*

5. *Condition (2) is sufficient for the required analytic continuation.*

We contend that

$$f(z) = w_l + \sum_{n=1}^{\infty} \left(\frac{\Delta_l^n}{n!} \cdot \prod_{j=0}^{n-1} (z - z_{l+j}) \right) \qquad (3)$$

is the required analytic function in a sufficiently small neighbourhood of z'. In the first place it is evident that $f(z)$ has exactly the required values w_{l+k} in the points z_{l+k} $(k = 0, 1, 2, \ldots)$, as

$$w_{l+k} = w_l + \sum_{n=1}^{k} \left(\frac{\Delta_l^n}{n!} \cdot \prod_{j=0}^{n-1} (z_{l+k} - z_{l+j}) \right) \qquad (k = 1, 2, \ldots)$$

according to the interpolation-formula of NEWTON. Further the series (3) is uniformly convergent in a neighbourhood of z'. For it is possible to find an $m \geqq n_0$ and a neighbourhood of z' such that for all points z of that neighbourhood and for all points z_{l+m+l} $(l = 1, 2, \ldots)$

$$|z - z_{l+m+l}| < \delta \qquad (\delta \text{ arbitrarily small}, > 0). \qquad (4)$$

The absolute value of (3) then is smaller than or equal to

$$\left| w_l + \sum_{n=1}^{m+1} \left(\frac{\Delta_l^n}{n!} \cdot \prod_{j=0}^{n=1} (z - z_{l+j}) \right) \right| + \prod_{s=0}^{m} |z - z_{l+s}| \cdot$$

$$\cdot \left| \sum_{n=m+2}^{\infty} \left(\frac{\Delta_l^n}{n!} \cdot \prod_{j=m+1}^{n+1} (z - z_{l+j}) \right) \right|.$$

Further, in connection with (2) and (4),

$$\left| \sum_{n=m+2}^{\infty} \left(\frac{\Delta_l^n}{n!} \cdot \prod_{j=m+1}^{n-1} (z - z_{l+j}) \right) \right| \leqq \sum_{n=m+2}^{\infty} r^n \cdot \delta^{n-m-1} = \frac{1}{\delta^{m+1}} \cdot \sum_{n=m+2}^{\infty} (r \cdot \delta)^n.$$

We now choose $\delta < 1/r$, from which follows that the last series is convergent and so the given series (3) is uniformly convergent in a sufficiently small neighbourhood of z'. As all terms of the series (3) are analytic functions, $f(z)$ is (according to a well-known theorem of WEIERSTRASZ) an analytic function in that neighbourhood of z'.

6. For the second part of the proof of theorem I the mean value theorem of the theory of complex functions is of importance. This theorem may be formulated as follows:

Theorem II (HANS FREUDENTHAL). *$f(z)$ be defined and analytic in a convex region G. Consider all values of the derivate $f'(z)$ and the convex closure G' of the corresponding points. We now contend that all difference-quotients $\frac{f(z_1) - f(z_2)}{z_1 - z_2}$ (where z_1 and z_2 belong to G) belong to G'* [1]).

To our purpose we have to generalize this theorem for higher difference-quotients. We shall prove

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[1]) And even: this difference-quotient belongs to the convex closure of the values $f'(z')$, where z' runs through the segment (z_1, z_2).

Theorem III. *$f(z)$ be defined and analytic in a convex region G. Consider all values of the n^{th} derivate $f^{(n)}(z)$ on G (n is an arbitrarily chosen but fixed natural number) and the convex closure $G^{(n)}$ of the corresponding points. We now contend that all n^{th} difference-quotients $[z_0 z_1 \dots z_n](f)$, where $z_0, z_1, \dots, z_n$ are arbitrary points belonging to G, belong to $G^{(n)}$ 2) 3).*

Proof. We start with two remarks:

1°. Be $v_0, v_1, \dots, v_n$ $(n+1)$ arbitrary complex numbers, and so as well points situated in the complex plane (for shortness we shall often identify the point of the complex plane with the corresponding complex number), then point v is situated within the convex closure of $v_0, v_1, \dots, v_n$ only if

$$\left.\begin{array}{r} v = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_n v_n \\ \lambda_i \text{ real with } 0 \leqq \lambda_i \leqq 1 \text{ for } i = 0, 1, \dots n \text{; and} \\ \lambda_0 + \lambda_1 + \dots + \lambda_n = 1 \end{array}\right\}. \tag{5}$$

2°. We may, as known, write the n^{th} difference-quotient

$$[z_0 z_1 \dots z_n](f),$$

which in this proof is denoted by $[z_0 z_1 \dots z_n]$, in the following symmetrical form:

$$[z_0 z_1 \dots z_n] = [z_0 z_1 \dots z_n](f) = n! \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)}. \tag{6}$$

Let us first consider the case where $n = 2$. We take three arbitrary points z_0, z_1 and z_2 of G (fig. 1) and consider the middles p_0, p_1 and p_2

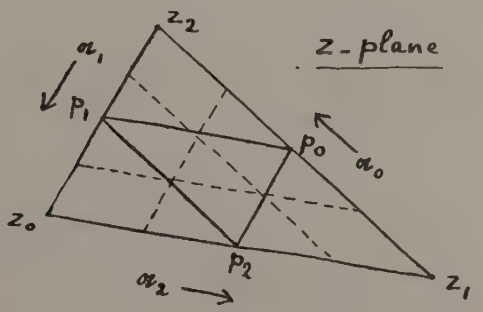

Fig. 1.

of the sides of the corresponding triangle. We shall prove that the following

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2) And even: this difference-quotient belongs to the convex closure of the values $f^{(n)}(z')$, where z' runs through all values of the convex closure of the points $z_0, z_1, \dots, z_n$.

3) By specializing the FREUDENTHAL proof and ours for the real axis one reaches an (especially for higher derivates) fairly short proof of the (extended) mean value theorem of the common calculus.

equation holds true for $[z_0 z_1 z_2]$:

$$[z_0\, z_1\, z_2] = \tfrac{1}{4}[p_0\, p_1\, p_2] + \tfrac{1}{4}[z_0\, p_1\, p_2] + \tfrac{1}{4}[p_0\, z_1\, p_2] + \tfrac{1}{4}[p_0\, p_1\, z_2]. \quad (7)$$

For the equation (7), which we must prove, we may write (according to (6)):

$$\frac{f(z_0)}{(z_0-z_1)(z_0-z_2)} + \frac{f(z_1)}{(z_1-z_0)(z_1-z_2)} + \frac{f(z_2)}{(z_2-z_0)(z_2-z_1)} =$$

$$= \frac{1}{4}\left\{\frac{f(p_0)}{(p_0-p_1)(p_0-p_2)} + \ldots + \frac{f(z_2)}{(z_2-p_0)(z_2-p_1)}\right\}.$$

If we put $z_2 - z_1 = a_0$, $z_0 - z_2 = a_1$ and $z_1 - z_0 = a_2$, then the coefficient of $f(z_0)$ on the left is $-\dfrac{1}{a_2 \,.\, a_1}$, and on the right $-\frac{1}{4}\dfrac{1}{\frac{1}{2}a_2 \,.\, \frac{1}{2}a_1}$; so these coefficients are equal. The coefficient of $f(p_0)$ on the right is

$$-\frac{1}{a_2.a_1} - \frac{1}{a_1.a_0} - \frac{1}{a_2.a_0} = -\frac{a_0+a_2}{a_0\,a_1\,a_2} - \frac{1}{a_2.a_0} = \frac{a_1}{a_0\,a_1\,a_2} - \frac{1}{a_0\,a_2} = 0.$$

By changing the letters for the rest of the coefficients (7) is generally proved.

Geometrically (7) may be interpreted thus, where we suppose the second difference-quotients to be points in the plane of $f''(z)$: the difference-quotient $[z_0 z_1 z_2]$ is situated in the centre of gravity of the four points $[p_0 p_1 p_2]$, $[z_0 p_1 p_2]$, $[p_0 z_1 p_2]$ and $[p_0 p_1 z_2]$ (fig. 2). Therefore $[z_0 z_1 z_2]$ certainly lies within the convex closure of these four points. After this we

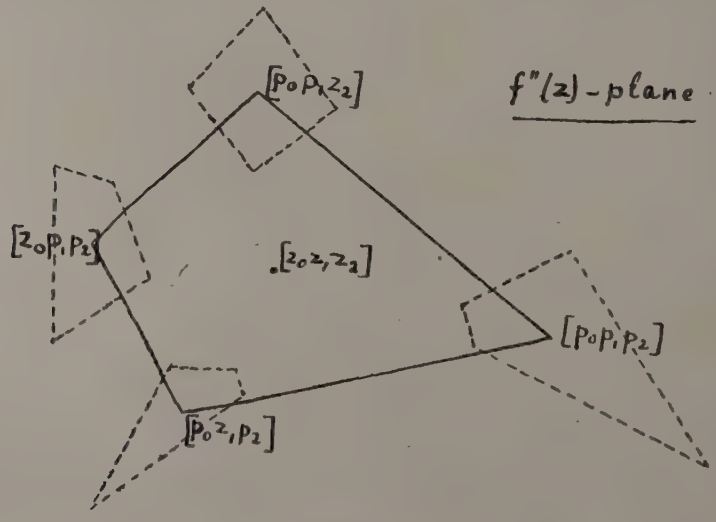

Fig. 2.

subdivide each of the four triangles of fig. 1 in the same way into four triangles, namely by taking the middles of the sides. In each subdivided triangle there is again a relation analogical to (7), in other words the

four difference-quotients, each of which is defined by the vertices of a triangle, again group themselves "around" the corresponding difference-quotient (fig. 2). Therefore $[z_0 z_1 z_2]$ certainly lies within the convex closure of the sixteen points which were formed by the second division. This process may be continued infinitely.

We shall now try to find a relation like (7) and an infinitely continued division for the n^{th} difference-quotients with $n > 2$. Let us first consider the case $n = 3$. Though the four points z_0, z_1, z_2 and z_3, on which a third difference-quotient is defined, lie in the complex plane, it is better to imagine these points to be spacial, as in that way the generalisation may be

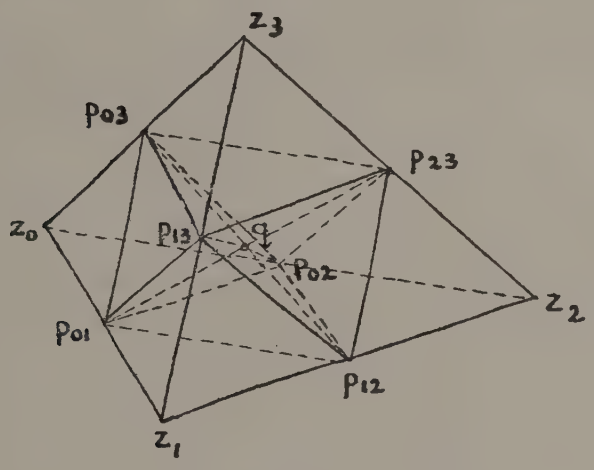

Fig. 3.

more easily realized. One divides every triangle of the tetrahedron $z_0 z_1 z_2 z_3$ (fig. 3) in the same way as in fig. 1 and joins all thus formed points p_{ij}. Three of these joining lines meet in the centre of gravity q. One easily proves — by comparing the coefficients on both sides — that the following relation holds true:

$$\left.\begin{aligned}[z_0 z_1 z_2 z_3] = \frac{1}{2^3}[z_0 p_{01} p_{02} p_{03}] + \frac{1}{2^3}[z_1 p_{01} p_{12} p_{13}] + \\ + \frac{1}{2^3}[z_2 p_{02} p_{12} p_{23}] + \frac{1}{2^3}[z_3 p_{03} p_{13} p_{23}] + \\ + \frac{1}{2^4}[q p_{01} p_{12} p_{13}] + \frac{1}{2^4}[q p_{01} p_{02} p_{12}] + \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots + \frac{1}{2^4}[q p_{03} p_{13} p_{23}]\end{aligned}\right\} \quad (7').$$

In the right side of this relation we find twelve terms corresponding with the twelve tetrahedra into which $z_0 z_1 z_2 z_3$ is divided. In connection with

(5) the point $[z_0 z_1 z_2 z_3]$ lies in the $f''(z)$-plane within the convex closure defined by the twelve difference-quotients of the right side. Each of the twelve tetrahedra is now likewise divided into twelve tetrahedra, and this process is infinitely continued. Always after the m^{th} division $[z_0 z_1 z_2 z_3]$ lies within the convex closure of the 12^m corresponding difference-quotients. By induction it is now possible, though complicated, to generally determine the division of the n-dimensional simplex $z_0 z_1 \dots z_n$, corresponding with the n^{th} difference-quotient $[z_0 z_1 \dots z_n]$ and the corresponding relation $(7^{(n)})$. Because the general formulas are complicated we shall not enter further into these, but shall close this subject by the following remarks. It appears that one must distinguish between odd and even dimension. For the n-simplices with odd n the centre of gravity always acts as point of division, while for the n-simplices with even n this is not the case (in case $n = 2$ one had a "centre-simplex" $p_0 p_1 p_2$, for $n = 4$ one finds a "centre-simplex" $q_0 q_1 q_2 q_3 q_4$, where the points q_i are the centres of gravity of the five tetrahedra by which the 4-simplex $z_0 z_1 z_2 z_3 z_4$ is bounded, etc.). In case $n = 3$ one may also attain one's purpose by a simpler division, namely a division into eight tetrahedra, where instead of the inner eight tetrahedra of fig. 3 we have the four tetrahedra $p_{01} p_{12} p_{02} p_{13}$, $p_{13} p_{02} p_{12} p_{23}$, $p_{01} p_{02} p_{23} p_{13}$ and $p_{03} p_{02} p_{13} p_{23}$. The point q now does not appear in this division. This division is, however, no longer symmetrical and I do not think it will be easy to generalize this division, although it is simpler, for a general n.

We denote $[z_0 z_1 \dots z_n]$ by $\triangle_1^n$, and have stated that the corresponding n-simplex is then subdivided into k_n n-simplices, where we denote the n^{th} difference-quotient of such an n-simplex by $\triangle_{1\,a_1}^n$ $(a_1 = 1, 2, \dots, k_n)$. An equation of the character $(7')$ holds true:

$$\triangle_1^n = \sum_{a_1=1}^{k_n} \lambda_{a_1} \triangle_{1a_1}^n \qquad (0 < \lambda_{a_1} < 1 \,;\; \sum_{a_1=1}^{k_n} \lambda_{a_1} = 1). \qquad (7^{(n)})$$

From now on, if no mistake is possible, we shall denote by $\triangle^n$ not only the n^{th} difference-quotients but also the corresponding n-simplices.

Each of the simplices $\triangle_{1\,a_1}^n$ is again subdivided in the same way into k_n n-simplices $\triangle_{1a_1a_2}^n$ $(a_2 = 1, 2, \dots, k_n)$, and this division is infinitely continued. Thus in general:

$$\triangle_{1a_1a_2\dots a_j}^n = \sum_{a_{j+1}=1}^{k_n} \lambda_{a_{j+1}} \cdot \triangle_{1a_1a_2,\dots a_j\,a_{j+1}}^n \qquad (j = 1, 2, \dots) \qquad (7^{(n)})$$

with the conditions

$$0 < \lambda_{a_{j+1}} < 1 \quad ; \quad \sum_{a_{j+1}=1}^{k_n} \lambda_{a_{j+1}} = 1.$$

The diameter of a sequence of monotonicly decreasing sets of n-simplices

$$\triangle_1^n \supset \triangle_{1\,a_1}^n \supset \triangle_{1a_1a_2}^n \supset \dots \qquad (8)$$

apparently tends to 0, in other words the sets of (8) have one point $d_{1\,a_1\,a_2\,a_3\ldots}$ as intersection; so the sets of (8) converge to that point. The corresponding n^{th} difference-quotients then tend to the n^{th} derivate $f^{(n)}(d_{1\,a_1\,a_2\ldots})$:

$$\lim_{j\to\infty} \Delta^n_{1a_1\,a_2\ldots a_j} = f^{(n)}(d_{1a_1\,a_2\,a_3\ldots.}).$$

We now consider the $f^{(n)}(z)$-plane, where the values $\Delta^n_{1\,a_1\,a_2\ldots a_j}$ are introduced as points (comp. fig. 2). From $(7^{(n)})$ it follows that point Δ^n_1 lies within the convex closure of the points $\Delta^n_{1\,a_1\,a_2\ldots\,a_j}$ $(j = 1, 2, \ldots, i;\ a_j = 1, 2, \ldots, k_n)$. We still have to prove that point Δ^n_1 lies within the convex closure V of the set of points $f^{(n)}(z)$, where z runs through the convex closure of $z_0, z_1, \ldots, z_n$, i.e., the n-simplex Δ^n_1. Suppose on the contrary that *point* Δ^n_1 lay outside V. V is, as the convex closure of the continuous image of a bounded closed set, bounded and closed. Between point Δ^n_1 and V there is a certain distance 2ε. Consider an ε-neighbourhood V_ε of the set V. This apparently is again a convex set while point Δ^n_1 is at a distance ε of V_ε.

The *points* $\Delta^n_{1\,a_1}$ $(a_1 = 1, 2, \ldots, k_n)$ of the first division have a convex closure to which point Δ^n_1 belongs according to the afore-said. Therefore there exists at least one point $\Delta^n_{1\,b_1}$ lying outside V_ε. With the simplex $\Delta^n_{1\,b_1}$ corresponds a division into k_n n-simplices $\Delta^n_{1b_1\,a_2}$ $(a_2 = 1, 2, \ldots, k_n)$. For at least one of these points again the corresponding point $\Delta^n_{1b_1\,b_2}$ lies outside V_ε. By infinitely continuing this process we find a sequence of points $\Delta^n_1, \Delta^n_{1\,b_1}, \Delta^n_{1b_1\,b_2}, \ldots\ldots$ such that for the corresponding n-simplices an equation of the character (8) holds true, in other words, this sequence of points converges to a point determining a value $f^{(n)}(d_{1\,b_1\,b_2\ldots})$. Here $d_{1\,b_1\,b_2\ldots}$ apparently belongs to the convex closure of $z_0\,z_1\,z_2\ldots z_n$. But t hen $f^{(n)}(d_{1b_1b_2\ldots})$ bust belong to V, which is impossible as the points Δ^n_1 $\Delta^n_{1\,b_1}$, ..., converge to a point of the boundary of V_ε or outside V_ε. Thus the required contradiction is reached, by which theorem III (and then by specialization also theorem II) has been proved.

7. Following directly from the mean value theorem derived in 6. is

Theorem IV. *Be $f(z)$ defined and analytic in a convex region G. We consider the n^{th} difference-quotient Δ^n $(n = 1, 2, \ldots)$ defined on $(n+1)$ arbitrary points of G. Now*

$$|\Delta^n| \leqq |f^{(n)}(z')|, \tag{9}$$

where z' is a suitably chosen point of G [4]). Or in other words: the upper bound of $|\Delta^n|$ in G is smaller than or equal to the upper bound of $|f^{(n)}(z)|$ in G.

Proof. Point Δ^n in the complex plane of the points $f^{(n)}(z)$ lies, accor-

[4]) z' even is a point of the convex closure of the $(n+1)$ points. — REMARK. As G is a region the $=$-mark in (9) may even be left out.

ding to theorem III, on the joining line of two points $f^{(n)}(p)$ and $f^{(n)}(q)$, where p and q are suitably chosen points of G. In the triangle determined by the points 0, $f^{(n)}(p)$, $f^{(n)}(q)$ the distance $|\triangle^n|$ between 0 and $\triangle^n$ is smaller than or equal to [5]) $|f^{(n)}(p)|$ or $|f^{(n)}(q)|$ by which the theorem has already been proved.

8. *Condition* (2) *mentioned in theorem I is also necessary.* $f(z)$ is a given function, which certainly is one-valued and analytic in a circle-region around point z'. According to an extension of a well-known inequality of CAUCHY

$$|f^{(n)}(z)| \leqq \frac{2\,n!\,M}{\varrho^n}.$$

Here z is an arbitrary point of a (sufficiently small) circle-region C (within the given circle-region) with z' for centre, and ϱ for radius; M is the maximum of $|f(z)|$ on the circumference of a circle with z' as centre and radius 2ϱ. According to (9) now for all points of C

$$|\triangle^n| \leqq |f^{(n)}(z')| \leqq \frac{2n!\,M}{\varrho^n}. \tag{10}$$

If $2M \leqq 1$, then (10) is $\leqq n!\,r^n$ where $1/\varrho = r$. If $2M > 1$, then (10) is also $\leqq n!\,r^n$, where $2M/\varrho = r$.

Thus in both cases the required condition (2) holds true as one may take i so great that all points $z_i, z_{i+1}, \ldots$, on which $\triangle_i^n$ is defined, belong to the circle-region C.

By this we have also proved the following

Theorem V. *If $f(z)$ be analytic in a point z', we may write $f(z)$ — besides the power-series-expansion in z' — by an expansion of the shape*

$$f(z) = f(z_1) + \sum_{n=1}^{\infty}\left(\frac{\triangle^n}{n!}\cdot\prod_{j=1}^{n}(z - z_j)\right)$$

where $z_1, z_2, z_3, \ldots$ is an arbitrary sequence of points converging to z'; these points have to lie within a sufficiently small neighbourhood of z' [6]). *$\triangle^n$ is the difference-quotient $[z_1\,z_2 \ldots z_{n+1}](f)$.*

As this expansion is a generalization of the interpolation-formula of NEWTON (for infinitely many terms) one might call this the NEWTON-*expansion of $f(z)$* to the sequence $z_1, z_2, \ldots$.

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[5]) When p and q coincide.

[6]) For that neighbourhood we may always take the above-mentioned circle-region C. In this region the series in question is certainly converging.

Mathematics. — *The foundation of the invariant theory of linear systems of curves on an algebraic surface.* By B. L. VAN DER WAERDEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 26, 1946.)

The birational transformation of a linear system of curves on a surface can easily be so defined that the transform of a system without fixed components is again a system without fixed components. But according to this definition the notion of a "complete system" (VOLLSCHAR), that is of a linear system which is not contained in a larger one, is not invariant. For instance, a Cremona transformation of the well-known type with 3 singular points:

$$x'_1 : x'_2 : x'_3 = x_2 x_3 : x_3 x_1 : x_1 x_2$$

transforms the complete system of all straight lines of the plane into a system of conics with three basic points, which is not complete, because it is contained in the larger system of all conics of the plane.

In order to establish the invariance of the notion of a complete system, the Italian geometers have introduced the notion of virtual multiplicity of an assigned base-point of a linear system. The system of conics with 3 base-points, obtained by the above transformation, is complete, if these base-points are *assigned* with *virtual multiplicity* 1, that means, if only those larger systems are let into consideration, whose curves have in these points at least these multiplicities.

But it is not enough to consider ordinary base points with assigned multiplicities; one has to consider also "infinitely near" base points. For instance, the transformation

$$x'_1 : x'_2 : x'_3 = x_2 x_3 : x_1 x_3 : x_2^2$$

transforms the system of all straight lines of the plane into a system of conics with two ordinary base points P and Q and a "neighbour point" P_1, infinitely near to P. The conics of the system have at least multiplicity 1 at P and Q, and the sum of their multiplicities at P and P_1 is at least 2, some of them having a double point at P and others passing through P and P_1. With these assigned virtual multiplicities the system is complete.

The notions of neighbour points and multiplicities in them can be strictly defined, according to NOETHER, ENRIQUES, and ZARISKI, but the analysis is very complicated and the extension to more than 2 dimensions seems hardly possible. Therefore it is desirable to find a simpler notion, by which the analysis of neighbour points is avoidable and which can easily be generalised. This simpler notion turns out to be the notion of valuation.

Let ξ be a general point of an algebraic surface F. Rational functions of ξ are quotients of forms $f(\xi)/g(\xi)$ of equal degrees. They form a field, invariant on birational transformation. Now consider valuations w of this field, whose values $w(\eta)$ are real numbers, with the well-known properties

$$w(\eta\zeta) = w(\eta) + w(\zeta)$$
$$w(\eta + \zeta) \geq \text{Min}\{w(\eta), w(\zeta)\}$$
$$w(a) = 0 \text{ for constant } a$$

Any such valuation defines a *valuation ideal* $\mathfrak{p}$, consisting of those elements η whose values are positive, situated in the *valuation ring* $\mathfrak{o}$, which consists of the elements η of non-negative value, and a *rest class field* $\mathfrak{o}/\mathfrak{p}$, which is a field of algebraic functions of 0 or 1 variable.

This number 0 or 1 is called the *dimension* of the valuation. In the more general case, when F is replaced by a variety of r dimensions, the dimension of the valuation can be any one of the numbers 0, 1, ..., $(r-1)$. We restrict ourselves, however, to valuations of dimension $(r-1)$. As ZARISKI has shown, these valuations are discrete, that is, they can be normed in such a way that the values $w(\eta)$ are whole rational numbers.

Any valuation w can be extended to forms f of arbitrary degree g, by the definition

$$w(f) = \underset{j}{\text{Max.}}\, w(\xi_j^{-g} f(\xi)).$$

The values $w(f)$, so defined, are non-negative whole numbers. Secondly, if the surface F is free from singular curves, the valuation w can be extended to curves C on F, the value $w(C)$ being defined as the minimum of all values $w(f)$ for all forms f, whose intersections with the surface contain the curve C. These values $w(C)$ are again non-negative, and the value $w(C + D)$ for a reducible curve $C + D$ is the sum of the values $w(C) + w(D)$. Moreover, the value $w(C)$ of any curve C of a linear system of curves is at least equal to the minimum of the values of the basic curves of the system.

An example of such a valuation is the multiplicity $w(C)$ of the curve C in a point P of F. Another example is the sum of the multiplicities of C in P and in a neighbour point P_1 of P. This sum can also be defined as the minimum of the multiplicities of intersection of C with an arbitrary curve branch passing by P and P_1. In the same way a valuation can be defined for any neighbour point of P, also to those of higher order, by means of branches passing through them.

On comparing these examples with those given before for assigned virtual multiplicities of base points of linear systems of conics, one sees that the notion of virtual multiplicity can be defined just as well with the aid of valuations without making use of the concept of infinitely near points. For instead of assigning virtual multiplicities to base points and their neighbour points, one can assign "virtual (minimal) values" w_0 for the values $w(C)$ of the curves C of a linear system in certain valuations w.

In order to work out this idea, we must introduce a distinction between "higher" and "lower" valuations. To any valuation w corresponds an irreducible manifold I on F, such that all forms f containing I have a positive value $w(f)$, and all forms f not containing I have the value 0. The dimension of I can be any one of the numbers 0, 1, ..., d, where d is the dimension of the valuation (hence in our case $d = r-1$). If the dimension of I is $(r-1)$, the valuation is called a *higher valuation*, if it is less, a *lower* one. The higher valuations are also called *prime divisors* ("Kurvenprimteiler" in the terminology of H. W. E. JUNG). In the case of a surface F the higher valuations corresponds to irreducible curves I on F: if C contains the curve I with multiplicity m, the value $w(C)$ is m. The lower valuations correspond to points P on F and their neighbour points P_1, P_2 ...: the value $w(C)$ is the minimum multiplicity of intersection of the curve C with an arbitrary curve branch passing through the neighbour point in question.

It is a generally adopted convention, not to assign virtual values w_0 in higher valuations, but only in lower valuations w. This means geometrically, that no fixed components are assigned for linear systems of curves, but only basic points and their neighbour points. But even these lower virtual values w_0 cannot be assigned arbitrarily, for there is a very useful general theorem in the Italian theory, to the effect that any complete linear system with assigned virtual values can be represented as the difference of two linear systems without fixed components, whose virtual values are equal to their effective ones, the *effective value* of a linear system in any valuation w being defined as the minimum of the values $w(C)$ for all curves of the system. In order to verify this theorem it is necessary to restrict the virtual values to those which can be represented as differences of the effective values of two linear systems $|C_1|$ and $|C_2|$ without fixed components, so that the equation

$$w_0 = \text{Min } w(C_1) - \text{Min } w(C_2)$$

holds for all valuations w. Adopting this restriction, the theorem just mentioned can be proved and generalised to r dimensions.

As I said before, the aim of the theory is birational invariance. Now the notion of valuation is invariant by itself, and so is the restriction to valuations of dimension $(r-1)$. The distinction between higher and lower valuations, however, is not invariant: any lower valuation can be transformed birationally into a higher one. The transformation of a linear system is governed in the Italian theory by the following rules:

1°. If a basic point with virtual multiplicity w_0 and effective multiplicity w_1 is transformed into a curve, this curve is to be taken into the transformed system as a fixed component with multiplicity $m = w_1 - w_0$.

2°. If, conversely, a fixed component of the linear system with multiplicity m is transformed into a point, the virtual multiplicity of this point is defined by the equation $m = w_1 - w_0$.

3°. The multiplicities of all other fixed components remain unchanged.

4°. For all basic points and neighbour points which are not transformed into curves the differences $w_1 - w_0$ remain unchanged.

These rules can be combined into one general principle of invariance, valid for all higher and lower valuations:

The difference $w_1 - w_0$ between effective and virtual value of a linear system in any valuation shall remain unchanged upon birational transformation.

This principle can be extended without any change to r dimensions. The virtual value in a higher valuation being zero, the effective multiplicities of all fixed components of the transformed system are determined by the principle of invariance. The variable components are simply the transforms of the variable components of the original system. The fixed and variable components of all curves of the transformed system being thus determined, the effective values w_1 are fixed, and as the differences $w_1 - w_0$ are again determined by the principle of invariance, the virtual values w_0 are fixed also.

If a linear system $|C|$ is contained in a larger system $|D|$ with the same virtual multiplicities, the transformed system $|C'|$ is contained in $|D'|$. From this theorem follows at once the invariance of the notion of a complete system.

The notions sum and difference of complete systems are also invariant on birational transformation, if their virtual values are defined as sums and differences of the virtual values of the component systems. The complete systems with assigned virtual values form upon addition an abelian semi-group, which can be extended to a group by the adjunction of "virtual" differences $|C| - |D|$, to which may or may not correspond effective linear systems.

The proofs of the theorems announced in this note are contained in a manuscript, which I hope is not lost and shall be published some time.

Mathematics. — *On the G-function.* I. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 26, 1946.)

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§ 1. Definition of the function $G^{m,n}_{p,q}(z)$.

Suppose that m, n, p and q are integers with

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q;$$

suppose further that the number z satisfies the inequalities

$$z \neq 0 \text{ and } |z| < 1 \text{ if } q = p,$$
$$z \neq 0 \text{ if } q > p;$$

moreover that the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fulfil the condition

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m). \quad (1)$$

Then the function $G^{m,n}_{p,q}(z)$ is defined as follows [1])

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds. \quad (2)$$

The contour C runs from $\infty - i\tau$ to $\infty + i\tau$ (τ is a positive number) and encloses all the poles

$$b_j, b_j + 1, b_j + 2, \ldots \qquad (j = 1, \ldots, m), \quad (3)$$

but none of the poles

$$a_j - 1, a_j - 2, a_j - 3, \ldots \qquad (j = 1, \ldots, n) \quad (4)$$

of the integrand [2]).

It follows from the asymptotic expansion of the Gamma function that the integral (2) is convergent and independent of τ.

If $m = 0$, the integrand in (2) is analytic inside C. Hence we find

$$G^{0,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = 0. \quad (5)$$

More generally: Suppose $m \geqq 0$ and

$$b_j - b_h \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, m;\ h = 1, \ldots, m;\ j \neq h); \quad (6)$$

then the integrand has simple poles at the points (3) and the value of the integral (2) is equal to minus the sum of the residues of the integrand at these poles. By evaluating these residues we obtain [3])

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[1]) If $p = 0$, the function G is denoted by $G^{m,0}_{0,q}(z \mid b_1, \ldots, b_q)$.

[2]) Such a contour C can always be drawn, since, because of (1), none of the poles (3) coincides with one of the poles (4).

[3]) We employ the usual notation for the generalized hypergeometric function:

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{h=0}^{\infty} \frac{z^h \prod_{j=1}^{p} \{\alpha_j (\alpha_j + 1) \ldots (\alpha_j + h - 1)\}}{h! \prod_{j=1}^{q} \{\beta_j (\beta_j + 1) \ldots (\beta_j + h - 1)\}}.$$

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\sum_{h=1}^{m}\frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-b_h)\prod_{j=1}^{n}\Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)}z^{b_h}\times \left.\vphantom{\frac{1}{1}}\right\} \quad (7)$$

$$\times {}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;1+b_h-b_1,\ldots^{*}\ldots,1+b_h-b_q;(-1)^{p-m-n}z);$$

the asterisk denotes that the number $1+b_h-b_h$ is to be omitted in the sequence $1+b_h-b_1,\ \ldots,\ 1+b_h-b_q$. This expansion holds, provided that a_j and b_j satisfy not only (1) but also the additional condition (6) [4].

I may remark here that in my former papers [5] the function $G_{p,q}^{m,n}(z)$ has been defined by means of (7).

From (7) it is apparent that $G_{p,q}^{m,n}(z)$ is a many-valued function of z with a branch-point at $z=0$.

The function $G_{p,q}^{m,n}(z)$ is a symmetric function of $a_1,\ \ldots,\ a_n$, of $a_{n+1},\ \ldots,\ a_p$, of $b_1,\ \ldots,\ b_m$ and of $b_{m+1},\ \ldots,\ b_q$; this follows immediately from the definition (2).

When there is no risk of ambiguity, we shall in our notation omit the parameters $a_1,\ \ldots,\ a_p$ and $b_1,\ \ldots,\ b_q$. The function G defined by (2) will then be denoted by $G_{p,q}^{m,n}(z)$. When the parameters are not explicitly mentioned, it will always be supposed that they are $a_1,\ \ldots,\ a_p$ and $b_1,\ \ldots,\ b_q$. For instance by the symbols $G_{p,q}^{q,0}(z)$ and $G_{p,q}^{k,l}(z)$ we shall denote the functions

$$G_{p,q}^{q,0}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right),\ \text{respect.}\ G_{p,q}^{k,l}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right).$$

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[4]) The generalized hypergeometric function ${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$ is meaningless if $\beta_j=0,-1,-2,\ldots\ (1\leqq j\leqq q)$. The function

$$\frac{1}{\prod_{j=s+1}^{q}\Gamma(\beta_j)}\,{}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)\qquad(0\leqq s\leqq q)$$

is only meaningless when $\beta_j=0,-1,-2,\ldots\ (1\leqq j\leqq s)$. If one or more of the numbers $\beta_j\ (s+1\leqq j\leqq q)$ is equal to zero or a negative integer, the last function is in the usual way defined as follows:

$$\frac{1}{\prod_{j=s+1}^{q}\Gamma(\beta_j)}\,{}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$$

$$=\sum_{h=0}^{\infty}\frac{z^h\prod_{j=1}^{p}\{\alpha_j(\alpha_j+1)\ldots(\alpha_j+h-1)\}}{h!\prod_{j=1}^{s}\{\beta_j(\beta_j+1)\ldots(\beta_j+h-1)\}\prod_{j=s+1}^{q}\Gamma(\beta_j+h)};$$

when $b_j=0,-1,-2,\ldots\ (s+1\leqq j\leqq q)$, then some initial terms of the series on the right-hand side of this relation vanish.

[5]) MEIJER, [18] — [25].

Now let

$$q \geqq 1,\ 0 \leqq l-1 \leqq n \leqq p \leqq q \text{ and } 0 \leqq k \leqq q;$$

we consider

$$G_{p,q}^{k,l-1}(z) = G_{p,q}^{k,l-1}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right).$$

In this function we change the order of the parameters $a_1, \ldots, a_p$ in a special way; we replace the sequence

$$a_1, \ldots, a_p \text{ by } a_{n-l+2}, \ldots, a_p, a_1, \ldots, a_{n-l+1}.$$

Then we obtain

$$G_{p,q}^{k,l-1}\left(z \,\middle|\, \begin{matrix} a_{n-l+2}, \ldots, a_p, a_1, \ldots, a_{n-l+1} \\ b_1, \ldots, b_q \end{matrix}\right). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This function we will, briefly, denote by $G_{p,q}^{k,l-1,n}(z)$.

We now suppose

$$1 \leqq l \leqq n \leqq p \leqq q,\ 1 \leqq t \leqq n-l+1,\ 0 \leqq k \leqq q$$

and consider

$$G_{p,q}^{k,l}(z) = G_{p,q}^{k,l}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right).$$

In this function we change the order of the parameters $a_1, \ldots, a_p$ in the following way: we replace the sequence

$$a_1, \ldots, a_p \text{ by } a_t, a_{n-l+2}, \ldots, a_p, a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{n-l+1}.$$

Then we get

$$G_{p,q}^{k,l}\left(z \,\middle|\, \begin{matrix} a_t, a_{n-l+2}, \ldots, a_p, a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{n-l+1} \\ b_1, \ldots, b_q \end{matrix}\right). \quad . \quad (9)$$

This function shall be denoted by $G_{p,q}^{k,l,n}(z \| a_t)$.

Since $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_{\lambda+1}, \ldots, a_p$, the functions (8) and (9) are independent of n if $l = 1$. Hence we find

$$G_{p,q}^{k,0,n}(z) = G_{p,q}^{k,0}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = G_{p,q}^{k,0}(z) \quad . \quad . \quad . \quad (10)$$

and

$$G_{p,q}^{k,1,n}(z \,\|\, a_t) = G_{p,q}^{k,1}\left(z \,\middle|\, \begin{matrix} a_t, a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right).$$

The function

$$G_{p,q}^{k,1}\left(z \,\middle|\, \begin{matrix} a_t, a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \qquad (1 \leqq t \leqq p)$$

shall henceforth be denoted by $G_{p,q}^{k,1}(z \| a_t)$. So we obtain

$$G_{p,q}^{k,1,n}(z \,\|\, a_t) = G_{p,q}^{k,1}(z \,\|\, a_t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

§ 2. **Asymptotic expansions of** BARNES' **type for** $G_{p,q}^{m,n}(z)$ $(q>p)$.

In some cases an asymptotic expansion of $G_{p,q}^{m,n}(z)$ $(q>p)$ for large values of $|z|$ can be deduced by means of the methods of BARNES [6]. I will communicate here the results. First I give two definitions.

Definition 1. *Suppose that p, q and t are integers with $1 \leqq t \leqq p < q$; suppose further that the numbers a_t and $b_1, \ldots, b_q$ satisfy the condition*

$$a_t - b_j \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots q) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Then, for brevity, we formally [7]) write

$$E_{p,q}(z \,|\, a_t) = z^{-1+a_t} \sum_{h=0}^{\infty} \frac{(-z)^{-h} \prod_{j=1}^{q} \Gamma(1 + b_j - a_t + h)}{h! \prod_{\substack{j=1 \\ j \neq t}}^{p} \Gamma(1 + a_j - a_t + h)} \quad . \quad . \quad (13)$$

If the divergent series on the right-hand side of (13) is written by means of the hypergeometric notation, the formal expansion (13) takes the form

$$E_{p,q}(z \,|\, a_t) = \frac{z^{-1+a_t} \prod_{j=1}^{q} \Gamma(1 + b_j - a_t)}{\prod_{\substack{j=1 \\ j \neq t}}^{p} \Gamma(1 + a_j - a_t)} \, {}_qF_{p-1}\left(\begin{matrix} 1 + b_1 - a_t, \ldots, 1 + b_q - a_t \\ 1 + a_1 - a_t, \ldots^{*} \ldots, 1 + a_p - a_t \end{matrix}; -z^{-1}\right) \quad . \quad . \quad (14)$$

the asterisk denoting that the number $1 + a_t - a_t$ is to be omitted in the sequence $1 + a_1 - a_t, \ldots, 1 + a_p - a_t$.

From this definition it follows

$$E_{p,q}(z\, e^{2\gamma\pi i} \,|\, a_t) = e^{2\gamma\pi i a_t} \, E_{p,q}(z \,|\, a_t) \qquad (\gamma = 0, \pm 1, \pm 2, \ldots). \quad (15)$$

Definition 2. *Suppose that m, n, q and t are integers with*

$$1 \leqq t \leqq n \text{ and } 0 \leqq m \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_n$ satisfy the condition

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, n; j \neq t).$$

Then the coefficient $\triangle_q^{m,n}(t)$ is defined as follows:

$$\triangle_q^{m,n}(t) = \pi^{m+n-q-1} \frac{\prod_{j=m+1}^{q} \sin(b_j - a_t)\pi}{\prod_{\substack{j=1 \\ j \neq t}}^{n} \sin(a_j - a_t)\pi} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This coefficient may also be written in the form [8])

$$\triangle_q^{m,n}(t) = (-1)^{m+n-q-1} \frac{\prod_{\substack{j=1 \\ j \neq t}}^{n} \{\Gamma(a_t - a_j)\,\Gamma(1 + a_j - a_t)\}}{\prod_{j=m+1}^{q} \{\Gamma(a_t - b_j)\,\Gamma(1 + b_j - a_t)\}} \quad . \quad . \quad (17)$$

---

[6]) BARNES, [3]. Some of BARNES' formulae are also proved by WINKLER, [34] and MACROBERT [14].

[7]) The series on the right-hand side of (13) is divergent because of $q > p$.

[8]) (17) follows from (16) by means of

$$\frac{\pi}{\sin z\pi} = -\Gamma(-z)\,\Gamma(1 + z).$$

The results obtained by BARNES can now be stated as follows [9]):

Theorem A. *The function* $G_{p,q}^{q,1}(z\|a_t)$ *with* $1 \leqq t \leqq p < q$ *admits for large values of* $|z|$ *with* $|\arg z| < (\frac{1}{2}q - \frac{1}{2}p + 1)\pi$ *the asymptotic expansion* [10])

$$G_{p,q}^{q,1}(z\|a_t) \backsim E_{p,q}(z\|a_t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

More generally [11]):

Theorem B. *Suppose that* m, n, p *and* q *are integers with*

$$1 \leqq n \leqq p < q, \quad 1 \leqq m \leqq q \text{ and } m + n > \tfrac{1}{2}p + \tfrac{1}{2}q;$$

suppose further that the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *satisfy the conditions*

$$a_t - b_j \neq 1, 2, 3, \ldots \quad (t = 1, \ldots, n; j = 1, \ldots, m), \quad . \quad . \quad (19)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \quad (j = 1, \ldots, n; t = 1, \ldots, n; j \neq t). \quad (20)$$

Then for large values of $|z|$ *with* $|\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ *the following asymptotic expansion holds* [12])

$$G_{p,q}^{m,n}(z) \backsim \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle_q^{m,n}(t)\, E_{p,q}(z\, e^{(q-m-n+1)\pi i} \| a_t) \quad . \quad (21)$$

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[9]) BARNES, [2], 295—296 and [3], 65. The formulae of BARNES are written in another notation. The symbols $G_{p,q}^{m,n}$, $E_{p,q}$ and $\triangle_q^{m,n}$ do not occur in his papers.

[10]) In (18) it is of course supposed that the numbers a_t and $b_1, \ldots, b_q$ satisfy the condition (12).

Formula (18) is not true uniformly for $|\arg z| < (\frac{1}{2}q - \frac{1}{2}p + 1)\pi$; it is true uniformly only for $|\arg z| \leqq (\frac{1}{2}q - \frac{1}{2}p + 1)\pi - \delta$ for every $\delta > 0$. A similar remark holds with regard to the asymptotic expansions (21) and (22).

[11]) BARNES, [3], 70. The formulae (21) and (18) are proved by evaluating the residues of the integrand in (2) at the poles outside the contour C.

The result of BARNES with regard to formula (21) is not entirely complete; he seems to consider only the case with $m \geqq q - p$.

[12]) The expansion of the term

$$e^{(m+n-q-1)\pi i a_t} \triangle_q^{m,n}(t)\, E_{p,q}(z\, e^{(q-m-n+1)\pi i} \| a_t)$$

can according to (17) and (13) be written in the form

$$\frac{e^{(1+a_t)} \prod_{\substack{j=1 \\ j \neq t}}^{n} \Gamma(a_t - a_j)}{\prod_{j=m+1}^{q} \Gamma(a_t - b_j)} \sum_{h=0}^{\infty} \frac{\{(-1)^{q-m-n} z\}^{-h} \prod_{j=1}^{m} \Gamma(1 + b_j - a_t + h) \prod_{j=m+1}^{q} \{(1 + b_j - a_t)(2 + b_j - a_t) \ldots (h + b_j - a_t)\}}{h! \prod_{\substack{j=1 \\ j \neq t}}^{n} \{(1 + a_j - a_t)(2 + a_j - a_t) \ldots (h + a_j - a_t)\} \prod_{j=n+1}^{p} \Gamma(1 + a_j - a_t + h)}.$$

The coefficients in this divergent series are significant, if the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfy the conditions (19) and (20).

BARNES has also investigated the behaviour of $G_{p,q}^{q,0}(z)$ for $|z|$ tending to infinity. His result runs as follows [13]:

The function $G_{p,q}^{q,0}(z)$ possesses for large values of $|z|$ an asymptotic expansion of the form

$$G_{p,q}^{q,0}(z) \backsim \left[\exp\left((p-q)\, z^{\frac{1}{q-p}}\right)\right] z^{\vartheta} \left\{ \frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{z^{\frac{1}{q-p}}} + \frac{M_2}{z^{\frac{2}{q-p}}} + \ldots \right\} \quad (22)$$

where

$$\vartheta = \frac{1}{q-p}\left\{ \tfrac{1}{2}(p-q+1) + \sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h \right\} \quad . \quad . \quad . \quad . \quad (23)$$

This expansion holds for $|\arg z| < (q-p+\varepsilon)\pi$; herein is

$$\varepsilon = \tfrac{1}{2} \text{ if } q = p+1, \quad \varepsilon = 1 \text{ if } q \geqq p+2. \quad . \quad . \quad . \quad . \quad (24)$$

The coefficients M_1, M_2, ... do not depend on z, but are complicated functions of the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ which I will not record here [14].

In what follows we shall, for brevity, formally write

$$H_{p,q}(z) = \left[\exp\left((p-q)\, z^{\frac{1}{q-p}}\right)\right] z^{\vartheta} \left\{ \frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{z^{\frac{1}{q-p}}} + \frac{M_2}{z^{\frac{2}{q-p}}} + \ldots \right\}, \quad (25)$$

where ϑ is defined by (23).

The results contained in (22) and (23) can then be formulated as follows:

Theorem C. *The function $G_{p,q}^{q,0}(z)$ possesses for large values of $|z|$ with $|\arg z| < (q-p+\varepsilon)\pi$ the asymptotic expansion*

$$G_{p,q}^{q,0}(z) \backsim H_{p,q}(z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It appears from (25) and (23) that the formal function $H_{p,q}(z)$ satisfies the relation

$$(-1)^{p-q+1} \exp\left\{ 2\pi i \left(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h \right) \right\} H_{p,q}\left(z\, e^{(2p-2q)\pi i}\right) = H_{p,q}(z). \quad (27)$$

§ 3. The analytic continuation of the function $G_{p,p}^{m,n}(z)$ (simple cases).

The function $G_{p,p}^{m,n}(z)$ has been defined in § 1 only for $|z|<1$. The analytic continuation of this function when $|z| \geqq 1$ will be denoted too by the symbol $G_{p,p}^{m,n}(z)$.

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[13] BARNES, [2], 296—297; [3], 80 and 108—110.
The proof of (22) is much more difficult than that of (18) and (21).

[14] The reader is referred to BARNES' paper [3].

Now it is well-known [15]) that the generalized hypergeometric function ${}_pF_{p-1}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_{p-1}; z)$ has a branch-point at $z = 1$, and that, if a cross-cut is made along the real axis of the z-plane from 1 to ∞, then the function is analytic and one-valued throughout the cut plane. Considering the function $G^{m,n}_{p,p}(z)$, which can by (7) be written as a linear combination of functions of the type $z^\lambda {}_pF_{p-1}((-1)^{p-m-n}z)$, we shall make a cross-cut in the z-plane along the real axis from $(-1)^{m+n-p}$ to $\infty \cdot (-1)^{m+n-p}$; we may expect that $G^{m,n}_{p,p}(z)$ is analytic (not one-valued in general) in the cut plane.

I will now shew that a somewhat more general result holds when $m + n \geqq p + 2$.

For this purpose I consider the integral

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{p}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s\,ds, \quad . \quad . \quad . \quad (28)$$

where $1 \leqq m \leqq p$, $1 \leqq n \leqq p$, $m + n \geqq p + 1$ and $a_j - b_h \neq 1, 2, 3, \ldots$ $(j = 1, \ldots, n; h = 1, \ldots, m)$.

The contour L runs from $-\infty i + \sigma$ to $\infty i + \sigma$ (σ is an arbitrary real number) and is curved, if necessary, so that the points

$$b_j, b_j + 1, b_j + 2, \ldots \qquad (j = 1, \ldots, m)$$

lie on the right and the points

$$a_j - 1, a_j - 2, a_j - 3, \ldots \qquad (j = 1, \ldots, n)$$

lie on the left of the contour.

The integral (28) is a function of z which is analytic for each value of z whether $|z|$ is greater than, equal to, or less than unity, provided that $z \neq 0$ and $|\arg z| < (m + n - p)\pi$ [16]).

If $|z| < 1$, we may [17]), without altering the value of the integral, bend round the contour L so as to coincide with the contour C defined in § 1. We therefore get in view of (2).

Theorem D. *The function $G^{m,n}_{p,p}(z)$ with $m + n \geqq p + 1$ is an analytic function of z in the sector $|\arg z| < (m + n - p)\pi$; this function may be represented by the integral* (28).

Now, when $m + n \geqq p + 2$, the point $(-1)^{m+n-p}$ lies in the interior of

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15) POCHHAMMER, [26]; WINKLER, [33]; MACROBERT, [16]. Comp. also MACROBERT, [17].

16) The reader may compare BAILEY, [1], 5—6 or WHITTAKER and WATSON, [32], 286—288, where a proof for a special case is elaborated. The proof of the general case runs along the same lines.

17) Comp. BAILEY, loc. cit. or WHITTAKER and WATSON, loc. cit..

the sector $|\arg z| < (m+n-p)\pi$. So, if we restrict ourselves to values of z for which $|\arg z| < (m+n-p)\pi$, the function $G^{m,n}_{p,p}(z)$ with $m+n \geqq p+2$ has no singularity at $z = (-1)^{m+n-p}$.

Hence, if $m+n \geqq p+2$ and at the same time $|\arg z| < (m+n-p)\pi$, the cross-cut from $(-1)^{m+n-p}$ to $\infty \cdot (-1)^{m+n-p}$ is superfluous.

In all other cases, in continuing the function $G^{m,n}_{p,p}(z)$ outside the circle $|z| = 1$, we will make a cross-cut along the real axis from $(-1)^{m+n-p}$ to $\infty \cdot (-1)^{m+n-p}$.

Replacing s by $-s$ in (28) we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{n}\Gamma(1-a_j-s)\prod_{j=1}^{m}\Gamma(b_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{p}\Gamma(1-b_j-s)}\left(\frac{1}{z}\right)^s ds.$$

On comparing this integral with (28) we find in virtue of theorem D that it is equal to

$$G^{n,m}_{p,p}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\ 1-a_1,\ldots,1-a_p\end{matrix}\right).$$

So we obtain

Theorem E. *The function $G^{m,n}_{p,p}(z)$ with $m+n \geqq p+1$ possesses in the sector $|\arg z| < (m+n-p)\pi$ the following analytic continuation outside the circle $|z| = 1$*

$$G^{m,n}_{p,p}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix}\right) = G^{n,m}_{p,p}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\ 1-a_1,\ldots,1-a_p\end{matrix}\right). \quad (29)$$

In order to obtain a convenient notation for the continuation of the function $G^{p,1}_{p,p}(z\|a_t)$ I give still another definition.

Definition 3. (Comp. Definition 1.) *If p and t are integers with $1 \leqq t \leqq p$ and if the numbers a_t and $b_1, \ldots, b_p$ satisfy the condition*

$$a_t - b_j \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, p),$$

then the function $E_{p,p}(z\,\|\,a_t)$ is defined as follows:

$$E_{p,p}(z\,\|\,a_t) = G^{1,p}_{p,p}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_p\\ 1-a_t,\,1-a_1,\ldots,1-a_{t-1},\,1-a_{t+1},\ldots,1-a_p\end{matrix}\right). \quad (30)$$

This function may because of (7) also be written (comp. (14))

$$E_{p,p}(z\,\|\,a_t) = \frac{z^{-1+a_t}\prod_{j=1}^{p}\Gamma(1+b_j-a_t)}{\prod_{\substack{j=1\\ j\neq t}}^{p}\Gamma(1+a_j-a_t)}\,{}_pF_{p-1}\left(\begin{matrix}1+b_1-a_t,\ldots,1+b_p-a_t;\\ 1+a_1-a_t,\ldots,\overset{*}{\ldots},1+a_p-a_t;\end{matrix}-z^{-1}\right); \quad (31)$$

the asterisk denotes, as usual, that the number $1 + a_t - a_t$ *is to be omitted in the sequence* $1 + a_1 - a_t, \ldots, 1 + a_p - a_t$ [18]).

Now it follows from (7), (17) and (31), if the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfy the conditions (19) and (20),

$$G_{p,p}^{n,m}\left(\frac{1}{z}\,\middle|\,\begin{matrix}1-b_1, \ldots, 1-b_p\\ 1-a_1, \ldots, 1-a_p\end{matrix}\right)$$

$$= \sum_{t=1}^{n} e^{(m+n-p-1)\pi i a_t} \Delta_p^{m,n}(t)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i} \,\|\, a_t).$$

Theorem E may therefore be stated in another way, namely

Theorem E*. (Comp. Theorem B.) *If* $m + n \geqq p + 1$ *and the numbers* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *satisfy the conditions* (19) *and* (20), *then the function* $G_{p,p}^{m,n}(z)$ *possesses in the sector* $|\arg z| < (m+n-p)\pi$ *an analytic continuation outside the unit-circle which can be expressed in the form*

$$G_{p,p}^{m,n}(z) = \sum_{t=1}^{n} e^{(m+n-p-1)\pi i a_t} \Delta_p^{m,n}(t)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i} \,\|\, a_t). \qquad (32)$$

The particular case with $m = p$ and $n = 1$ of formula (29) (which is, of course, also the corresponding particular case of (32)) is worth while to mention it separately; because of theorem E and (30) we obtain:

Theorem F. (Comp. Theorem A.) *The function* $G_{p,p}^{p,1}(z \,\|\, a_t)$ *possesses in the sector* $|\arg z| < \pi$ *the following analytic continuation outside the unit-circle*

$$G_{p,p}^{p,1}(z \,\|\, a_t) = E_{p,p}(z \,\|\, a_t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

This formula and the particular case with $m = 1$ and $n = p$ of (32) are well-known. For $p = 2$ the formulae are proved by GAUSS, KUMMER, GOURSAT, BARNES and VAN DER CORPUT [19]). The general case has been treated by THOMAE, WINKLER, MACROBERT and F. C. SMITH [20]).

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[18]) Comp. footnote [4]).

[19]) GAUSS, [10], 220; KUMMER, [13], 60; GOURSAT, [11], 29; BARNES, [5], 146—152; VAN DER CORPUT, [6]. Comp. also: FORSYTH, [8], Chapter VI; KLEIN, [12], § 10; WHITTAKER and WATSON, [32], § 14. 51. VAN DER CORPUT and WHITTAKER and WATSON give the formula in question with some misprints.

[20]) THOMAE, [29]; WINKLER, [33]; MACROBERT, [15]; SMITH, [27]. Comp. also MEIJER, [19], formula (40).

Mathematics. — *Primitief-symmetrische projectieve invarianten.* I. By P. G. MOLENAAR. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of November 24, 1945.)

Samenvatting.

Naar men weet, kan elke projectieve invariant $I(a)$, die van de graad t is in de coëfficiënten van de grondvorm $f = a_x^p$, door gelijkstelling van aequivalente coëfficiënten worden afgeleid uit een simultaaninvariant $I_1(a, b, \ldots, l)$, die lineair en homogeen is in elk der coëfficiëntenrijen van t aequivalente grondvormen

$$f_1 = a_x^p \quad f_2 = b_x^p \quad - \quad - \quad - \quad f_t = l_x^p.$$

Deze simultaaninvariant is in het algemeen niet symmetrisch in de coëfficiëntenrijen. Beschouwt men de grondvormen als niet-aequivalent (en dit is noodzakelijk bij het onderzoek van invarianten van gemengde vormen (§ 6)), dan is het wenselijk, deze invariant te splitsen in delen, die behoren tot primitieve symmetrieklassen.

Met behulp van hypercomplexe getallen, ontleend aan de volledig gereduceerde reguliere voorstelling ener permutatiegroep (§ 1) verkrijgt men primitieve karakteristieke eenheden, waardoor primitieve symmetrieklassen bepaald worden (§ 2). De invariant I_1 (en tevens elk van zijn isomeren) kan nu worden gesplitst in primitief-symmetrische invarianten (§ 3). Eén en ander wordt toegelicht, door invarianten van vier binaire cubische grondvormen te onderzoeken, waarbij symmetrische, anti-symmetrische en cyclisch-symmetrische invarianten tevoorschijn komen (§ 4). Het symmetrische deel levert bij gelijkstelling van de als aequivalent beschouwde coëfficiëntenrijen juist de invariant $I(a)$ van a_x^3 op. Het anti-symmetrische deel valt uiteen in een som van producten van primitief-symmetrische invarianten van lagere graad. De cyclisch-symmetrische delen vallen eveneens uiteen. Zij leveren de voorwaarden voor „cubisch-harmonische" ligging van puntentriples (§ 5).

Bij het opstellen van invarianten van een gemengde vorm $F = a_x^p \alpha_\xi^\pi$ moet men kunnen beschikken over de invariantensystemen van twee groepen van gelijksoortige grondvormen

$$a_x^p \quad b_x^p \quad - \quad - \quad l_x^p \text{ en } \alpha_\xi^\pi \quad \beta_\xi^\pi \quad - \quad - \quad \lambda_\xi^\pi.$$

Allereerst wordt bewezen, dat elke invariant van F symbolisch is voor te stellen door een product van haakfactoren

$$(ab\ldots)\ldots\ldots(\alpha\beta\ldots)\ldots\ldots$$

Haakfactoren $(a\,\alpha\ldots)$ behoeven dus niet voor te komen (§ 6). Voorts wordt uit een primitief-symmetrische invariant $L = (a, b, \ldots, l)$ en een primitief-symmetrische invariant $\Lambda = \Lambda\,(\alpha, \beta, \ldots, \lambda)$ door „contractie" een invariant I van F afgeleid, welke in het algemeen niet primitief-symmetrisch is. Door een KRONECKER's productvoorstelling volledig te reduceren, vindt men een splitsing van I in primitief-symmetrische invarianten (§ 7, 8).

Als voorbeeld wordt een invariant van de gemengde vorm $F = a_x^3\,\alpha_\xi^3$ onderzocht. De niet-aequivalente grondvormen

$$a_x^3\,\alpha_\xi^3 \quad , \quad b_x^3\,\beta_\xi^3 \quad , \quad c_x^3\,\gamma_\xi^3 \quad , \quad d_x^3\,\delta_\xi^3$$

leveren dan 3 symmetrische, 3 anti-symmetrische en 10 cyclisch-symmetrische invarianten (§ 9). De symmetrische worden nader beschouwd. Zij leveren na gelijkstelling van de als aequivalent beschouwde coëfficiënten drie vierdegraads invarianten van $F = a_x^3\,\alpha_\xi^3$ (§ 10). Tenslotte wordt voor twee gemengde vormen een overschuiving gedefinieerd. Hierdoor wordt wederom een symmetrische invariant van F verkregen (§ 11).

Er zij op gewezen, dat de splitsing in primitief-symmetrische invarianten berust op volledige reductie van de reguliere voorstelling. Deze is echter niet eenduidig. Wel kan men steeds reduceren tot irreducibele voorstellingen met rationale coëfficiënten.

§ 1. Laat

$$\mathfrak{S}_t = E, A, B, \ldots, G$$

de *symmetrische permutatiegroep* van t grootheden zijn [1]).

AB zij de permutatie A, gevolgd door de permutatie B. Voegt men aan de elementen van deze groep $n = t!$ basisvectoren of hypercomplexe eenheden

$$e_E,\ e_A,\ e_B, \ldots, e_G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

toe, met voorschrift $e_A\,e_B = e_{AB}$, dan wordt de vermenigvuldiging der vectoren of hypercomplexe getallen

$$\xi = \xi_E\,e_E + \xi_A\,e_A + \ldots + \xi_G\,e_G$$

gedefinieerd door

$$\left.\begin{aligned} \zeta = \xi\eta = \sum_P \xi_P\,e_P \sum_Q \eta_Q\,e_Q = \sum_{P,Q} \xi_P\,\eta_Q\,e_{PQ} = \sum_{P,R} \xi_P\,\eta_{P^{-1}R}\,e_R \cdots \\ \text{dus} \qquad \zeta_R = \sum_P \xi_P\,\eta_{P^{-1}R} = \sum_P \xi_{RP^{-1}}\,\eta_P \quad \cdots\cdots\cdots\cdots \end{aligned}\right\} . \quad (2)$$

De vectoren ξ vormen de groepenring of groepenruimte $\mathfrak{R}$.

De *reguliere voorstelling* Π_t van $\mathfrak{S}_t$ wordt verkregen, door de vectoren van $\mathfrak{R}$ rechts te vermenigvuldigen met de basisvectoren (1). Aan de

[1]) Zie P. G. MOLENAAR, Eindige Substitutiegroepen, pag. 38—40, Noordhoff, Groningen 1930.

permutatie S is dan toegevoegd de afbeelding van ξ naar $\bar{\xi}$:

$$\left.\begin{aligned} &\xi\, e_S = \bar{\xi} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ &\text{of in componenten:} \\ &\xi_P = \bar{\xi}_{PS} \quad (P = E, A, B, \ldots, G) \;.\;.\;.\;.\;.\;. \end{aligned}\right\} \quad (3)$$

Bevat $\mathfrak{S}_t$ r klassen van geconjugeerde elementen, dan zijn er juist r niet-aequivalente irreducibele voorstellingen

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_r$$

Deze zijn, tot op transformatie na, eenduidig bepaald.

Door volledige reductie van Π_t komen deze r voorstellingen alle voor de dag:

$$\Pi_t = n_1 \Gamma_1 + n_2 \Gamma_2 + \ldots + n_r \Gamma_r$$

n_σ geeft de multipliciteit aan, waartoe Γ_σ in Π_t voorkomt. Tevens is n_σ de dimensie van de voorstelling Γ_σ.

Bovendien is dus

$$n = n_1^2 + n_2^2 + \ldots + n_r^2$$

Zijn de irreducibele voorstellingen Γ_σ $(\sigma = 1, 2, \ldots, r)$ gegeven door de *unitaire* afbeeldingen van x naar $\bar{x}$.

$$x_i = e_i^{(\sigma)k}\, \bar{x}_k \quad , \quad x_i = a_i^{(\sigma)k}\, \bar{x}_k, \ldots, x_i = g_i^{(\sigma)k}\, \bar{x}_k \;^{2})$$

dan wordt de groepenruimte $\mathfrak{R}$ volledig onderspannen door de n vectoren

$$\zeta_i^{(\sigma)k} = \frac{n_\sigma}{n}\left(e_i^{(\sigma)k}\, e_E + a_i^{(\sigma)k}\, e_A + \ldots + g_i^{(\sigma)k}\, e_G\right) \;.\;.\;.\;. \quad (4)$$

$$(i, k = 1, 2, \ldots, n_\sigma \;\; ; \;\; \sigma = 1, 2, \ldots, r).$$

Verder is

$$e_S = \sum_{i,k}^{n_1} S_k^{(1)i}\, \zeta_i^{(1)k} + \sum_{i,k}^{n_2} S_k^{(2)i}\, \zeta_i^{(2)k} + \ldots + \sum_{i,k}^{n_r} S_k^{(r)i}\, \zeta_i^{(r)k} \;.\;.\;. \quad (5)$$

waarin $\| S_i^{(\sigma)k} \|$ de inverse matrix is van $\| s_i^{(\sigma)k} \|$.

$S_k^{(\sigma)i}$ is dus gelijk aan de algebraïsche minor van $s_i^{(\sigma)k}$ gedeeld door de determinant $| s_i^{(\sigma)k} |$.

Uit (5) volgt in het bijzonder nog

$$e_E = \sum_i^{n_1} \zeta_i^{(1)i} + \sum_i^{n_2} \zeta_i^{(2)i} + \ldots + \sum_i^{n_r} \zeta_i^{(r)i} \;.\;.\;.\;.\;.\;. \quad (6)$$

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2) Over gelijke boven- en benedenindices sommeren, tenzij het tegendeel wordt beweerd.

De nieuwe basisvectoren (4) voldoen aan de betrekkingen

$$\left.\begin{aligned} &\zeta^{l}_{i(\sigma)}\,\zeta^{k}_{l(\sigma)} = \zeta^{k}_{i(\sigma)} \quad \text{(niet sommeren over } l\text{)} \\ &\zeta^{l}_{i(\sigma)}\,\zeta^{k}_{s(\sigma)} = 0 \qquad (l \neq s) \\ &\zeta^{l}_{i(\sigma)}\,\zeta^{k}_{s(\varrho)} = 0 \qquad (\sigma \neq \varrho) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

De betrekkingen (5), (6) en (7) worden afgeleid met behulp van de orthogonaliteitsrelaties voor unitaire irreducibele voorstellingen. Gaat men echter door middel van de transformatie

$$P^{\lambda}_{i(\sigma)}\,s^{\mu}_{\lambda(\sigma)}\,p^{k}_{\mu(\sigma)} = \tilde{s}^{k}_{i(\sigma)}$$

over op niet-unitaire voorstellingen, dan is

$$P^{\lambda}_{i(\sigma)}\,\zeta^{\mu}_{\lambda(\sigma)}\,p^{k}_{\lambda(\sigma)} = \tilde{\zeta}^{k}_{i(\sigma)}$$

en dan controleert men gemakkelijk, dat de betrekkingen (5), (6) en (7) ook gelden voor niet-unitaire irreducibele voorstellingen.

De vectoren der ruimte $\mathfrak{R}_{i(\sigma)}$, onderspannen door

$$\zeta^{1}_{i(\sigma)},\ \zeta^{2}_{i(\sigma)},\ \ldots,\ \zeta^{n_\sigma}_{i(\sigma)}$$

vormen een rechtsideaal in $\mathfrak{R}$.

Binnen $\mathfrak{R}_{i(\sigma)}$ levert de reguliere voorstelling juist de irreducibele voorstelling Γ_σ. Is

$$x = \sum_{i,k}^{n_\sigma} x^{i}_{k(\sigma)}\,\zeta^{k}_{i(\sigma)}$$

dan luidt $x\,e_S = \bar{x}$ in componenten

$$\bar{x}^{i}_{k(\sigma)} = \sum_{l}^{n_\sigma} S^{l}_{k(\sigma)}\,x^{i}_{l(\sigma)} \quad \text{of} \quad x^{i}_{k(\sigma)} = \sum_{l}^{n_\sigma} s^{l}_{k(\sigma)}\,\bar{x}^{i}_{l(\sigma)} \quad (i \text{ vast};\ k = 1, 2, \ldots n_\sigma). \quad . \quad (8)$$

De voorstellingen Γ_σ binnen de ruimten

$$\mathfrak{R}_{1(\sigma)},\ \mathfrak{R}_{2(\sigma)},\ \ldots,\ \mathfrak{R}_{n_\sigma(\sigma)}$$

zijn dus congruent.

§ 2. Onder een *karakteristieke* of *idempotente* eenheid verstaat men een grootheid ε (hypercomplex getal of vector in $\mathfrak{R}$) die voldoet aan de betrekking

$$\varepsilon\,\varepsilon = \varepsilon.$$

Een karakteristieke eenheid bepaalt door middel van

$$\varepsilon\,\xi\,e_S = \bar{\xi} \quad (S = E, A, \ldots, G) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

binnen de ruimte $\varepsilon\,\xi$ (rechtsideaal) een voorstelling van $\mathfrak{S}_t$. Is deze voorstelling irreducibel, dan heet ε een *primitieve* karakteristieke eenheid.

Volgens § 1 (7) is $\zeta^{i}_{i(\sigma)}$ een karakteristieke eenheid. Het rechtsideaal $\zeta^{i}_{i(\sigma)}\,\xi$ valt samen met de ruimte $\mathfrak{R}_{i(\sigma)}$. De voorstelling (1), die door de reguliere voorstelling binnen deze ruimte wordt geleverd, is echter irreducibel (Γ_σ). $\zeta^{i}_{i(\sigma)}$ is derhalve een primitieve karakteristieke eenheid.

Voldoet ξ aan de betrekkingen

$$\alpha\,\xi = 0 \quad \beta\,\xi = 0 \ldots \varkappa\,\xi = 0 \quad (\alpha, \beta, \ldots, \varkappa \text{ constant})$$

dan zegt men, dat ξ behoort tot de *symmetrieklasse* $C(\alpha, \beta, \ldots, \varkappa)$ [3]). De vectoren ξ vormen dan een rechtsideaal in $\mathfrak{R}$ en zijn voor te stellen door

$$\xi = \varepsilon\,\xi'$$

waarin ε een karakteristieke eenheid en ξ' een veranderlijke vector in $\mathfrak{R}$ is. Is ε een primitieve karakteristieke eenheid, dan heet $C(\alpha, \beta, \ldots, \varkappa)$ een *primitieve* symmetrieklasse.

Deze stelling is om te keren. Behoort ξ tot een rechtsideaal

$$\xi = \gamma\,\xi'$$

(γ constant, ξ' variabel binnen $\mathfrak{R}$), dan behoort ξ tot een symmetrieklasse $C(\alpha, \beta, \ldots, \varkappa)$ en vult deze geheel. Bovendien is ξ dan voor te stellen door

$$\xi = \varepsilon_\gamma\,\xi'$$

waarin ε_γ een karakteristieke eenheid is.
Beschouw als voorbeeld de vectoren van het rechtsideaal

$$\xi = \zeta^{k}_{i(\sigma)}\,\xi' \quad (\sigma, i, k \text{ vast}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

De symmetrieklasse $C(\alpha, \beta, \ldots, \varkappa)$ wordt gevonden door de lineair-onafhankelijke oplossingen van de vergelijking

$$\alpha\,\zeta^{k}_{i(\sigma)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

te geven. Stelt men

$$\alpha = \sum_{\varrho=1}^{r} \sum_{p,q}^{n_\varrho} a^{p}_{q(\varrho)}\,\zeta^{q}_{p(\varrho)}$$

dan geeft (3)

$$\sum_{\varrho=1}^{r} \sum_{p,q}^{n_\varrho} a^{p}_{q(\varrho)}\,\zeta^{q}_{p(\varrho)}\,\zeta^{k}_{i(\sigma)} = 0$$

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[3]) Zie H. WEYL, Rendiconti di Palermo 48 (1924), pag. 29—36, of P. G. MOLENAAR, Eindige Substitutiegroepen, pag. 50—54.

dus volgens § 1 (7)

$$\sum_{p}^{n_\sigma} a_{l}^{p(\sigma)}\, \zeta_{p}^{k(\sigma)} = 0.$$

Hieruit volgt

$a_{l}^{s(\varrho)}$ = variabel voor $\varrho \neq \sigma$; l en s willekeurig.

$a_{l}^{s(\sigma)}$ = variabel voor $l \neq i$; s willekeurig.

$a_{l}^{s(\sigma)} = 0$; s willekeurig.

Voor $(\alpha, \beta, \ldots, \varkappa)$ kan men dus geven de basisvectoren

$$\zeta_{s}^{l(\varrho)}\ (\varrho \neq \sigma) \text{ en } \zeta_{s}^{1(\sigma)} \ldots \zeta_{s}^{i-1(\sigma)}\ \zeta_{s}^{i+1(\sigma)} \ldots \zeta_{s}^{n\sigma(\sigma)}.$$

Daar deze uitkomst onafhankelijk is van k, behoort het rechtsideaal

$$\xi = \zeta_{i}^{i(\sigma)}\, \xi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

tot dezelfde symmetrieklasse als het rechtsideaal (2).

$\zeta_{i}^{i(\sigma)}$ is echter een primitieve karakteristieke eenheid. Het rechtsideaal (2) of (4) behoort dus tot een primitieve symmetrieklasse en valt samen met $\mathfrak{R}_{i}^{(\sigma)}$.

§ 3. Laat $I(a, b, \ldots, l)$ een *simultaaninvariant* zijn van t gelijksoortige grondvormen

$$f_1 = a_x^p,\ f_2 = b_x^p, \ldots, f_t = l_x^p \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

die lineair en homogeen is in de coëfficiëntenrijen van elk dezer vormen. Hierin is a_x^p de symbolische notatie voor $a^{i_1 i_2 \ldots i_p}\, x_{i_1}\, x_{i_2} \ldots x_{i_p}$. Onderwerpt men de coëfficiëntenrijen $a^{ikl\ldots}, b^{ikl\ldots}, \ldots, l^{ikl\ldots}$ in I aan de permutatie P, dan ontstaat een vorm $I_P(a, b, \ldots, l)$, die wederom een invariant van (1) is, en die een *isomeer* van $I(a, b, \ldots, l) = I_E$ genoemd wordt. Aldus ontstaan $n = t!$ invarianten

$$I_E,\ I_A, \ldots, I_G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

die kunnen worden opgevat als componenten van een vector

$$I = I_E\, e_E + I_A\, e_A + \ldots + I_G\, e_G$$

in $\mathfrak{R}$. Noem I de representerende vector van $I(a, b, \ldots, l)$.

Tussen deze isomeren kunnen lineaire betrekkingen bestaan, b.v.

$$\alpha_E\, I_{E^{-1}} + \alpha_A\, I_{A^{-1}} + \ldots + \alpha_G\, I_{G^{-1}} \equiv 0 \qquad \{a, b, \ldots, l\}$$
$$\beta_E\, I_{E^{-1}} + \beta_A\, I_{A^{-1}} + \ldots + \beta_G\, I_{G^{-1}} \equiv 0 \qquad \{a, b, \ldots, l\}$$

enz.

Door $a, b, \ldots, l$ aan de permutatie R te onderwerpen, volgt hieruit

$$\alpha_E I_{E^{-1}R} + \alpha_A I_{A^{-1}R} + \ldots + \alpha_G I_{G^{-1}R} \equiv 0 \{a, b, \ldots, l\}$$

enz.

Volgens § 1(2) is dus

$$\alpha I = 0 \quad \beta I = 0 \ldots \varkappa I = 0 \ldots \ldots \ldots \quad (3)$$

De representerende vector behoort dan tot een symmetrieklasse $C(\alpha, \beta, \ldots, \varkappa)$. Kortheidshalve zeggen we, dat de invariant $I(a, b, \ldots, l)$ behoort tot deze symmetrieklasse.

Een belangrijke opgave uit de theorie der projectieve invarianten is het opstellen van een *kleinst volledig systeem* of *moduulbasis*. De op te nemen isomeren moeten lineair onafhankelijk zijn. Bovendien kan men eisen, dat deze invarianten behoren tot primitieve symmetrieklassen.

Dergelijke *primitief-symmetrische invarianten* verkrijgt men, door I te ontbinden langs de n basisvectoren $\zeta^{k}_{i(\sigma)}$.

Volgens § 1(6) is

$$I = e_E I = \sum_{i,\sigma} \zeta^{i}_{i(\sigma)} I = \sum_{i,\sigma} I^{i}_{i(\sigma)} \ldots \ldots \ldots \quad (4)$$

en volgens § 1(5)(7) is

$$I^{i}_{i(\sigma)} = \zeta^{i}_{i(\sigma)} I = \zeta^{i}_{i(\sigma)} \sum_{S} I_S e_S = \zeta^{i}_{i(\sigma)} \sum_{S} I_S \sum_{p,q,\varrho} S^{p}_{q(\varrho)} \zeta^{q}_{p(\varrho)} =$$

$$= \sum_{q} \sum_{S} S^{i}_{q(\sigma)} I_S \zeta^{q}_{i(\sigma)} = \sum_{q} \sum_{S} s^{i}_{q(\sigma)} I_{S^{-1}} \zeta^{q}_{i(\sigma)} \ldots \ldots \quad (5)$$

De vector $I^{i}_{i(\sigma)}$ ligt binnen de ruimte $\mathfrak{R}_{i(\sigma)}$ en heeft langs $\zeta^{q}_{i(\sigma)}$ de coördinaat $\sum_{S} s^{i}_{q(\sigma)} I_{S^{-1}}$.

Stelt men nu

$$L^{i}_{q(\sigma)} = \frac{n_\sigma}{n} \sum_{S} s^{i}_{q(\sigma)} I_{S^{-1}} \ldots \ldots \ldots \quad (6)$$

dan is

$$I = \sum_{i,q,\sigma} \frac{n}{n_\sigma} L^{i}_{q(\sigma)} \zeta^{q}_{i(\sigma)} \ldots \ldots \ldots \quad (7)$$

Uit (6) en § 1(4) volgt, dat

$$L^{i}_{q\,E(\sigma)}, \; L^{i}_{q\,A(\sigma)}, \ldots, L^{i}_{q\,G(\sigma)}$$

de componenten zijn van de vector $\zeta^{i}_{q(\sigma)} I$.

$L^{i}_{q(\sigma)}$ is dus een primitief-symmetrische invariant.

Voorts volgt uit (6), dat de $n_1^2 + n_2^2 + \ldots + n_r^2 = n$ invarianten $L^{i}_{q(\sigma)}$ zijn uit te drukken in de n isomeren $I_E, I_A, \ldots, I_G$.

De S-component van (7) geeft

$$I_S = \sum_{i,q,\sigma} s^{q}_{i(\sigma)} L^{i}_{q(\sigma)} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hieruit volgt, dat de isomeren $I_E, I_A, \ldots, I_G$ zijn uit te drukken in de invarianten $L^{i}_{q(\sigma)}$.

De isomeren kunnen als basisinvarianten dus vervangen worden door deze primitief-symmetrische invarianten.

Uit (8) volgt nog een splitsing voor de invariant

$$I(a, b, \ldots, l) = I_E = \sum_{i,\sigma} L^{i}_{i(\sigma)} = \sum_{i,\sigma} \frac{n_\sigma}{n} \sum_{S} s^{i}_{i(\sigma)} I_{S^{-1}} \quad . \quad . \quad . \quad (9)$$

Verder kan men de *lineair onafhankelijke* invarianten $L^{i}_{q(\sigma)}$ gemakkelijk opsporen. Bestaat er namelijk tussen de n invarianten $L^{i}_{q(\sigma)}$ een lineaire betrekking

$$\sum_{i,q,s} p^{q}_{i(\sigma)} L^{i}_{q(\sigma)} = 0 \qquad \{a, b, \ldots, l\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(waarin $p^{q}_{i(\sigma)}$ gewone complexe getallen zijn), dan is ook

$$\sum_{i,q,\sigma} p^{q}_{i(\sigma)} L^{i}_{q(\sigma)S} = 0 \qquad (S = E, A, \ldots, G)$$

dus

$$\sum_{i,q,\sigma} p^{q}_{i(\sigma)} \zeta^{i}_{q(\sigma)} I = 0$$

Door links-vermenigvuldiging met $\zeta^{m}_{l(\varrho)}$ (ϱ, l, m vast) volgt hieruit

$$\sum_{i,q,\sigma} p^{q}_{i(\sigma)} \zeta^{m}_{l(\varrho)} \zeta^{i}_{q(\sigma)} I = 0 \quad \text{of} \quad \sum_{l} p^{m}_{l(\varrho)} \zeta^{i}_{l(\varrho)} I = 0.$$

dus ook

$$\sum_{l} p^{m}_{l(\varrho)} L^{i}_{l(\varrho)} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

De lineaire betrekkingen (10), waarvan er meerdere kunnen bestaan, vallen dus uiteen in betrekkingen tussen primitief-symmetrische invarianten

met dezelfde klasse-index (ϱ) en dezelfde beneden-index l. Rangschikt men de invarianten met dezelfde klasse-index (ϱ) aldus:

$$\begin{matrix} L^{1}_{(\varrho)1} & L^{2}_{(\varrho)1} & \ldots & L^{n_\varrho}_{(\varrho)1} \\ L^{1}_{(\varrho)2} & L^{2}_{(\varrho)2} & \ldots & L^{n_\varrho}_{(\varrho)2} \\ \ldots & \ldots & \ldots & \ldots \\ L^{1}_{(\varrho)n_\varrho} & L^{2}_{(\varrho)n_\varrho} & \ldots & L^{n_\varrho}_{(\varrho)n_\varrho} \end{matrix}$$

dan kan men enige kolommen lineair homogeen uitdrukken in de overige kolommen. Invarianten ener zelfde kolom zijn lineair onafhankelijk.

§ 4. De vier binaire cubische vormen

$$f_1 = a_x^3 \quad f_2 = b_x^3 \quad f_3 = c_x^3 \quad f_4 = d_x^3$$

bezitten een simultaaninvariant [4])

$$R\,(abcd) = (ab)^2\,(cd)^2\,(ad)\,(bc) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Deze is noch symmetrisch, noch anti-symmetrisch in de coëfficiëntenrijen a^{ikl}, b^{ikl}, c^{ikl}, d^{ikl}. Het zal blijken, dat de 24 isomeren van deze invariant zijn uit te drukken door 4 primitief-symmetrische invarianten, waarvan de eerste symmetrisch en de tweede anti-symmetrisch is, terwijl de derde en vierde behoren tot hogere symmetrieklassen.

Uit (1) volgt onmiddellijk

$$R\,(abcd) = R\,(badc) = R\,(cdab) = R\,(dcba) \quad . \quad . \quad . \quad (2)$$

De vier isomeren, die uit $R\,(pqrs)$ ontstaan door de permutaties van de *vier*-groep van KLEIN $\mathfrak{V}_4$

$$(p)\,(q)\,(r)\,(s) \qquad (pq)\,(rs) \qquad (pr)\,(qs) \qquad (ps)\,(qr)$$

zijn dus aan elkaar gelijk.

Er zijn dus hoogstens zes onderling verschillende isomeren. Deze vindt men als volgt: onderwerp abc aan de zes permutaties der symmetrische permutatiegroep $\mathfrak{S}_3$

$$\left.\begin{matrix} E = \begin{pmatrix}123\\123\end{pmatrix} & A = \begin{pmatrix}123\\231\end{pmatrix} & B = \begin{pmatrix}123\\312\end{pmatrix} \\ C = \begin{pmatrix}123\\321\end{pmatrix} & D = \begin{pmatrix}123\\213\end{pmatrix} & F = \begin{pmatrix}123\\132\end{pmatrix} \end{matrix}\right\} \quad . \quad . \quad . \quad (3)$$

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[4]) Zie GORDAN–KERSCHENSTEINER, Vorlesungen über Invariantentheorie II, Leipzig (1887), pag. 167.

dan ontstaan uit $R\,(abcd)$ de isomeren

$$\left.\begin{aligned} R_E &= R\,(abcd) = (ab)^2\,(cd)^2\,(ad)\,(bc) \\ R_A &= R\,(bcad) = (bc)^2\,(ad)^2\,(bd)\,(ca) \\ R_B &= R\,(cabd) = (ca)^2\,(bd)^2\,(cd)\,(ab) \\ R_C &= R\,(cbad) = (cb)^2\,(ad)^2\,(cd)\,(ba) \\ R_D &= R\,(bacd) = (ba)^2\,(cd)^2\,(bd)\,(ac) \\ R_F &= R\,(acbd) = (ac)^2\,(bd)^2\,(ad)\,(cb) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Uit elk dezer isomeren verkrijgt men door de permutaties van $\mathfrak{V}_4$ vier gelijke isomeren. Deze verdeling van de 24 isomeren van R in zes groepen van vier correspondeert met de splitsing van $\mathfrak{S}_4$ in zes nevenklassen (restklassen) door middel van de normaaldeler $\mathfrak{V}_4$. De factorgroep $\mathfrak{S}_4/\mathfrak{V}_4$ is isomorph met $\mathfrak{S}_3$.

Tussen de zes isomeren (4) bestaan lineaire betrekkingen. Deze kan met alle vinden door *identiek omvormen* van haakfactoren [5]). Zo is

$$\begin{aligned} &(ab)^2\,(cd)^2\,(ad)\,(bc) = \\ &= (ab)\,(cd)\,(cb)\,(ad)\,(ad)\,(bc) - (ab)\,(cd)\,(ca)\,(bd)\,(ad)\,(bc) = \\ &= (cb)^2\,(ad)^2\,(cd)\,(ba) + (ab)\,(ac)\,(ad)\,(bc)\,(bd)\,(cd) \end{aligned}$$

dus

$$R_E = R_C + P_E \; . \; . \; .

Op analoge wijze vindt men nog

$$R_E - R_B - R_C + R_D = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

De 24 isomeren van R kunnen dus lineair worden uitgedrukt door 4 isomeren. Deze behoren echter niet tot primitieve symmetrieklassen.

Primitief-symmetrische basisinvarianten vindt men aldus: de reguliere voorstelling Π_3 van $\mathfrak{S}_3$ is volledig te reduceren tot drie irreducibele voorstellingen

$$\Pi_3 = \Gamma_1 + \Gamma_2 + 2\,\Gamma_3.$$

Hierin zijn Γ_1 (de identieke) en Γ_2 voorstellingen van de eerste graad met „matrices"

	E	A	B	C	D	F
Γ_1:	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|+1\|\|$
Γ_2:	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|+1\|\|$	$\|\|-1\|\|$	$\|\|-1\|\|$	$\|\|-1\|\|$

Γ_3 is een voorstelling van de tweede graad, die echter bepaald is tot op transformatie na. Zo bestaat er een voorstelling door unitaire matrices, wier coëfficiënten primitieve derde eenheidswortels zijn [6]), doch ook kan men niet-unitaire voorstellingen construeren, wier matrices bestaan uit rationale coëfficiënten [7]).

Onderwerpt men b.v. de coördinaten x_1, x_2 en x_3 aan de permutaties $E, A, \ldots, F$

$$E\begin{cases} x_1 = \bar{x}_1 \\ x_2 = \bar{x}_2 \\ x_3 = \bar{x}_3 \end{cases} \quad A\begin{cases} x_1 = \bar{x}_2 \\ x_2 = \bar{x}_3 \\ x_3 = \bar{x}_1 \end{cases} \quad . \quad . \quad . \quad . \quad F\begin{cases} x_1 = \bar{x}_1 \\ x_2 = \bar{x}_3 \\ x_3 = \bar{x}_2 \end{cases}$$

en stelt men de voor deze permutaties invariante som der coördinaten gelijk nul

$$x_1 + x_2 + x_3 = 0$$

dan kan men b.v. x_2 elimineren,

$$E\begin{cases} x_1 = \bar{x}_1 \\ x_3 = \bar{x}_3 \end{cases} \quad A\begin{cases} x_1 = -\bar{x}_1 - \bar{x}_3 \\ x_3 = \bar{x}_1 \end{cases} \quad . \quad . \quad . \quad . \quad F\begin{cases} x_1 = \bar{x}_1 \\ x_3 = -\bar{x}_1 - \bar{x}_3 \end{cases}$$

waardoor men een irreducibele voorstelling Γ_3 verkrijgt met matrices

$$\begin{array}{cccccc} E & A & B & C & D & F \\ \left\|\begin{matrix} +1 & 0 \\ 0 & +1 \end{matrix}\right\| & \left\|\begin{matrix} -1 & -1 \\ +1 & 0 \end{matrix}\right\| & \left\|\begin{matrix} 0 & +1 \\ -1 & -1 \end{matrix}\right\| & \left\|\begin{matrix} 0 & +1 \\ +1 & 0 \end{matrix}\right\| & \left\|\begin{matrix} -1 & -1 \\ 0 & +1 \end{matrix}\right\| & \left\|\begin{matrix} +1 & 0 \\ -1 & -1 \end{matrix}\right\| \end{array}. \quad (9)$$

[6]) Zie P. G. MOLENAAR, Eindige Substitutiegroepen, pag. 67.

[7]) Zie B. L. v. D. WAERDEN, Moderne Algebra II, pag. 203—207, Springer, Berlin.

Uit deze voorstellingen Γ_1, Γ_2 en Γ_3 vindt men volgens § 1(4) de hypercomplexe getallen

$$\left.\begin{aligned}
\zeta^{1}_{1(1)} &= \tfrac{1}{6}(e_E + e_A + e_B + e_C + e_D + e_F)\\
\zeta^{1}_{1(2)} &= \tfrac{1}{6}(e_E + e_A + e_B - e_C - e_D - e_F)\\
\zeta^{1}_{1(3)} &= \tfrac{1}{3}(e_E - e_A \qquad\quad - e_D + e_F)\\
\zeta^{2}_{1(3)} &= \tfrac{1}{3}(\quad - e_A + e_B + e_C - e_D \quad)\\
\zeta^{1}_{2(3)} &= \tfrac{1}{3}(\quad + e_A - e_B + e_C \qquad - e_F)\\
\zeta^{2}_{2(3)} &= \tfrac{1}{3}(e_E \qquad - e_B \qquad + e_D - e_F)
\end{aligned}\right\} \quad . \; . \; . \; . \quad (10)$$

en volgens § 3(6) de primitief-symmetrische invarianten

$$\left.\begin{aligned}
L^{1}_{1(1)} &= \tfrac{1}{6}(R_E + R_A + R_B + R_C + R_D + R_F)\\
L^{1}_{1(2)} &= \tfrac{1}{6}(R_E + R_A + R_B - R_C - R_D - R_F)\\
L^{1}_{1(3)} &= \tfrac{1}{3}(R_E \qquad - R_B \qquad - R_D + R_F)\\
L^{2}_{1(3)} &= \tfrac{1}{3}(\quad + R_A - R_B + R_C - R_D \quad)\\
L^{1}_{2(3)} &= \tfrac{1}{3}(\quad - R_A + R_B + R_C \qquad - R_F)\\
L^{2}_{2(3)} &= \tfrac{1}{3}(R_E - R_A \qquad\quad + R_D - R_F)
\end{aligned}\right\} \quad . \; . \; . \quad (10a)$$

Volgens § 3 verwachten we lineaire betrekkingen uitsluitend tussen invarianten met dezelfde klasse-index en dezelfde beneden-index. Inderdaad is volgens (10a), (7) en (8)

$$L^{1}_{1(3)} = L^{2}_{1(3)} \quad \text{en} \quad L^{1}_{2(3)} = L^{2}_{2(3)}.$$

Stelt men eenvoudigheidshalve

$$L^{1}_{1(1)} = L_{(1)} \quad L^{1}_{1(2)} = L_{(2)} \quad L^{1}_{1(3)} = L^{2}_{1(3)} = L_{1(3)} \quad \text{en} \quad L^{1}_{2(3)} = L^{2}_{2(3)} = L_{2(3)}$$

dan komen dus $L_{(1)}$, $L_{(2)}$, $L_{1(3)}$ en $L_{2(3)}$ in aanmerking voor opname in een kleinst volledig systeem. $L_{(1)}$ is symmetrisch, $L_{(2)}$ is anti-symmetrisch in de coëfficiëntenrijen.

$L_{1(3)}$ en $L_{2(3)}$ behoren resp. tot de primitieve symmetrieklassen

$$C_{1(3)} = C\,(\zeta^{1}_{1(1)}, \zeta^{1}_{1(2)}, \zeta^{2}_{1(3)}, \zeta^{2}_{2(3)}) \quad \text{en} \quad C_{2(3)} = C\,(\zeta^{1}_{1(1)}, \zeta^{1}_{1(2)}, \zeta^{1}_{1(3)}, \zeta^{1}_{2(3)}).$$

Volgens § 3(8) is tenslotte

$$\left.\begin{aligned} R_E &= L^{(1)} + L^{(2)} + L_1^{(3)} + L_2^{(3)} \\ R_A &= L^{(1)} + L^{(2)} \qquad - L_2^{(3)} \\ R_B &= L^{(1)} + L^{(2)} - L_1^{(3)} \\ R_C &= L^{(1)} - L^{(2)} + L_1^{(3)} + L_2^{(3)} \\ R_D &= L^{(1)} - L^{(2)} - L_1^{(3)} \\ R_F &= L^{(1)} - L^{(2)} \qquad - L_2^{(3)} \end{aligned}\right\} \quad \ldots\ldots \quad (11)$$

Biochemistry. — *Complex relations in the 2nd. phase of blood coagulation.* By C. VAN DER MEER. (Communicated by Prof. H. G. BUNGENBERG DE JONG.)

(Communicated at the meeting of November 24, 1945.)

1. *Introduction.*

For a detailed description of the theories of blood coagulation and a survey of the literature we refer to WÖHLISCH, Erg. Phys. **43**, 174, 1940 and HOWELL, Phys. Reviews **15**, 43, 1935.

The formation of thrombin from prothrombin is called the first phase of blood coagulation, while the second phase is that in which fibrin is formed from fibrinogen under the influence of thrombin. Then follows a third phase, in which there is retraction of the clot, the liquid being pressed out.

Little is known as yet as to the chemical composition of thrombin. Most investigators are of the opinion that it is a proteid or polypeptid.

Since A. SCHMIDT expressed his opinion that thrombin is a ferment, this theory has always had many adherents. Many investigators (a.o. WALDSCHMIDT LEITZ) have pronounced the opinion that thrombin is a proteolytic ferment, which causes the formation of fibrin through a slight decomposition of fibrinogen. WÖHLISCH has advanced the theory, based on a great many investigations, that thrombin is a denaturase of fibrinogen, i.e. a ferment that accelerates the normal denaturation of fibrinogen.

In addition to this enzyme theory a number of theories has been developed, according to which thrombin should bind with fibrinogen, flocculating with it. For this and other theories of the second phase we refer to the literature. Here we will only mention the theories of DEL BAERE [1]) and of GLAZKO and GREENBERG [2]), as there is a direct connection between them and the investigation described in this paper.

In our investigations we started from the viewpoint of the theory of complex relations as developed by BUNGENBERG DE JONG and his collaborators, of which we shall first give a short survey.

2. *Complex relations according to* BUNGENBERG DE JONG.

For a detailed description and discussion of the theory of complex relations we refer to the following surveys: BUNGENBERG DE JONG Kolloid Z. **79**, 223, 334, 1937; ibid. **80**, 221, 350, 1937; Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 776, 1938.

Here follows a short survey only of the most important phenomena, which were first of all observed in the investigations of so-called complex coacervates. Since the complex coacervate gelatin + gum arabic is at the

1) Bioch. Z. **296**, 38, (1932).
2) Am. J. Phys. **128**, 399, (1939).

moment the best known, we take this system as an example in this discussion.

For the formation of such a complex coacervate certain conditions must be fulfilled, while a coacervate, once it has been formed, possesses a number of properties the most significant of which we shall now mention.

a. pH.

Coacervation only occurs in a pH tract in which the gelatin is positively, the gum arabic negatively charged.

b. Mixing proportion.

When two isohydric sols of gelatin and gum arabic are mixed in varying proportions, we find on measuring the coacervation volumes or turbidities a curve with an optimum.

So there is a certain mixing proportion in which coacervation is strongest.

The electrophoretic charge of the coacervate drops is zero at or near the optimum; when there is a superabundance of gelatin it is positive, when there is an excess of gum arabic it is negative. The coacervates with a positive charge contain relatively more gelatin than the uncharged coacervates, whereas the negative ones have a proportionally higher gum arabic percentage. The water percentage of the coacervate is minimal in the optimum, increasing on both sides of it.

c. Suppression through neutral salts.

The coacervation can be suppressed through the addition of neutral salts. These salts arrange themselves according to the double valence rule, i.e. a valence rule for cations and one for anions. When the monovalent anion remains the same a salt with a bivalent cation will have a greater effect in an equivalent concentration than a salt with a monovalent cation, the effect of a trivalent cation will be greater than that of a bivalent one, etc.

The same thing goes for anions, when the cation remains the same. This may be symbolically indicated by $3 - 1 > 2 - 1 > 1 - 1$ and $1 - 3 > 1 - 2 > 1 - 1$.

d. Maximal salt resistance in the optimum.

The salt concentration necessary to obtain total suppression is highest in the optimum of an isohydric mixing series, and decreases as we move further away from the optimum; in other words, the salt resistance of the coacervate is greatest in the optimum [1].

e. Continuous valence rule on either side of the optimum [1] [2].

When we consider the influence of neutral salts on coacervate volume or turbidity on either side of the optimum, we find that, in low salt concentrations, the curves may arrange themselves in a continuous bundle. With negative coacervates the salts $n - 1$ will, compared with $1 - 1$, promote coacervation more strongly or suppress less strongly as n increases, and the salts $1 - n$ will suppress more strongly or promote less strongly.

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[1] BUNGENBERG DE JONG and DEKKER, Koll. Beihefte **43**, 143, (1935).

[2] BUNGENBERG DE JONG and VAN DER MEER, Proc. Ned. Akad. v. Wetensch., Amsterdam **45**, 490 (1942).

So we find: relatively stronger promotion 3—1 / 2—1 / 1—1 / 1—2 / 1—3 relatively stronger suppression. We will indicate this as the continuous valence rule of the cations.

With positieve coacervates the bundle is reversed: relatively stronger promotion 1—3 / 1—2 / 1—1 / 2—1 / 3—1 relatively stronger suppression, the continuous valence rule of the anions.

In higher salt concentrations we again find the double valence rule.

On measuring the electrophoretic charge we find with all coacervates, positive, uncharged or negative, the continuous valence rule: relative positivation 3—1 / 2—1 / 1—1 / 1—2 / 1—3 relative negativation.

f. Shifting of the optimum through neutral salts.

Neutral salts of type 1—1 will generally not affect the position of the optimum in a mixing series. Salts with a high-valence cation and monovalent anion shift the optimum to an excess of the negative component, salts with a high-valence anion and monovalent cation shift the optimum to higher concentrations of the positive component, and the more so as the valence of the high-valence ion is greater [1]).

All these phenomena, however, are by no means restricted to coacervates only. Various flocculations, gels, sols, fibrils, films etc., show them under certain circumstances. These systems are then called complex systems and the general phenomenon is referred to as the occurrence of complex relations.

3. *The influence of neutral salts on the second phase of blood coagulation.*

When investigating the effect of neutral salts on the isolated second phase of coagulation, the authors originally confined themselves to the salts NaCl and $CaCl_2$ (lit. see WÖHLISCH Erg. Phys. '29 and Erg. Phys. '40 and also WEITNAUER, GRÜNING and WÖHLISCH Bioch. Z **307**, 325, '41), sometimes also other earth alkali metals (WEITNAUER see WÖHLISCH [2])). GLAZKO and GREENBERG [3]) investigated the influence of salts with polyvalent anion and monovalent cation on the second phase and found an increase of the impeding effect with the valence. They also found that high-valence cations ($Th(NO_3)_4$) have a strongly impeding effect, whereas monovalent cations hardly have such an effect at all.

In view of the great importance of the arrangement of ions according to their valence in recognizing complex relations according to BUNGENBERG DE JONG, we compared the effect of neutral salts with 1-, 2- resp. 3-valent cation and 1-valent anion on the one hand, and neutral salts with 1-, 2-, resp. 3-valent anion and 1-valent cation on the other hand.

The fibrinogen used was prepared as follows:

To 200 cc 0.3 % oxalated cows plasma was added 40 cc of a $Mg(OH)_2$

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[1]) BUNGENBERG DE JONG and HOSKAM. Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 59 (1942).

[2]) Erg. Phys. **43**, 174 (1940).

[3]) l.c.

suspension (SMITH, WARNER and BRINKHOUS [1])). After half an hour's stirring the liquid was centrifugalized during 20 minutes. With 0.5 N HCl (10—15 cc) the pH was brought to ca. 7,4. After this saturated NaCl was added to a concentration of 72.5 % saturated. Centrifigalizing during 20 minutes after 5 minutes' standing. The precipitation was loosened with aq. dest. and as a whole brought over in 100 cc aq. dest. After solution centrifugalizing for another half hour. The advantage of this method is that only NaCl is used for salting out, so that with the tests on the effects of salts there are no additional high-valence ions in the system.

This fibrinogen has a concentration of ca. 0.4 %, ca. 10 % of which is dispurification and it contains averagely 1,4 % NaCl. It was used exclusively on the day after the preparation.

Preparation of thrombin.

1 L 0.3 % oxalated cows plasma was added to 10 L aq. dest. plus 385 cc 1 % acetic acid. After a few drops of toluol were added it was left for a week at room temperature. The precipitate was dried with acetone. Before use 10 gr. powder was extrahated during 2 hours with 150 cc 0.9 % NaCl, at room temperature, centrifugalized and filtered. After one night again centrifugalized.

The experiments were made at 37° C. In determining the coagulation time all the reagents necessary, except fibrinogen, are placed in a test tube in a thermostat of 37° C during 1 minute. Then the fibrinogen, which is also in the thermostat, is added while a stopwatch is pressed. After thorough mixing the tube is sunk into the waterbath, after which every 5 seconds the tube is placed in a horizontal position, to see if coagulation has set in. The criterion is the occurrence of an evident clot or of fibrin threads. In the graphs 1000/t in seconds is set out as coagulation velocity.

The salts used were NaCl (1—1), $CaCl_2$ (2—1), Co $(NH_3)_6$ Cl_3 Luteocobalt chlorid (3—1), $Na_2S_2O_3$ (1—2) and K_3 Fe $(CN)_6$ (1—3).

None of these salts gives flocculation with thrombin or fibrinogen only.

Arrangement:

0,25 cc fibrinogen, containing 1,4 % NaCl.

1 cc thrombin, containing 0.9 % NaCl.

0.5 cc aq. dest., resp. salt solution of the concentration required.

So this system contains a basis concentration of 123 m. aequ. NaCl. See fig. 1.

When we compare the action of the different salts, it appears that in order to cause a certain degree of impediment a lower concentration in aequivalents of $CaCl_2$ (2—1) is necessary than of NaCl (1—1). Luteo-Cl_3 (3—1) acts in even lower concentration than $CaCl_2$. According to the strength of their effect the salts are therefore arranged according to 3—1 > 2—1 > 1—1, the valence rule of cations. At the same time we see, however, that $K_3Fe(CN)_6$ (1—3) has a stronger effect than $Na_2S_2O_3$ (1—2), while the effect of the latter is greater than of NaCl (1—1), so

[1]) Am. J. Phys. **107**, 63 (1931).

that at the same time the valence rule for anions (1—3 > 1—2 > 1—1) appears.

So we find the double valence rule.

We also saw, under 2, that the occurrence of the double valence rule is a positive indication of the presence of complex relations in the sence of BUNGENBERG DE JONG.

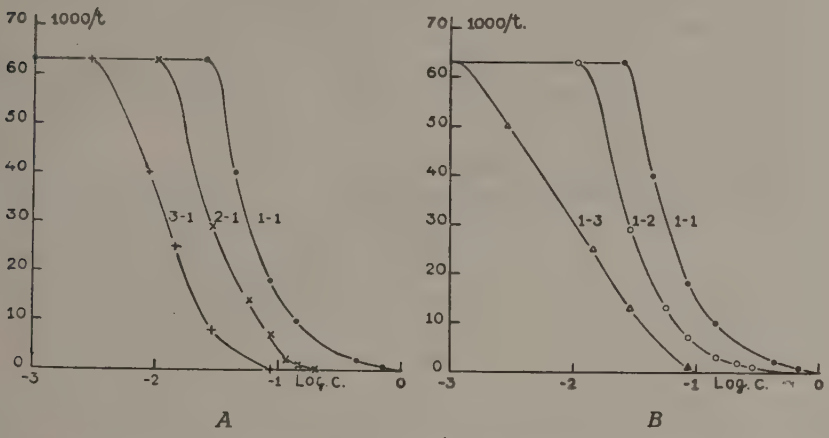

Fig. 1. Impeding effect of neutral salts on the second phase. *A.* valence rule of cations *B.* valence rule of anions.

absciss: log. conc. salt in aequ. p. Litre.
ordinate: coagulation velocity 1000/t in sec.

4. *Optimal mixing proportion.*

We have come to recognize the existence of an optimal mixing proportion as one of the principal criteria for the existence of complex relations, while an excess of one of the components can partially or entirely suppress the complex formed. So it was important to check our coagulation system for this phenomenon.

Thrombin in increasing concentration was added for this purpose to a constant quantity of fibrinogen.

Several investigators (DUCLAUX, ARTHUS, RETTGER, MELLANBY, BLEIBTREU, QUICK [1]), ASTRUP and DARLING [2])) have shown the existence of a direct proportionality between thrombin concentration and coagulation velocity. But nearly all these investigators find deviations from this rule with low and high concentrations of thrombin so that the rule seems to be restricted to a certain tract.

At the other side FULD, MADSON and WALBUM, KUGELMASS [1]) find a certain other law.

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[1]) For lit. see WÖHLISCH Erg. Phys. 1929 and 1940.

[2]) Acta Phys. Scand. 2, 22 (1941).

STROMBERG, TSUNOO [1]) found that in an extensive concentration tract, none of these laws persist throughout.

EAGLE and WÖHLISCH, DIEBOLD and KIDERLEN [1]) likewise found deviations from the inverse proportionality.

RETTGER, STROMBERG found that the addition of more thrombin to a certain thrombin concentration does not result in shortening of the coagulation time, so there is a certain concentration section where the coagulation time is independent of the thrombin concentration.

Here follows a series of experiments which, in order to exclude any pH differences between thrombin and fibrinogen was made with a 1/15 M phosphate buffer of pH 6.81.

This experiment was made as follows:

On preparing the fibrinogen was dissolved in 1/15 M phosphate buffer pH 6.81 instead of in aq. dest. This solution still contains 1.47 % NaCl.

The thrombin was also dissolved in this phosphate buffer. On checking with an indicator it was seen that the pH's of these two solutions were about the same.

Arrangement:

x cc thrombin undiluted, resp. 10 × diluted with 1/15 M phosphate buffer Ph 6.81.

$(2-x)$ cc. 1/15 M phosphate buffer Ph 6.81.

1 cc buffer, as above.

0.5 cc fibrinogen, undiluted resp. diluted with 1/15 M phosphate buffer Ph 6.81, containing 1.47% NaCl.

All this at 37° C.

The result is seen in fig. 2, where log. % thrombin (in which $x = 2$ cc undiluted thrombin is supposed to be 100 %) is set out against the coagulation velocity as 1000/t in seconds.

So we see at first, that on increasing the thrombin concentration, the coagulation time decreases, (the velocity increases) then to remain constant in a certain concentration tract.

When more thrombin is added we see that the coagulation time increases again (the velocity decreases) the result being a curve with a fairly pronounced optimum.

The same curves result from experiments with rabbit citrate plasma instead of the fibrinogen solution, similarly with fibrinogen prepared according to ASTRUP and DARLING's method [2]).

An experiment was made with the usual thrombin preparation which had been purified as much as possible [3]) and the usual fibrinogen.

Here again we found the same curve with an optimum.

These experiments further show that on a decrease of the fibrinogen

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[1]) For lit. see WÖHLISCH Erg. Phys. 1929 and 1940.

[2]) Acta Phys. Scand. **4**, 45 (1942).

[3]) Later on we hope to give a detailed description in a more extensive publication.

concentration the optimum is also reached with lower thrombin concentrations (see table 1).

TABLE 1.

Conc. fibrinog.	Log. % thromb. in optimum
1	1.83?
1/2	1.61
1/4	1.42
1/8	1.26
1/16	1.12

This points to a certain extent of proportionality between fibrinogen and thrombin in the optimum.

In order to detect the regularity in this shifting we set out the data in a

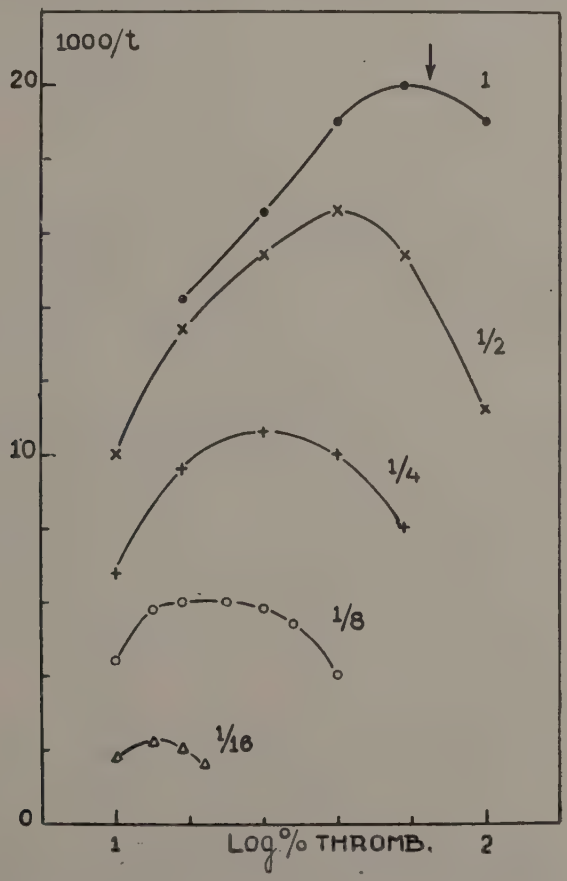

Fig. 2. Optimal mixing proportion of fibrinogen and thrombin. Shifting of the optimum to lower thrombin concentrations on diluting the fibrinogen 2, 4, 8 and 16 times. **absciss:** log. c.c. added thrombin stock solution in %, 2 c.c. being taken as 100%. **ordinate:** coagulation velocity 1000/t in sec.

graph in which the absciss indicates the quantity of fibrinogen in an arbitrary measure and the ordinate the quantity of thrombin in % (arbitrarily 2 cc is supposed to be 100%), which react together in the optimum. The points belonging to the different curves of fig. 2 lie in a straight line intersecting the ordinate at a finite value. The line obtained shows that in the optima a certain quantity of thrombin reacts to a certain quantity of fibrinogen and that when the quantity of fibrinogen is doubled the quantity of thrombin is also doubled. But there is always a certain amount of thrombin which does not take part in the reaction, but which, as it were, is present in the equilibrium liquid and which in this case is 10.4% of the quantity of thrombin present in 2 cc of liquid (point of intersection with ordinate) or about 1/7 of the total quantity which in the optimum reacts with 0.5 cc undiluted fibrinogen.

We may remark that also when plasma is used instead of fibrinogen the optimum shifts on dilution to lower thrombin concentrations.

So these experiments show us an optimal mixing proportion between thrombin and fibrinogen, while in case of lower fibrinogen concentrations kept constant, ever less thrombin is needed to reach the optimum, two phenomena in keeping with what we know of complexes.

5. *Maximal salt resistance in the optimum.*

We have seen under 2 *d* that it was found in a complex coacervate, that the resistance to neutral salts, measured as that concentration, which can exactly suppress the coacervate totally, is not equally great at all points of the mixing proportion curve. The resistance is seen to be greatest

TABLE 2.

Coag. time sec.	Dilution of thrombin added	Salt conc. in m. aequ. p. L. needed to obtain the coag. time of column 1				
		NaCl	$CaCl_2$	Luteo Cl_3	$Na_2S_2O_3$	$K_3Fe(CN)_6$
100	30 ×	48	31	16	22	4
	3 ×	150	73	25	73	33
	undil.	80	33	19	53	15
200	30 ×	93	60	21	45	16
	3 ×	240	94	36	105	48
	undil.	115	56	26	74	22
500	30 ×	170	91	26	115	30
	3 ×	400	107	54	210	71
	undil.	160	85	43	105	39
1000	30 ×	263	112	28	195	40
	3 ×	479	145	72	347 [1]	96
	undil.	214	93	63	145	63

[1]) By extrapolation. For 700 sec. the conc. is 288 m. aequ.

in the optimum, to decrease on both sides of it. As we have seen that in our system of thrombin plus fibrinogen there is also a curve with an optimum, the neutral salt resistance must also in this system be greatest in the optimum and less on either side of it.

At three points of the curve, the optimum and one point on each of the branches, we determined the coagulation times under the influence of the five neutral salts discussed with the experiments concerning the double valence rule. The three thrombin concentrations used were: undiluted (excess of thrombin), 3 × diluted (optimum) and 30 × diluted (excess of fibrinogen). The coagulation times of the blanks were 40 seconds for undiluted thrombin, 16 seconds for 3 × diluted thrombin, 60 seconds for 30 × diluted thrombin.

The data of table 2 were taken, partially through interpolation, from the curves obtained in this way. They show that in order to obtain a certain degree of suppression [1]), it may be said of each of the salts that the highest concentration is necessary in the optimum, while on either side of it this degree of suppression is reached with lower salt concentrations.

This entirely agrees with what is known of complexes.

6. *Shifting of the optimum through neutral salts.*

If the curves we have seen in fig. 2 do indeed point to the existence of complex relations, between thrombin and fibrinogen, we must also expect shifting of the optimum to higher or lower thrombin concentrations under the influence of added neutral salts.

In the arrangement used in these experiments, in which the fibrinogen concentration was kept constant and the thrombin concentration varies, we shall expect shifting to lower thrombin concentrations with polyvalent cations, if thrombin is the *positive* component, and with polyvalent anions

TABLE 3.

Exp.		Log. % thromb. in optimum
1	$CaCl_2$ 66.7 m.a.	1.34
	Blank	1.56
	$Na_2S_2O_3$ 66.7 m.a.	1.78
2	$CaCl_2$ 86 m.a.	0 38
	Blank	0.65
	Liquoid 2.3 m gr%	1.11

[1]) We can hardly — with this system — speak of total suppression as one would, for instance, have to wait 24 hours to see if coagulation yet occurs, so that it is better to compare certain coagulation times as corresponding with a certain degree of suppression. For the matter of that we did not take total suppression as a criterion in the experiments with coacervates, but a turbidity, which after being left for 30 minutes still gave 2% absorption in Moll's extinctiometer, in the arrangement used.

we shall expect shifting to higher thrombin concentrations. When, on the other hand, thrombin is the *negative* component, more thrombin is needed with polyvalent cations whereas polyvalent anions lower the optimal thrombin quantity.

The results of the first experiment are found in table 3 exp. 1. The final concentration of $CaCl_2$ and $Na_2S_2O_3$ was 66.7 m. aequ. p. L.

So there actually proves to be a certain amount of shifting. $CaCl_2$ shifts the optimum to lower thrombin concentrations, $Na_2S_2O_3$ to higher thrombin concentrations. In the terms of complex relations this means that of the thrombin-fibrinogen complex the thrombin carries the *positive* charges needed, the fibrinogen the necessary *negative* charges.

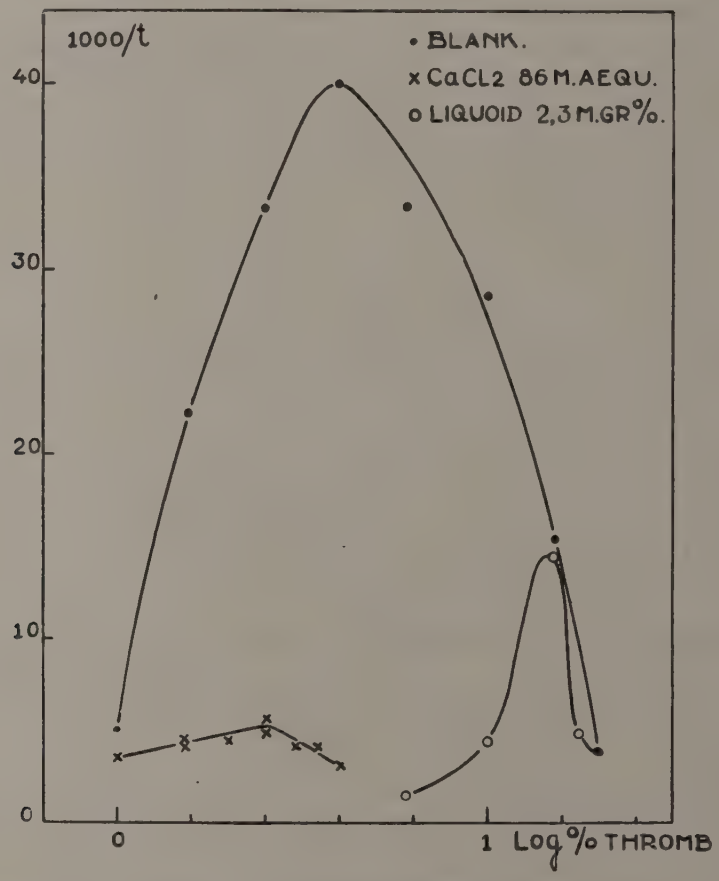

Fig. 3. Shifting of the optimum through $CaCl_2$ and liquoid.
absciss: Log c.c. added thrombine stock solution in %. 2 c.c. = 100 %.
ordinate: coagulation velocity 1000/t in sec.

Another experiment with $CaCl_2$ and $K_3\ Fe(CN)_6$ shows the same shifting.

A number of experiments was further made with the well-known anti-coagulation expedient liquoid of ROCHE, the formula of which is not entirely

known, it is true, but which is stated to be a polyanethol sodium solfonate so that it gives a high valence negative ion. This always gave evident shifting.

A result is seen in fig. 3 and table 3 exp. 2; 2,3 mg% final concentration liquoid shifts the optimum from 0,65 to 1,11 in the log., while 86 m. aequ. p. litre final concentration $CaCl_2$ gives the optimum at 0.38 in the log.

Summarizing we may say that the above experiments have given us important data. These shiftings of the optimum agree with our expectations that there are complex relations between thrombin and fibrinogen, they have considerably strengthened our presumption that there is indeed such a complex relation between thrombin and fibrinogen, moreover they have shown that the positive charges of this complex are to be found in the thrombin, while the negative charges must be found in the fibrinogen.

7. Continuous valence rule.

Now that we have seen, that in the thrombin-fibrinogen complex, the thrombin contains the positive charges, and the fibrinogen the negative ones, it follows that the two branches of the curve, found on either side of the optimum, obtained by adding an increasing quantity of thrombin to a constant quantity of fibrinogen, represent complexes of different charges, namely negatively charged with thrombin concentrations lower than the optimum, and positively charged complexes with thrombin concentrations higher than optimal, while in the vicinity of the optimum the charge must be zero. It seemed worth while to determine if in our thrombin-fibrinogen system the continuous valence rule could also be found on either side of the optimum.

Experiments made on this score showed that for the manifestation of this salt sequence it is essential that the total salt concentration in the system (so the basis concentration of NaCl) should be low. It was also found that, especially on the side of the optimum where there is an excess of thrombin, the reproducibility is sometimes bad, as here the curve often shows a sudden descent, making it impossible to select a point of experimentation too far removed from the optimum.

Here follows the result of an experiment where the reproducibility was good.

A thrombin-fibrinogen curve was made beforehand and then the effect of salts was determined at two points.

The first point was approximately in the optimum, the second on the branch which was expected to be positive. See fig. 4 *A* and *B*.

The first point (*A*) shows us the continuous valence rule of cations, as we expect in a negatively charged complex [1]), at the second point (*B*) this has passed into the double valence rule, which appears in uncharged

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[1]) In higher salt concentrations we again see the double valence rule for the impediment of coagulation.

complexes. The explanation of these facts may be that the point of reversal of charge does not coincide exactly with the optimum, but it is also possible that the complex is, indeed, formed by the negative groups of the fibrinogen and the positive groups of the thrombin, but that as an ampholyte the thrombin has so many negative groups that as a whole it is still negatively charged, or at least that it cannot give the complex a positive charge.

Although we have not seen that an excess of thrombin can give a *positive* complex, these experiments do show that the addition of thrombin can make the whole more positive.

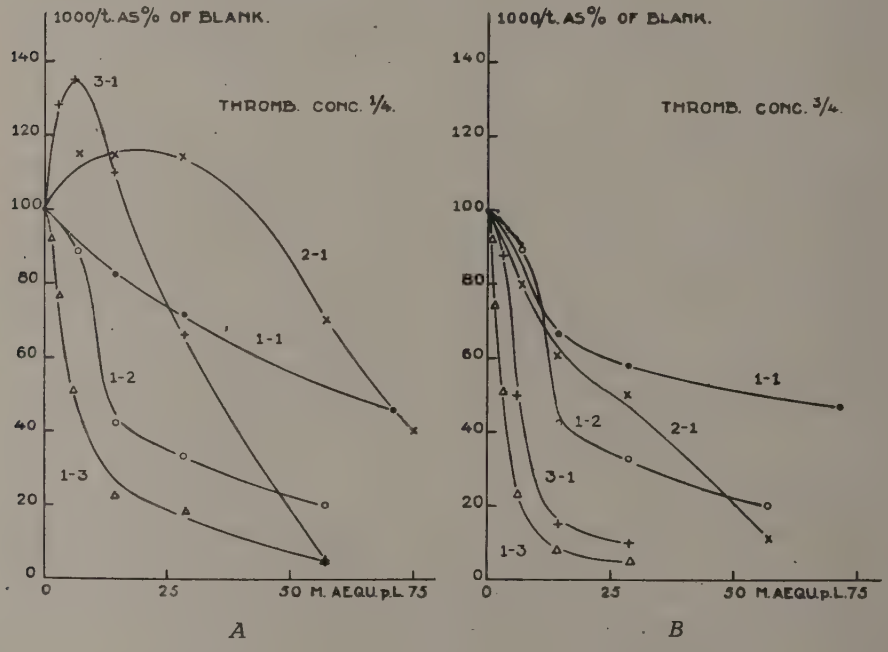

Fig. 4. *A.* continuous valence rule of cations, point of experimentation nearly in the optimum.

B. double valence rule, point of experimentation at excess of thrombin.

absciss: salt concentration in m. aequ. p. Litre.
ordinate: coagulation velocity 1000/t in sec.

8. *Conclusion.*

In studying the coagulation time of the 2nd. phase of blood coagulation under various circumstances, we have seen appearing a number of laws as we know them in complex coacervates, or more generally, complex systems according to BUNGENBERG DE JONG. So complex relations must be of decisive significance in this 2nd. phase.

As we have seen that in this complex thrombin is the positive component, we should expect that the I.E.P. of this substance lies at a pH $>$ ca 7.3. The I.E.P., however, is usually found at pH ca. 5. The investigations of

TSUNOO [1]) show the probability that, with the usual determinations, the thrombin is always carried away by negative dispurifications. It is also very well possible that the thrombin in toto is negative, but that it contains a number of positive groups, which are adjusted specifically to the negative groups of the fibrinogen; so that, as it were, a complex is formed locally, which determines the second phase of blood coagulation and of which we have found the manifestations in our experiments, but that the clot as a whole remains negative owing to an excess of negative groups, present in the thrombin molecule.

Finally this complex conception does not account for the irreversibility of the clot. A great number of positive substances can make fibrinogen flocculate, or even "clot", but then the products are always reversible which is never the case with real coagulation. So after the thrombin has been attached another process is to occur to make the fibrin irreversible and it seems best to follow the denaturase theory of WÖHLISCH. As is seen from the experiments, the attachment of thrombin according to complex relations is determinative, then the thrombin has an effect of denaturizing the fibrinogen, producing an irreversible clot. Owing to this change the thrombin loses its affinity to the fibrin and is partially detached again, which accounts for the fact that after coagulation thrombin is still found free in the liquid pressed out. It is possible that the detachment of the thrombin does not occur until the retraction (or the artificial retraction by stirring or pressing out) of the clot.

As in our experiments the onset of coagulation was only noted, i.e. the moment at which a minimal quantity of fibrin has been formed, sufficient to form a distinct clot or floccule, we did not in our experiments find any confirmation of the latter possibility. The phenomena we observe are dominated in the first place by the primary binding of the thrombin to the fibrinogen according to complex relations.

I may here express my deep-felt thanks to Prof. Dr. H. G. BUNGENBERG DE JONG, director of the laboratory of Medical Chemistry at Leiden, and Dr. W. A. L. DEKKER, concervator in this laboratory, for their highly appreciated assistance and support, and to Prof. Dr. J. JONGBLOED, director of the Physiological laboratory at Utrecht for his kind interest and willingness which have made it possible for me to continue this investigation in his laboratory.

Summary.

1. In studying a number of influences on the coagulation time of the second phase of blood coagulation we found a number of phenomena which agree with the data known from the study of complex relations by BUNGENBERG DE JONG c.s.

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[1]) Pfl. Arch. **210**, 343, (1925).

These phenomena are the following:

a. Neutral salts arrange themselves in the impeding effect on the 2nd. phase of blood coagulation according to the double valence rule 3—1 > 2—1 > 1—1 and 1—3 > 1—2 > 1—1.

b. With a fibrinogen quantity kept constant and a varying thrombin concentration the coagulation velocity passes through an optimum.

c. This optimum shifts to lower thrombin concentrations when the fibrinogen concentration is lowered.

d. The resistance to neutral salts is greatest in the optimum, to decrease on removal from the optimum.

e. Neutral salts with a polyvalent cation and a monovalent anion shift the optimum to lower thrombin concentrations. Neutral salts with a polyvalent anion and a monovalent cation show shifting to higher thrombin concentrations.

f. In the optimum there is a continuous valence rule of the cations 3—1 / 2—1 / 1—1 / 1—2 / 1—3, the addition of more thrombin gives rise to a double valence rule. It has not been possible with an excess of thrombin to show a double valence rule of anions 1—3 / 1—2 / 1—1 / 2—1 / 3—1.

2. This has given rise to the notion that there is complex formation in the second phase of blood coagulation, the positive charges being provided by the thrombin, the negative ones by the fibrinogen.

3. It is probable that after the attachment of the thrombin to the fibrinogen according to complex relations, the thrombin still affects the fibrinogen, in the sense of the denaturase theory of WÖHLISCH.

Physiology. — *De beteekenis van groeihormoon en schildklierhormoon voor de groei van de ingewanden* (II). By J. H. GAARENSTROOM. (Uit het Pharmacologisch Laboratorium der Rijksuniversiteit te Leiden, Beheerend Directeur Dr. S. E. DE JONGH). (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of January 26, 1946.)

De splanchnomegalie (ingewandsgroei buiten verhouding tot die van de rest van het lichaam), welke het gevolg is van de toediening van hypophyse-extracten, wordt, zooals wij in een vroeger beschreven onderzoek [1]) konden aantoonen, niet veroorzaakt door het in de extracten aanwezige groeihormoon. Integendeel, na toediening van behoorlijk gezuiverd groeihormoon aan ratten van de dag der hypophysectomie af, blijven de ingewanden in groei ten achter. De toediening van schildklierpoeder geeft, ook nog na de hypophysectomie, een vergrooting van de inwendige organen (SWANN [2]), KRET en DE JONGH [3])); het was dus voor de hand liggend, het in de hypophyse-extracten aanwezige thyreotrope hormoon voor de splanchnomegalie verantwoordelijk te stellen. Verdere analyse bracht aan het licht, dat de omvang der ingewanden nauw correleert met de behoefte aan functie van deze organen. De invloed van de schildklier moet, aldus gezien, bestaan in het doen stijgen van het stofwisselingspeil, wat meer functie eischt, en ook groeihormoon moet dan tenslotte toch een indirecte werking op de ingewandsgrootte uitoefenen, daar de veroorzaakte lichaamsgroei meer eischen aan de inwendige organen gaat stellen.

Het boven vermelde achterblijven in groei van de ingewanden, als groeihormoon aan kort te voren gehypophysectomeerde ratten wordt gegeven, moet naar onze meening toegeschreven worden aan het atrophisch worden van de schildklier. De negatieve invloed hiervan uitgaande moet dan de positieve, veroorzaakt door de, door groeihormoon gestimuleerde lichaamsgroei, tegengaan.

Het leek nu van belang te onderzoeken, welke invloed groeihormoon op de groei van de ingewanden uitoefent, wanneer de beoordeeling van deze groei niet wordt vertroebeld door een, gedurende de proef, wisselend peil aan schildklierhormoon Daartoe werd een proef gedaan met (volwassen) ratten, die 3 weken tevoren waren gehypophysectomeerd en zoodoende gelegenheid hadden gekregen hun schildklier volkomen atrophisch te doen worden *).

*) Over een klein deel van dit materiaal werd vroeger reeds bericht (BOERÉ en GAARENSTROOM [4])); dit was echter voor het nu nagestreefde doel veel te gering.

Een groep dezer dieren werd direct afgemaakt, een tweede groep gedurende 14 dagen met 20 E groeihormoon *) (vrij van thyreotroop hormoon) behandeld, terwijl bij de laatste groep slechts het oplosmiddel ingespoten werd. Bij de sectie werd het gewicht bepaald van lever, nier en m. gastrocnemius. Een indruk van het skeletgewicht werd verder verkregen door het gezamenlijke gewicht van tibia, fibula, femur, radius, ulna en humerus (nadat deze waren gedroogd) te bepalen.

In tabel I zijn de uitkomsten samengevat. *Alle gewichten zijn uitgedrukt per 100 gram lichaamsgewicht bij hypophysectomie, ten einde vergelijkbare waarden te verkrijgen.* Het lichaamsgewicht bij hypophysectomie wordt afzonderlijk opgegeven.

TABEL I.

Groep	Aantal	Gem. gew. bij hyp. ect.	Sectie na	Behandeling 3e—5e week	Gem. eindgewichten			
					van:	Per 100 gr. gew. bij hyp.	Toeneming in % t.o.v. aanv. contrôles	Toeneming in % t.o.v. eind contrôles
I aanvangs-contrôles	12	212 gr.	3 weken		lichaam	80 gr.		
					lever	2.81 „		
					nier	235 mgr.		
					m. gastrocn.	446 „		
					„skelet"	427 „		
II eind-contrôles	14	216 gr.	5 weken		lichaam	74 gr.	—8	
					lever	2.80 „	0	
					nier	237 mgr.	1	
					m. gastrocn.	409 „	—9	
					„skelet"	424 „	—1	
III groei-hormoon dieren	12	210 gr.	5 weken	20 E groeihormoon per dag	lichaam	92 gr.	15	24
					lever	3.56 „	27	27
					nier	298 mgr.	27	26
					m. gastrocn.	503 „	13	24
					„skelet"	429 „	0	0

Vergelijkt men de getallen van de met groeihormoon ingespoten dieren met die van de „aanvangscontrôles", dus eigenlijk met de „uitgangstoestand" (d.w.z. 3 weken na hypophysectomie), dan ziet men dat lichaamsgewicht zoowel als spiergewicht met ongeveer 15 % zijn toegenomen. Voor de lever en nier bedraagt deze toeneming aanzienlijk meer, n.l. 27 %. Dit laatste is opmerkelijk en zou toch weer op een afzonderlijke „ingewandsvergrootende" invloed van groeihormoon, zelfs als dit vrij is van thyreotroop hormoon, kunnen wijzen. De resultaten komen evenwel in een ander licht te staan, als ook de „eindcontrôles" in de beoordeeling

*) Welwillend afgestaan door Dr. E. DINGEMANSE te Amsterdam. Droogrest $1\,E = 50\,\gamma$.

worden betrokken. Bij deze dieren ziet men een achteruitgang van 8 en 9 % t.o.v. de „aanvangscontrôles", voor zoover het lichaamsgewicht en spiergewicht betreft, het gewicht van lever en nier daalt echter in het geheel niet meer. Waarom dit laatste niet gebeurt, blijft onzeker (men zou kunnen denken aan een grootere activiteit der ingewanden, door dat na het opgebruiken der vetreserves het lichaamseiwit moet worden aangetast), doch het verklaart de geringere stijging van het spiergewicht vergeleken met het ingewandsgewicht, bij de met groeihormoon behandelde ratten. Wat de spier betreft moest het groeihormoon immers een achteruitgang tegenhouden, hetgeen bij de lever en nier niet het geval was. Betrekt men de toeneming der gewichten op die gevonden bij de eindcontrôles, dan bedraagt het verschil voor lichaam en spier 24 %, voor lever en nier 27 %, dus practisch evenveel.

Kleine verschillen tusschen ingewands- en spiergroei kunnen verder nog worden veroorzaakt door de grootere eischen, die door het tot zich nemen van meer voedsel der met groeihormoon behandelden, aan de ingewanden worden gesteld.

De slotsom is, dat bij laag, niet wisselend peil aan schildklierhormoon, de gewichtstoeneming der ingewanden door groeihormoon correleert met die van het lichaamsgewicht en hieraan dus zeer wel ondergeschikt kan zijn.

De aandacht moet nog gevestigd worden op het in alle diergroepen nauwkeurig gelijk blijven van het skeletgewicht. Kennelijk gaat de doode botmassa, zooals reeds vroeger beschreven (BOERÉ en GAARENSTROOM) door hypophysectomie niet in gewicht achteruit, terwijl toeneming door groeihormoon eveneens onmogelijk is, omdat de epiphysairschijf zich in de eerste drie weken na de hypophysectomie heeft gesloten.

Het is wellicht van belang hier nog de aandacht te vestigen op een vraagstuk, dat ons bij de bewerking van het bovenomschreven getallenmateriaal bezig hield. Bij een proef met gehypophysectomeerde ratten sterft gedurende de proefperiode gewoonlijk een deel der dieren. Aan het einde houdt men slechts rekening met de nog levende. Wordt nu, zooals in ons geval, als contrôle tusschentijds een aantal ratten afgemaakt, dan bevinden zich hieronder dieren die niet (*A*) en dieren die wèl (*B*) het tot het einde van de proef zouden hebben uitgehouden. Voor een exacte beoordeeling zou het nu gewenscht zijn alleen met de laatste rekening te houden. Het is immers niet uitgesloten, dat tusschen groep *A* en *B* verschil bestaat; b.v. het lichaamsgewicht en dus misschien ook het orgaangewicht, ligt bij de lang levende doorgaans hooger dan bij de snel stervende. *A* en *B* uit elkaar houden is echter onmogelijk, wil men genoemde fout dus niet maken, dan is een, langs een omweg soms te berekenen correctiefactor noodzakelijk.

Deze correctiefactor was gebaseerd op het feit, dat de invloed van de genoemde omstandigheid wel te bepalen is voor het lichaamsgewicht.

Wij bepaalden eerst de achteruitgang in lichaamsgewicht van een groep

(van 55) ratten (boven 150 gram), die niet behandeld werden en 3 weken na de hypophysectomie werden afgemaakt, daarna geschiedde hetzelfde voor een groep (van 41), die na 5 weken was geseceerd. Deze gewichten waren, uitgedrukt in % van het gewicht bij hypophysectomie, resp. 83 en 79. Dezelfde getallen van 22 ratten, op welke na 5 weken sectie werd verricht en waarbij na 3 weken het gewicht was bepaald, waren 85 (na 3 weken) en 77 (na 5 weken). Corrigeert men deze laatste waarden (voor een goede vergelijking met de eerstgenoemde) door hen beide met $\frac{79}{77}$ te vermenigvuldigen, dan verkrijgt men 87 en 79. Met andere woorden het lichaamsgewicht van gehypophysectomeerde ratten, die na 5 weken nog leven, is na de 3e week slechts 13 % gedaaald, terwijl dat van willekeurige hypophyselooze ratten na 3 weken met 17 % (tot 83) was verminderd. Iets dergelijks moet gelden voor het orgaangewicht; wil men dus het orgaangewicht van 3 weken tevoren gehypophysectomeerde ratten, die tot 5 weken na deze operatie blijven leven, weten, dan is, aangenomen dat het orgaangewicht na hypophysectomie parallel met het lichaamsgewicht daalt, een correctie van $\frac{87}{83}$ noodig. De daling van het orgaangewicht is meestal grooter dan die van het lichaamsgewicht, althans in de eerste weken na hypophysectomie, eigenlijk is de correctie dus ingewikkelder. Men zou in ons geval behalve met $\frac{87}{83}$ nog moeten vermenigvuldigen met de breuk:

$$\frac{\text{gem. achteruitgang orgaangewicht in 3 weken (in \%)}}{\text{gem. achteruitgang lich.gewicht in 3 weken (17 \%)}}$$

en dus ook de gemiddelde daling van het orgaangewicht bij een groot aantal ratten moeten kennen.

In ons geval bleek de toepassing van genoemde correctie niet van groote invloed te zijn, weshalve deze werd achterwege gelaten. Dat deze invloed ditmaal zoo gering was, was echter naar onze meening toeval en wij achten het daarom gewenscht op dit feit kort de aandacht te vestigen.

Samenvatting:

De invloed van groeihormoon op de groei van de ingewanden bij de hypophyselooze rat, is pas zuiver te beoordeelen als variaties in de activiteit van de eveneens ingewandsgroei gevende schildklier, zijn uitgesloten. Laat men de schildklier eerst atrophisch worden dan is deze voorwaarde vervuld. Wanneer men, met doelmatige contrôleproeven, de beoordeeling zoo volledig mogelijk maakt, blijkt de ingewandsgroei door groeihormoon parallel met die van het geheele lichaam te verloopen. Het aannemen van een bijzonder aangrijpingspunt in de inwendige organen is dus niet noodig.

Summary:

The effect of growth hormone on the growth of the bowels in hypophysectomized rats can only be judged correctly when fluctuations in the

production of thyroid hormone (which also affects bowel weight) are excluded. This condition is fulfilled when some weeks after hypophysectomy the thyroid gland has become atrophic. Under these circumstances and when suitable control experiments are made, the growth of the bowels runs parallel with that of the whole body. So the assumption of a particular point of attack of growth hormone at the bowels is not necessary.

Résumé:

L'influence de l'hormone de croissance sur la croissance des viscères du rat hypophysectomisé, ne peut être jugée de façon précise que lorsque des variations dans l'activité de la glande thyroïde (celle-ci donnant également lieu à la croissance des viscères) sont exclues. Cette condition s'accomplie en laissant d'abord atrophier la glande thyroïde. Lorsque l'on rend, au moyen d'expériences de contrôle, le jugement aussi complèt que possible, il paraît que la croissance des viscères provoquée par l'hormone de croissance est parallèle à celle du corps entier. Il n'est donc pas nécessaire d'admettre un point de saisissement dans les organes internes.

Zusammenfassung:

Der Einflusz des Wachstumshormons auf das Wachstum der Eingeweide (verfolgt wurden Leber und Niere hypophyseloser Ratten), ist nur dann einwandfrei zu bewerten, wenn Schwankungen in der Leistung der, gleichfalls Eingeweidewachstum hervorrufende, Schilddrüse ausgeschlossen sind. Diese Bedingung wird erfüllt, indem man die Schildrüse vor dem Versuch atrophisch werden läszt. Wenn man darüber hinaus, mittels zweckmäsziger Kontrolversuche, die Beurteilung möglichst vervollständigt, so stellt sich heraus, dasz das Eingeweidewachstum parallel mit dem Körperwachstum verläuft. Einen dortigen besonderen Angriffspunkt braucht man also nicht an zu nehmen.

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Aerodynamics. — *On the transmission of sound waves through a shock wave.* By J. M. BURGERS. (Mededeling No. 47 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of February 23, 1946.)

1. *Introduction.* — The problem considered in the following pages is related to the one treated in a paper on the propagation of certain pressure disturbances in an ideal gas [1]). As in that paper wave propagation in one dimension only is considered, while it is assumed that the gas carrying the waves obeys the ideal gas laws, internal friction, heat conductivity and heat radiation being neglected. It was shown in the paper mentioned that when two shock waves meet, or when a shock wave is met by an expansion wave, special frontiers will appear at which there is a change of the entropy of the gas. It will be the object of the present lines to show that a similar phenomenon occurs when a shock wave is met by sound waves.

In order to fix ideas it is convenient to take in view the case of a stationary shock wave, forming the boundary between two regions (comp.

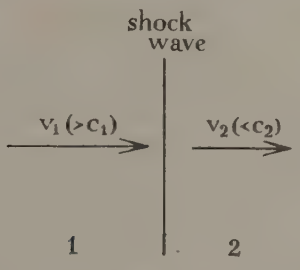

Fig. 1

fig. 1), in one of which the gas moves with the supersonic velocity v_1 directed towards the shock wave, whereas in the other region the gas moves away from the shock wave with the subsonic velocity v_2. Indicating the pressure, density, temperature and the velocity of sound in the gas on one side of the shock wave resp. by p_1, ϱ_1, T_1, c_1, and on the other side resp. by p_2, ϱ_2, T_2, c_2, we have the following well known relations:

$$\left.\begin{aligned} \varrho_1 v_1 &= \varrho_2 v_2 \\ p_1 + \varrho_1 v_1^2 &= p_2 + \varrho_2 v_2^2 \\ \frac{k R T_1}{k-1} + \frac{v_1^2}{2} &= \frac{k R T_2}{k-1} + \frac{v_2^2}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

[1]) See: J. M. BURGERS, Over de eendimensionale voortplanting van drukstoringen in een ideaal gas, Versl. Nederl. Akademie v. Wetensch., afd. Natuurkunde, **52**, 476, 560 (1943). — The present problem was suggested to me by professor S. GOLDSTEIN at Manchester during a discussion in February 1946.

Together with the formulae:

$$\left.\begin{aligned} p_1/\varrho_1 &= RT_1 = c_1^2/k \\ p_2/\varrho_2 &= RT_2 = c_2^2/k \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

we have sufficient equations to calculate *e.g.* v_2, p_2, ϱ_2, T_2 and c_2 when v_1, p_1 and ϱ_1 have been given.

Next suppose that a sound wave is generated in the region 1 and is propagated relatively to the gas with the velocity c_1 towards the shock wave. Then a transmitted sound wave will appear in the region beyond the shock wave. It is impossible, however, that a reflected sound wave makes its appearance in region 1, as every reflected wave, although relatively to the gas it would travel away from the boundary with the velocity c_1, nevertheless is carried back into the boundary in consequence of the fact that the gas itself moves with a velocity exceeding c_1. Now at the boundary usually two conditions are observed, one referring to the continuity of mass transport, the other one referring to the pressures. In ordinary cases these two conditions suffice to determine the amplitudes of both the transmitted and the reflected waves. When no reflected waves are present, it seems to be impossible to satisfy both boundary conditions. This dilemma is solved by the circumstance that beyond the shock wave (*i.e.* in the region 2) not only a sound wave is propagated, but that at the same time in this region is produced a field of spatial inequalities in the entropy, which inequalities are transported by the moving gas with the velocity v_2 and which in a certain sense take the part of a second wave motion.

The physical cause of this fact is to be found in the circumstance that a shock wave introduces a change of the entropy of the gas which moves through the wave front. In the case of an undisturbed stationary shock wave this change of entropy is independent of the time. As soon, however, as sound waves are superimposed upon the shock wave, the change of entropy of the gas passing through it is no longer constant, and a periodic field of entropy changes makes its appearance.

Actually the situation is still more complicated. In reality there are *three* boundary conditions to be fulfilled, corresponding to equations (1) for the stationary shock wave, the third equation referring to the energy. Owing to the periodic changes of state of the gas (changes of velocity, of pressure and of density) produced on both sides of the shock wave, the position of the latter will not remain fixed, but will oscillate. The amplitude of the oscillations forms a third unknown in the problem, and it thus becomes possible to satisfy the three boundary conditions.

A slightly different problem is obtained when the original sound waves should have been produced in the region where the gas is moving with the subsonic velocity (*i.e.* in the region 2). In that case sound waves clearly cannot be transmitted into the region on the other side, as again every wave motion on that side inevitably will be carried into the boundary. Hence in this case we shall have a reflected sound wave only. At the same

time, however, there appears a field of periodic inequalities of entropy on the same side as where the reflected wave is formed, which field again is carried along by the moving gas with the velocity v_2. Also in this case the shock wave itself will oscillate. Again there are three unknowns, which can be determined in such a way that the three boundary conditions will be satisfied.

2. *Equations for the sound waves and the entropy waves.* — In deriving a set of equations which will enable us to describe the peculiar system of wave motion produced, we must keep in mind that in the domain where the gas moves with the supersonic velocity, the state of the gas will always be homogeneous, so that here pressure, density, temperature and entropy will have constant values. Hence in this region we have to do with ordinary sound waves, propagating themselves with the well known velocity of sound, deduced on the supposition that the changes of state of the gas are not only adiabatic, but at the same time isentropic.

In the region on the other side of the shock wave this latter supposition is untenable. In order to deduce an appropriate system of equations, we start from the equations of motion:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} \\ \frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} &= -\varrho\frac{\partial u}{\partial x} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

To these equations must be added the equation of energy, which expresses that energy is neither lost nor gained, and that the work which is performed by the pressure in changing the volume of an element, must find its equivalent in a change of the interior energy of the gas $U = RT/(k-1)$. This equation takes the form:

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = \frac{p}{\varrho^2}\left(\frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x}\right). \quad \ldots\ldots \quad (4)$$

Together with the gas law:

$$p/\varrho = RT$$

we have four equations, connecting the four variables u, p, ϱ, T.

It is convenient to make a change of variables and to use as dependent variables the velocity of the gas u, the quantity c defined as $\sqrt{kp/\varrho}$, and the quantity $\theta = p/\varrho^k$. The latter quantity is connected with the entropy S per unit mass by means of the relation:

$$S = \frac{R}{k-1} \ln \theta + \text{constant}.$$

The temperature can be eliminated altogether, as $T = c^2/kR$. The resulting equations obtain the form [2]):

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}\left(c + \frac{k-1}{2} u\right) = \frac{c}{2k}\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\} ln\, \theta \quad . \quad (6)$$

$$\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}\left(c - \frac{k-1}{2} u\right) = \frac{c}{2k}\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\} ln\, \theta. \quad . \quad (7)$$

In applying these equations to the case of sound waves we may consider the deviations of θ, u and c from their normal constant values as small quantities of the first order. When all quantities of higher order are neglected, the factors u and c before a differential operator can be treated as constants. Writing $\delta\theta$, δc, δu resp. for the deviations of $ln\ \theta$, c, u from their normal values, the equations reduce to:

$$\frac{\partial \delta\theta}{\partial t} + u \frac{\partial \delta\theta}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}\left(\delta c + \frac{k-1}{2} \delta u\right) = \frac{c}{2k}\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\} \delta\theta. \quad (9)$$

$$\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}\left(\delta c - \frac{k-1}{2} \delta u\right) = \frac{c}{2k}\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\} \delta\theta \quad . \quad (10)$$

These equations are linear in the unknowns and can be used in describing the propagation of the combined system of sound and entropy waves of small amplitude.

3. *Solution of equations* (8)—(10) *for the case of sound waves transmitted through a shock wave.* — The form of the differential operators occurring in eqs. (8)—(10) induces us to look for solutions which are functions resp. of the following combinations of the independent variables:

$$x - ut \ ; \ x - (u+c)\,t \ ; \ x - (u-c)\,t.$$

The first expression refers to phenomena which are simply carried along by the gas, moving with the velocity u; the second and third expressions refer to phenomena which, apart from being carried along by the gas with the velocity u, moreover show a propagation relatively to the gas with the velocity c, either in the same direction as u, or in the opposite direction.

In the case of the transmission of sound waves from the supersonic region into the subsonic region there clearly cannot appear waves moving with the velocity c against the motion of the gas (such waves might appear

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[2]) These equations have been given on page 561 of the paper mentioned in footnote [1]) (p. 273 above).

only in the case where obstructions would be present in the field at some distance downstream from the shock wave, which could produce a reflection travelling towards the shock wave). Hence we must discard the combination $x-(u-c)t$, and restrict ourselves to expressions depending upon $x-ut$ and $x-(u+c)t$.

The following solution seems appropriate to this case:

$$\delta\theta = \varepsilon(x-ut) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\delta c + \frac{k-1}{2}\delta u = \frac{c}{2k} \cdot \varepsilon(x-ut) + (k-1) \cdot \varphi(x-ut-ct) \quad . \quad (12)$$

$$\delta c - \frac{k-1}{2}\delta u = \frac{c}{2k} \cdot \varepsilon(x-ut) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where ε and φ provisionally are two arbitrary functions, afterwards to be fixed by means of the boundary conditions to be satisfied at the shock wave. From (12) and (13) we deduce:

$$\delta c = \frac{c}{2k} \cdot \varepsilon(x-ut) + \frac{k-1}{2} \cdot \varphi(x-ut-ct) \quad . \quad . \quad . \quad (14)$$

$$\delta u = \varphi(x-ut-ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The corresponding values of the deviations of the pressure, the density and the temperature are given by:

$$\delta p = \varrho c \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\delta \varrho = -\frac{\varrho}{k}\varepsilon + \frac{\varrho}{c}\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\delta T = T\left(\frac{\varepsilon}{k} + \frac{k-1}{c}\varphi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the case of simple harmonic sound waves, the functions ε and φ will become simple harmonic functions of their respective arguments, both having the same frequency with respect to the time as the original sound waves.

It will be seen from the expressions (15) and (16) that there will be a simple train of waves both in the velocity and in the pressure, propagated with the velocity c relatively to the moving gas. According to (11) there will also be a simple train of waves in the entropy field, carried along by the gas with the velocity u (and thus being stationary with respect to the gas). In the density and in the temperature on the other hand a double system of waves makes itself felt, which can give rise to a system of beats.

In the case of sound waves originating in the subsonic domain and originally travelling towards the shock wave, where a reflection takes place, the solution assumes the form:

$$\delta\theta = \varepsilon\,(x-ut) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

$$\delta c + \frac{k-1}{2}\,\delta u = \frac{c}{2k}\,.\,\varepsilon\,(x-ut) + (k-1)\,.\,\varphi\,(x-ut-ct) \quad . \quad (12a)$$

$$\delta c - \frac{k-1}{2}\,\delta u = \frac{c}{2k}\,.\,\varepsilon\,(x-ut) - (k-1)\,.\,\Phi\,(x-ut+ct) \quad . \quad (13a)$$

from which:

$$\delta c = \frac{c}{2k}\,\varepsilon + \frac{k-1}{2}\,\varphi - \frac{k-1}{2}\,\Phi \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

$$\delta u = \varphi + \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

and further:

$$\delta p = \varrho\, c\,(\varphi - \Phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

$$\delta \varrho = -\frac{\varrho}{k}\,\varepsilon + \frac{\varrho}{c}\,(\varphi - \Phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

$$\delta T = T\left\{\frac{\varepsilon}{k} + \frac{k-1}{c}\,(\varphi - \Phi)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

The function $\Phi(x - ut + ct)$ introduced here describes the original sound waves, travelling relatively to the gas with the velocity c in the opposite direction of the flow velocity u; hence Φ is to be considered as a given function.

As equations (11)—(18a) all refer to the domain where the flow is subsonic, the flow velocity u introduced here corresponds to v_2 in fig. 1, while c corresponds to c_2; similarly for ϱ must be read ϱ_2 and for T : T_2.

It is convenient to add to these results the corresponding expressions for the sound waves which in the first case are present in the supersonic region (in the second case sound waves are absent from this region). They have the form:

$$\delta\theta = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11b)$$

$$\delta c + \frac{k-1}{2}\,\delta u = (k-1)\,.\,F\,(x-ut-ct) \quad . \quad . \quad . \quad . \quad (12b)$$

$$\delta c - \frac{k-1}{2}\,\delta u = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13b)$$

from which:

$$\delta c = \frac{k-1}{2}\,F\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14b)$$

$$\delta u = F\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15b)$$

$$\delta p = \varrho\, c F\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16b)$$

$$\delta \varrho = \frac{\varrho}{c}\,F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17b)$$

$$\delta T = \frac{k-1}{c}\,TF\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18b)$$

Here F represents a given function, while now u, c, ϱ, T resp. correspond to v_1, c_1, ϱ_1, T_1.

4. *The motion of the shock wave front itself and the boundary conditions.* — We write ξ for the velocity of the shock wave front, which velocity in the case of a shock wave disturbed by sound waves will be a small quantity of the first order.

The conditions to be fulfilled at the shock wave can be deduced from eqs. (1) by subtracting the amount ξ from all velocities, simultaneously paying regard to the changes of the various quantities, described by δu, $\delta\varrho$ etc. In this way the following set of equations is obtained [3]):

$$(\varrho_1 + \delta\varrho_1)(v_1 + \delta u_1 - \xi) = (\varrho_2 + \delta\varrho_2)(v_2 + \delta u_2 - \xi)$$

$$(p_1 + \delta p_1) + (\varrho_1 + \delta\varrho_1)(v_1 + \delta u_1 - \xi)^2 = (p_2 + \delta p_2) + (\varrho_2 + \delta\varrho_2)(v_2 + \delta u_2 - \xi)^2$$

$$\frac{kR}{k-1}(T_1 + \delta T_1) + \frac{(v_1 + \delta u_1 - \xi)^2}{2} = \frac{kR}{k-1}(T_2 + \delta T_2) + \frac{(v_2 + \delta u_2 - \xi)^2}{2}.$$

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[3]) The equations also can be deduced from equations (3) and (4) of the text above by making a change to coordinates t_1, x_1 in such a way that $t_1 = t$; $x_1 = x - \psi(t)$, where $\psi(t)$ provisionally is an arbitrary function of t, which is independent of x. We write: $\psi'(t) = \xi$. The equations then assume the form (with a change of order):

$$\frac{\partial\varrho}{\partial t_1} + (u-\xi)\frac{\partial\varrho}{\partial x_1} = -\varrho\frac{\partial u}{\partial x_1}$$

$$\frac{\partial u}{\partial t_1} + (u-\xi)\frac{\partial u}{\partial x_1} = -\frac{1}{\varrho}\frac{\partial p}{\partial x_1}$$

$$\frac{\partial U}{\partial t_1} + (u-\xi)\frac{\partial U}{\partial x_1} = \frac{p}{\varrho^2}\left\{\frac{\partial\varrho}{\partial t_1} + (u-\xi)\frac{\partial\varrho}{\partial x_1}\right\}.$$

We multiply the second and third one by ϱ; moreover the third equation is transformed by introducing the expression for U, eliminating the derivatives of ϱ bij means of the first one and adding to it the second one multiplied by $\varrho(u-\xi)$. In this way we arrive at the following system:

$$\frac{\partial\varrho}{\partial t_1} + (u-\xi)\frac{\partial\varrho}{\partial x_1} + \varrho\frac{\partial u}{\partial x_1} = 0 \quad \ldots\ldots\ldots \text{(I)}$$

$$\varrho\frac{\partial u}{\partial t_1} + \varrho(u-\xi)\frac{\partial u}{\partial x_1} + \frac{\partial p}{\partial x_1} = 0 \quad \ldots\ldots\ldots \text{(II)}$$

$$\left.\begin{aligned}\frac{R\varrho}{k-1}\frac{\partial T}{\partial t_1} + \varrho(u-\xi)\frac{\partial u}{\partial t_1} + \frac{R}{k-1}\varrho(u-\xi)\frac{\partial T}{\partial x_1} + p\frac{\partial u}{\partial x_1} + \\ + (u-\xi)\frac{\partial p}{\partial x_1} + \varrho(u-\xi)^2\frac{\partial u}{\partial x_1} = 0\end{aligned}\right\} \quad \ldots \text{(III)}$$

When we restrict to the case of sound waves originating in the region 1 (the supersonic region), and for $\delta u_1, \delta \varrho_1, \ldots \delta u_2, \ldots$ introduce the appropriate expressions obtained from eqs. (14*b*)—(18*b*) and from eqs. (14)—(18) respectively, neglecting quantities of higher order of magnitude than the first and striking out all quantities of zero order on the ground that they already satisfy eqs. (1), the following results are obtained:

$$\frac{\varrho_1 v_1}{c_1} F + \varrho_1 (F - \xi) = - \frac{\varrho_2 v_2}{k} \varepsilon + \frac{\varrho_2 v_2}{c_2} \varphi + \varrho_2 (\varphi - \xi) \quad \ldots \quad (19)$$

$$\left.\begin{aligned} \varrho_1 c_1 F + \frac{\varrho_1 v_1^2}{c_1} F + 2 \varrho_1 v_1 (F - \xi) = \\ = \varrho_2 c_2 \varphi - \frac{\varrho_2 v_2^2}{k} \varepsilon + \frac{\varrho_2 v_2^2}{c_2} \varphi + 2 \varrho_2 v_2 (\varphi - \xi) \end{aligned}\right\} \quad \ldots \quad (20)$$

$$\frac{k R T_1}{c_1} F + v_1 (F - \xi) = \frac{R T_2}{k - 1} \varepsilon + \frac{k R T_2}{c_2} \varphi + v_2 (\varphi - \xi). \quad \ldots \quad (21)$$

In these equations we must take the values of the functions F, φ and ε at the point $x = 0$; hence F, φ and ε, like ξ, in these equations are functions of the time t only. In the case of simple harmonic functions they all can be taken as having the same time factor, *e.g.* $\sin \nu (t - t_0)$; as this factor drops out, the equations in reality determine the ratio of the amplitudes of the waves.

By making use of the first line of eqs. (1) in order to eliminate ϱ_1 and

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These equations are still perfectly general. In the case of a shock wave the derivatives with respect to x_1 take very large values. The assumption now is introduced that a suitable function $\psi(t)$ can be found, such that $\partial \varrho / \partial t_1$, $\partial u / \partial t_1$, $\partial T / \partial t_1$ *all three remain finite* when the domain of steep variation of ϱ, u, p and T with respect to x_1 is narrowed down without limit. When the equations are multiplied by dx_1 and are integrated over the domain of steep variation, the contributions to be derived from the terms with time derivatives can then be neglected, so that attention only is to be given to the contributions deriving from the derivatives with respect to x_1.

The first equation then gives:

$$\varrho (u - \xi) = \text{constant}. \quad \ldots \quad \text{(IV)}$$

Introducing this result into the second equation we obtain upon integration:

$$\varrho (u - \xi)^2 + p = \text{constant}. \quad \ldots \quad \text{(V)}$$

The third one gives:

$$\varrho (u - \xi) \frac{RT}{k - 1} + (u - \xi) p + \frac{\varrho (u - \xi)^3}{2} = \text{constant},$$

or, after division by $\varrho (u - \xi)$ and making use of $p/\varrho = RT$:

$$\frac{k R T}{k - 1} + \frac{(u - \xi)^2}{2} = \text{constant}. \quad \ldots \quad \text{(VI)}$$

ϱ_2, and of eqs. (2) in order to eliminate the temperatures, we find, after some reductions:

$$\left.\begin{aligned} F\frac{c_1+v_1}{v_1^2}\left\{c_1(c_1+v_1)+\left(k+\frac{v_1}{c_1}\right)v_1(v_1-v_2)\right\} = \\ = \varphi\frac{c_2+v_2}{v_2^2}\{c_2(c_2+v_2)+(k-1)v_2(v_1-v_2)\}\end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

$$\varepsilon\frac{v_2}{k} = -F\frac{(c_1+v_1)^2}{c_1 v_1}+\varphi\frac{(c_2+v_2)^2}{c_2 v_2} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$(v_1-v_2)\xi = F\frac{(c_1+v_1)(c_1+v_1-v_2)}{c_1}-\varphi\frac{(c_2+v_2)v_1}{v_2} \quad . \quad . \quad . \quad (24)$$

The first one of these formulae determines the ratio of the transmitted wave φ to the original wave F [4]; having found this ratio the values of ε and ξ can be obtained by means of the other two equations.

In the case of waves originating in the subsonic domain, which are reflected at the shock wave, a similar set of expressions is obtained. In this case the equations corresponding resp. with (19)—(21) are found by substituting zero for F, and $\varphi+\Phi$ for φ in terms referring to δu, while $\varphi-\Phi$ must be substituted for φ in terms referring to δp, $\delta\varrho$, δT. The final equations take the form:

$$\left.\begin{aligned} \Phi(c_2-v_2)\{c_2(c_2-v_2)+(k-1)v_2(v_1-v_2)\} = \\ = \varphi(c_2+v_2)\{c_2(c_2+v_2)+(k-1)v_2(v_1-v_2)\}\end{aligned}\right\} \quad . \quad . \quad (22a)$$

$$\varepsilon\frac{v_2}{k} = -\Phi\frac{(c_2-v_2)^2}{c_2 v_2}+\varphi\frac{(c_2+v_2)^2}{c_2 v_2} \quad . \quad . \quad . \quad . \quad (23a)$$

$$(v_1-v_2)\xi = \Phi\frac{(c_2-v_2)v_1}{v_2}-\varphi\frac{(c_2+v_2)v_1}{v_2} \quad . \quad . \quad . \quad . \quad (24a)$$

London, February 1946.

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[4]) In order to obtain this equation in the form given above use has been made of the relation:

$$c_1^2/v_1+k v_1 = c_2^2/v_2+k v_2,$$

which follows from the second one of eqs. (1). The final result is not wholly symmetrical. It will be observed, however, that when $c_1=c_2$, $v_1=v_2$, the equation gives $F=\varphi$, as the unsymmetrical terms then reduce to zero.

Mathematics. — *On Unities and Dimensions.* II. By H. B. DORGELO and J. A. SCHOUTEN.

(Communicated at the meeting of February 23, 1946.)

If in threedimensional space a negative or positive definite fundamental tensor $g_{\lambda\varkappa}$ is introduced it is possible to consider orthogonal coordinate systems. Denoting orthogonal coordinates by ξ^1, ξ^2, ξ^3 [1]) we have

$$g_{11} = g_{22} = g_{33} = \mp 1: \quad g_{ih} = 0; \quad i \neq h; \quad h, i = 1, 2, 3. \quad . \quad (26)$$

The quantity

$$\mathfrak{g} \overset{\text{def}}{=} |\operatorname{Det}(g_{\lambda\varkappa})| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

is a scalar density of weight + 2, having with respect to every *orthogonal* coordinatesystem the value + 1. For orthogonal coordinatesystems we have $\sqrt{g} = \pm 1$ and accordingly the difference between *ordinary quantities and densities vanishes.* This identification can be got by multiplying with $\mathfrak{g}^{1/2}$ or $\mathfrak{g}^{-1/2}$ respectively. Accordingly we get the following table of identification (in orthogonal components)

$s = g = q$; $v^1 = f^1$; $w_1 = f_1$; $f^{23} = w_{23}$; $h_{23} = w_{23}$; $p^{123} = \mathfrak{v}^{123}$, $q_{123} = \delta_{123}$ cycl 1,2,3

volume = scalar volume = scalar

Fig. 2

From fig. 2 we see that the identification is brought about by means of the unity volume fixed by $\mathfrak{g}$. For instance to derive the arrow of $v^\varkappa$ from the tube of $\mathfrak{h}^\varkappa$ we have to construct two parallel planes cutting the unit of volume from the tube. Then $v^\varkappa$ fits precisely between these planes and its direction and orientation is that of the tube. Using the fundamental tensor only as far as the unit of volume is concerned we have got four different quantities. In physical publications these are often called (in the order of fig. 2) *polar vector, polar bivector, axial bivector and axial vector.* But the fundamental tensor fixes also the notion "perpendicular" and this gives rise to more identifications. The fundamental tensor establishes a one to one correspondence between co- and contravariant vectors and bivectors according to the formulas

$$\left.\begin{aligned} w_\lambda &= g_{\lambda\varkappa}\, v^\varkappa; \quad v^\varkappa = g^{\varkappa\lambda}\, w_\lambda \\ h_{\lambda\varkappa} &= g_{\lambda\varrho}\, g_{\varkappa\sigma}\, f^{\varrho\sigma}; \quad f^{\varkappa\lambda} = g^{\varkappa\varrho}\, g^{\lambda\sigma}\, h_{\varrho\sigma} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (28)$$

[1]) We always use greek indices running from *1* to *3* (*4*) for general coordinates and latin indices running from 1 to 3 (4) for orthogonal coordinates.

Hence we get the identifications of fig. 3 (orthogonal components):

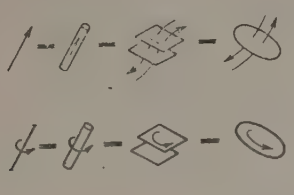

Fig. 3 [2])

Now there remain only two quantities, in physical publications often called *polar vector* and *axial vector* according to their simplest geometrical representation.

The difference between these two quantities depending only on the difference between inner and outer orientation, it is possible to get a last identification by introducing next to $g_{\lambda\varkappa}$ a definite screwsense in space. Then we get the identification

$$v^1 = \mathfrak{h}^1 = \mp w_1 = \mp \mathfrak{f}_1 = \mathfrak{v}_{23} = h_{23} = \mathfrak{w}^{23} = f^{23}; \quad \text{cycl. } 1, 2, 3. \quad (29)$$

leaving only *one* quantity, the "vector", having eight different geometrical representations. It has to be remarked that these different identifications have not to be mixed up (as is often done) but that at each stage it has to be absolutely clear what we have introduced: either only the unit volume or the whole fundamental tensor or the fundamental tensor and a screwsense.

The following algebraic operations (corresponding to the scalar and the vectorial product of vectors in ordinary vectoranalysis) are invariant with *general* coordinate transformations:

1. The transvection of a contravariant and a covariant vector:

$$v \,.\, w = v^\lambda w_\lambda; \quad \lambda = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

2. The alternating product of two contravariant or covariant vectors:

$$\bar{u} = \bar{v} \times \bar{w}; \quad u^{\varkappa\lambda} = 2\, v^{[\varkappa} w^{\lambda]}; \quad u_{\varkappa\lambda} = 2\, v_{[\varkappa} w_{\lambda]}; \; \varkappa, \lambda = 1, 2, 3; \quad . \quad (31)$$

In orthogonal components the scalar product can be written in the following ways

$$\left.\begin{aligned} \bar{v} \,.\, \bar{w} &= \mp v^1 w^1 + \text{cycl.} = v^1 w_1 + \text{cycl.} = \mp v_1 w_1 + \text{cycl.} = \\ &= \mp v^{23} w^1 + \text{cycl.} = v^{23} w_1 + \text{cycl.} = \mp v_{23} w^1 + \text{cycl.} = \\ &\qquad\qquad = v_{23} w_1 + \text{cycl} = \\ &= \mp v^{23} w^{23} + \text{cycl.} = \mp v^{23} w_{23} + \text{cycl.} = \mp v_{23} w_{23} + \text{cycl.} \end{aligned}\right\} \quad (32)$$

For the vectorial product we have in orthogonal components the forms

$$u^1 = u^{23} = v^2 w^3 - v^3 w^2 = v_{31} w^3 - v_{12} w^2 = v^{31} w^{12} - v^{12} w^{31}. \quad (33)$$

and all the forms that can be deduced from these by raising and lowering of indices with the aid of the fundamental tensor.

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[2]) In the figures with two arrows the arrow to the left belongs to the case of a negativ definite fundamental tensor and the arrow to the right to the other case.

The following differential operations (corresponding to the operations grad, div en rot in ordinary vector analysis) are invariant with *general* coordinate transformations:

1. the *gradient* of a scalar s:

$$w_\lambda = \partial_\lambda s; \quad \partial_\lambda \overset{\text{def}}{=} \frac{\partial}{\partial \xi^\lambda}; \quad \lambda = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad (34)$$

2. the *rotation* of a covariant vector:

$$h_{\mu\lambda} = 2\,\partial_{[\mu} w_{\lambda]}; \quad \lambda, \mu = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

3. the *rotation* of a covariant bivector:

$$q_{\omega\mu\lambda} = 3\,\partial_{[\omega} h_{\mu\lambda]}; \quad \omega, \mu, \lambda = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad (36)$$

4. the *divergence* of a contravariant vectordensity of weight + 1:

$$\mathfrak{p} = \partial_\mu \mathfrak{h}^\mu; \quad \mu = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

5. the *divergence* of a contravariant bivectordensity of weight + 1:

$$\mathfrak{h}^\varkappa = \partial_\mu \mathfrak{w}^{\mu\varkappa}; \quad \varkappa, \mu = 1, 2, 3; \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

6. the *divergence* of a contravariant trivectordensity of weight + 1:

$$\mathfrak{w}^{\lambda\varkappa} = \partial_\mu \mathfrak{S}^{\mu\lambda\varkappa}; \quad \varkappa, \lambda, \mu = 1, 2, 3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The equations

$$2\,\partial_{[\mu} w_{\lambda]} = 0 \quad \text{and} \quad \partial_\mu \mathfrak{w}^{\mu\varkappa} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

express the invariant property, that the double-planes of the field (constructed on an infinitesimal scale) fit together to form a system of double-surfaces, filling the whole space.

The equations

$$3\,\partial_{[\omega} h_{\mu\lambda]} = 0 \quad \text{and} \quad \partial_\mu \mathfrak{h}^\mu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

express the invariant property, that the tubes of the field (constructed on an infinitesimal scale fit together, filling the whole space.

§ 4. *The electromagnetic equations independent of the choice of the electromagnetic unities.*

The electromagnetic equations are mostly formulated with respect to ordinary orthogonal coordinates and certain well chosen electromagnetic units. In general this is very convenient but the special choice of coordinates and units has as a consequence that the difference between vectors, bivectors and densities gets lost and that the equations do not give a clear insight into the possibilities concerning the choice of the electromagnetic units. In this section the equations will be made independent of the choice

of these units. The units of mass, length and time may be arbitrarily chosen. We start from the equations

$$\left.\begin{array}{ll} a) \quad \nabla \times \mathbf{F} + \alpha \dot{\mathbf{B}} = 0 & e) \quad \mathbf{K} = \theta\, Q\, \mathbf{F} \\ b) \quad \nabla . \mathbf{B} = 0 & f) \quad F = \dfrac{\iota}{\varepsilon} \dfrac{Q}{r^2} \\ c) \quad \beta \dot{\mathbf{D}} - \gamma \nabla \times \mathbf{H} = -\varrho\, \mathbf{u} & g) \quad \mathbf{B} = \varphi\, \varepsilon\, \mathbf{F} \\ d) \quad \zeta \nabla . \mathbf{D} = \mp \varrho\ ^{3}) & h) \quad \mathbf{B} = \psi\, \mu\, \mathbf{H} \end{array}\right\} \quad . \quad . \quad (4.1)$$

where

$$\left.\begin{array}{ll} \mathbf{F} = \text{electric fieldstrength} & \varrho = \text{electric charge density} \\ \mathbf{D} = \text{electric displacement} & Q = \text{electric charge} \\ \mathbf{H} = \text{magnetic fieldstrength} & \varepsilon = \text{dielectric constant} \\ \mathbf{B} = \text{magnetic induction} & (\varepsilon = \varepsilon_0 \text{ in vacuum}) \\ & \mu = \text{permeability} \\ & (\mu = \mu_0 \text{ in vacuum}) \end{array}\right\} (4.1a)$$

$\mathbf{K} =$ force; $\mathbf{u} =$ velocity of electric charge

and where α, β, γ, ζ, θ, ι, φ and ψ are eight constant parameters depending on the choice of the electromagnetic units. The fundamental units of mass, length and time being chosen, the electromagnetic units can be fixed by the following seven assumptions:

1. *Unit of* **F**. The unit of F is the field strength exerting the force a on the charge 1.

2. *Unit of* **Q**. Two positive unit charges on a distance 1 exert on each other a force h'^{-1}.

3. *Unit of* **D**. D has the value h on a distance 1 of a unit charge in vacuum.

4. *Unit of* **H**. The unit of **H** is the fieldstrength on a distance 1 of an infinitely long straight conductor carrying a current $\frac{1}{2}\, k$.

5. *Unit of* **B**. If the current of magnetic induction through a closed curve changes with k' units in the unit of time, the electromotoric force in the curve (i.e. the integral of the electric fieldstrength along this curve) has the value 1.

6. *Unit of* ε. The unit of ε is $\dfrac{\varepsilon_0}{p}$

7. *Unit of* μ. The unit of μ is $\dfrac{\mu_0}{p'}$

Every one of these assumptions contains one constant, that can be fixed in an arbitrary way. In consideration of (4. 1) we get from them

$$\theta = a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

$$\iota = \frac{\varepsilon_0}{a\, h'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

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[3]) The sign — belongs to a negative definite fundamental tensor and the sign + to a positive definite one: $\mathbf{v} . \mathbf{w} = \mp\, v^1 w^1 \mp v^2 w^2 \mp v^3 w^3$.

$$\zeta = \frac{1}{4\pi h};\quad \varphi\,\varepsilon_0 = a\,h\,h';\quad \beta = \frac{1}{4\pi h} \quad . \quad . \quad . \quad . \quad (4.4)$$

$$\gamma = \frac{k}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

$$\alpha = \frac{1}{k'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

$$\varepsilon_0 = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

$$\mu_0 = p' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

From the fact that the propagation of electromagnetic phenomena has the velocity c in vacuum it can be deduced that

$$c^2 = \frac{h\,k\,k'}{\varphi\,\varepsilon_0\,\psi\,\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

or

$$\psi = \frac{k\,k'}{a\,h'\,p'\,c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.10)$$

Hence the eight parameters α, β, γ, ζ, θ, ι, φ and ψ and the values of ε_0 and μ_0 can now be expressed in terms of the seven constants a, k, k', h, h', p and p':

$$\left.\begin{array}{ll} \alpha = \dfrac{1}{k'} & \iota = \dfrac{p}{a\,h'} \\ \beta = \dfrac{1}{4\pi h} & \varphi = \dfrac{a\,h\,h'}{p} \\ \gamma = \dfrac{k}{4\pi} & \psi = \dfrac{k\,k'}{a\,h'\,p'\,c^2} \\ \zeta = \dfrac{1}{4\pi h} & \varepsilon_0 = p \\ \theta = a & \mu_0 = p' \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4.11)$$

That implies that there have to exist just three relations between the eight parameters and ε_0 and μ_0. In fact it is readily proved that

$$\beta = \zeta = \frac{1}{4\pi\,\iota\,\varphi} \quad ; \quad \varphi\,\psi = \frac{\gamma}{\alpha\,\beta\,\varepsilon_0\,\mu_0\,c^2}. \quad . \quad . \quad . \quad (4.12)$$

The equations (4.1) now take the form

$$\left.\begin{array}{lll} a) & k'\,\nabla \times \mathbf{F} + \dot{\mathbf{B}} = 0 & e) \quad \mathbf{K} = a\,\mathbf{Q}\,\mathbf{F} \\ b) & \nabla \,.\, \mathbf{B} = 0 & f) \quad F = \dfrac{p}{a\,h'}\,\dfrac{1}{\varepsilon}\,\dfrac{\mathbf{Q}}{r^2} \\ c) & \dfrac{1}{4\pi h}\,\dot{\mathbf{D}} - \dfrac{k}{4\pi}\,\nabla \times \mathbf{H} = -\varrho\,\mathbf{u} & g) \quad \mathbf{D} = \dfrac{a\,h\,h'}{p}\,\varepsilon\,\mathbf{F} \\ d) & \dfrac{1}{4\pi h}\,\nabla \,.\, \mathbf{D} = \mp\varrho & h) \quad \mathbf{B} = \dfrac{k\,k'}{a\,h'\,p'\,c^2}\,\mu\,\mathbf{H} \end{array}\right\} \quad . \quad (4.13)$$

The choice of the seven constants a, k, k', h, h', p and p' is entirely free. It is allowed to give them a dimension, i.e. to take them dependent on the choice of the fundamental units of mass, length and time. To every choice belongs a system of electromagnetic units.

If the fundamental units of mass, length and time are multiplied by m^{-1}, l^{-1} and t^{-1} and the constants a, k, k', h, h', p and p' by α', $\varkappa$, $\varkappa'$, χ, χ', ω and ω' respectively, it follows from (4. 13) that the units of **F**, **D**, **H**, **B**, Q, ε and μ get the following factors:

$$\left.\begin{array}{ll} \mathbf{F} : m^{-1/2} l^{1/2} t \chi'^{1/2} \alpha' & Q : m^{-1/2} l^{-3/2} t \chi'^{-1/2} \\ \mathbf{D} : m^{-1/2} l^{1/2} t \chi^{-1} \chi'^{-1/2} & \varepsilon : \omega^{-1} \\ \mathbf{H} : m^{-1/2} l^{-1/2} t^2 \varkappa \chi'^{-1/2} & \mu : \omega'^{-1} \\ \mathbf{B} : m^{-1/2} l^{3/2} \varkappa'^{-1} \chi'^{-1/2} \alpha' & \end{array}\right\} \quad . \quad . \quad (4.14)$$

If the factor of the unit of charge is denoted by q^{-1}, it is possible to eliminate χ' and to express the factors in terms of m, l, t, q, α', $\varkappa$, $\varkappa'$, χ, ω and ω':

$$\left.\begin{array}{ll} \mathbf{F} : m^{-1} l^{-1} t^2 q \alpha' & Q : q^{-1} \\ \mathbf{D} : l^2 q^{-1} \chi^{-1} & \varepsilon : \omega^{-1} \\ \mathbf{H} : l t q^{-1} \varkappa & \mu : \omega'^{-1} \\ \mathbf{B} : m^{-1} t q \varkappa'^{-1} \alpha' & \end{array}\right\} \quad . \quad . \quad . \quad . \quad (4.15)$$

and these expressions do not contain any fractional exponents. Instead of the unit of charge it would have been possible to take the unit of **F**, **D**, **H** or **B** to get rid of fractional exponents.

In the following four wellknown systems the seven constants have the values:

	Electromagnetic c.g.s	*Electrostatic* c.g.s.	*Gauss* c.g.s.	*Giorgi* m. kg. s
a	1	1	1	1
k	1	1	1	1
k'	1	1	c	1
h	1	1	c	1
h'	$1/c^2$	1	1	$10^7/c^2$
p	$1/c^2$	1	1	$10^7/c^2$
p'	1	$1/c^2$	1	10^{-7}
	(c in $^{cm}/_s$)	(c in $^{cm}/_s$)	(c in $^{cm}/_s$)	(c in $^{M}/_s$).

(4.16)

In each of these systems the constants are chosen in such a way that φ en ψ in (4. 1, g, h) have dimension [1]:

$$\varphi = \frac{a h h'}{p} = 1 ; \qquad \psi = \frac{k k'}{a h' p c^2} = 1 \quad . \quad . \quad . \quad . \quad (4.17)$$

The units belonging to these four systems are

	Electromagnetic c.g.s	*Electrostatic* c.g.s.	*Gauss* c.g.s.	*Giorgi* m. kg. s.
Force	1 dyne	1 dyne	1 dyne	1 Newton $= 10^5$ dyne
Work	1 erg	1 erg	1 erg	1 Joule $= 10^7$ erg
Q	10 coulomb	$^{10}/_{c}$ coulomb	$^{10}/_{c}$ coulomb	1 coulomb
F	10^{-8} $^{\text{volt}}/_{\text{cm}}$	$10^{-8} \cdot c$ $^{\text{volt}}/_{\text{cm}}$	$10^{-8} \cdot c$ $^{\text{volt}}/_{\text{cm}}$	1 $^{\text{volt}}/_{\text{m}}$
$\mathbf{D}$	$\frac{1}{4\pi} \cdot 10$ $^{\text{coulomb}}/_{\text{cm}^2}$	$\frac{1}{4\pi} \cdot \frac{10}{c}$ $^{\text{coulomb}}/_{\text{cm}^2}$	$\frac{1}{4\pi} \cdot \frac{10}{c}$ $^{\text{coulomb}}/_{\text{cm}^2}$	$\frac{1}{4\pi}$ $^{\text{coulomb}}/_{\text{m}^2}$
$\mathbf{H}$	1 oerstedt	c^{-1} oerstedt	1 oerstedt	10^{-3} oerstedt
$\mathbf{B}$	1 gauss	c gauss	1 gauss	10^{-4} gauss
ε	$c^2 \varepsilon_0$	ε_0	ε_0	$10^{-7} c^2 \varepsilon_0$
μ	μ_0	$c^2 \mu_0$	μ_0	$10^7 \mu_0$
E	10^{-8} volt	$10^{-8} \cdot c$ volt	$10^{-8} \cdot c$ volt	1 volt
I	10 ampère	$^{10}/_{c}$ ampère	$^{10}/_{c}$ ampère	1 ampère
R	10^{-9} ohm	$10^{-9} \cdot c^2$ ohm	$10^{-9} \cdot c^2$ ohm	1 ohm
	(c in $^{\text{cm}}/_{\text{s}}$)	(c in $^{\text{cm}}/_{\text{s}}$)	(c in $^{\text{cm}}/_{\text{s}}$)	(c in $^{\text{m}}/_{\text{s}}$)

. (4. 18)

and the equations (4. 13) take the form

	Electromagn., electrostat., Giorgi	*Gauss*
a)	$\nabla \times \mathbf{F} + \dot{\mathbf{B}} = 0$	$c \nabla \times \mathbf{F} + \dot{\mathbf{B}} = 0$
b)	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
c)	$\frac{1}{4\pi} \mathbf{D} - \frac{1}{4\pi} \nabla \times \mathbf{H} = -\varrho \mathbf{u}$	$\frac{1}{4\pi} \mathbf{D} - \frac{c}{4\pi} \nabla \times \mathbf{H} = -\varrho \mathbf{u}$
d)	$\frac{1}{4\pi} \nabla \cdot \mathbf{D} = \mp \varrho$	$\frac{1}{4\pi} \nabla \cdot \mathbf{D} = \mp \varrho$
e)	$\mathbf{K} = Q \mathbf{F}$	$\mathbf{K} = Q \mathbf{F}$
f)	$F = \frac{1}{\varepsilon} \frac{Q}{r^2}$	$F = \frac{1}{\varepsilon} \frac{Q}{r^2}$
g)	$\mathbf{D} = \varepsilon \mathbf{F}$	$\mathbf{D} = \varepsilon \mathbf{F}$
h)	$\mathbf{B} = \mu \mathbf{H}$	$\mathbf{B} = \mu \mathbf{H}$

(4. 19)

In every one of these four systems it is inconvenient that there occurs a factor 4π in (c) and (d). Comparing (4. 19) with (4. 13) we see that it is possible to get rid of this factor by starting from (4. 19) and choosing α', $\varkappa$, $\varkappa'$, χ, χ', ω and ω' in such a way that

$$a) \quad \varkappa = 4\pi; \qquad b) \quad \chi = \frac{1}{4\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4. 20)$$

But from (4. 13) we see that then (g) and (h) also change. Now we make the condition that the factor 4π vanishes in all formulas but for (f) and

that in (f) a factor 4π comes in the denominator of the right member. Then from (4. 13, *a*, *c*, *d*, *e*) it follows that

$$\alpha' = 1; \quad \varkappa = 4\pi; \quad \varkappa' = 1; \quad \chi = \frac{1}{4\pi}. \quad . \quad . \quad . \quad (4.\,21a)$$

and from (4. 13, *f*, *g*, *h*) that

$$\chi' = 4\pi\omega; \quad \omega\omega' = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.\,21b)$$

Considering only simple values of ω, ω' and χ' there are only two possibilities:

First method of rationalization (GAUSS):

$$\alpha' = \varkappa' = 1; \; \varkappa = 4\pi; \; \chi = \frac{1}{4\pi}; \; \chi' = 4\pi; \; \omega = \omega' = 1. \quad (4.\,22a)$$

Second method of rationalization:

$$\alpha' = \varkappa' = 1; \; \varkappa = 4\pi; \; \chi = \frac{1}{4\pi}; \; \chi' = 1; \; \omega = \frac{1}{4\pi}; \; \omega' = 4\pi. \quad (4.\,22b)$$

With both methods the values of φ and ψ in (4. 1 *g*, *h*) remain 1. With the method of GAUSS all units except those of ε and μ get a factor $\sqrt{4\pi}$ or $1 : \sqrt{4\pi}$. This is highly objectionable. Using the second method, only the units of **D**, **H**, ε and μ change and the unit of **H** gets the factor 4π. Because of

$$4\pi \text{ oerstedt} = 10 \, \frac{\text{ampère windings}}{\text{cm}}. \quad . \quad . \quad . \quad . \quad . \quad (4.\,23)$$

the new unit of **H** is certainly better than the old one for all practical purposes. For the new units of **D**, **H**, ε and μ we get for the second method of rationalization

	Electromagnetic c. g. s.	*Electrostatic* c. g. s.	*Gauss* c. g. s.	*Giorgi* m. kg. s.
D	10 coulomb/cm²	$\frac{10}{c}$ coulomb/cm²	$\frac{10}{c}$ coulomb/cm²	1 coulomb/m²
H	10 amp. w./cm	$\frac{10}{c}$ amp. w./cm	10 amp. w./cm	1 amp. w/m
ε	$4\pi c^2 \varepsilon_0$	$4\pi \varepsilon_0$	$4\pi \varepsilon_0$	$4\pi \cdot 10^{-7} c^2 \varepsilon_0$
μ	$\frac{1}{4\pi} \mu_0$	$\frac{1}{4\pi} c^2 \mu_0$	$\frac{1}{4\pi} \mu_0$	$\frac{1}{4\pi} \cdot 10^7 \mu_0$
	(c in cm/s)	(c in cm/s)		(c in m/s)

(4. 24)

and for both methods of rationalization the equations take the form

	Electromagn., electrostat., Giorgi	*Gauss*
a)	$\nabla \times \mathbf{F} + \dot{\mathbf{B}} = 0$	$c\,\nabla \times \mathbf{F} + \dot{\mathbf{B}} = 0$
b)	$\nabla \,.\, \mathbf{B} = 0$	$\nabla \,.\, \mathbf{B} = 0$
c)	$\mathbf{D} - \nabla \times \mathbf{H} = -\varrho\,\mathbf{u}$	$\mathbf{D} - c\,\nabla \times \mathbf{H} = -\varrho\,\mathbf{u}$
d)	$\nabla \,.\, \mathbf{D} = \mp\varrho$	$\nabla \,.\, \mathbf{D} = \mp\varrho$
e)	$\mathbf{K} = Q\,\mathbf{F}$	$\mathbf{K} = Q\,\mathbf{F}$
f)	$F = \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}$	$F = \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}$
g)	$\mathbf{D} = \varepsilon\,\mathbf{F}$	$\mathbf{D} = \varepsilon\,\mathbf{F}$
h)	$\mathbf{B} = \mu\,\mathbf{H}$	$\mathbf{B} = \mu\,\mathbf{H}$

(4. 25)

According to (4. 24) with the second method of rationalization ε_0 and μ_0 take the values

	Electromagnetic	*Electrostatic*	*Gauss*	*Giorgi*
	c. g. s.	*c. g. s.*	*c. g. s.*	*m. kg. s.*
ε_0	$1/4\pi c^2$	$1/4\pi$	$1/4\pi$	$10^7/4\pi c^2$
μ_0	4π	$4\pi/c^2$	4π	$4\pi \,.\, 10^{-7}$
	(c in $^{cm}/_{s}$)	(c in $^{cm}/_{s}$)		(c in $^{m}/_{s}$)

. (4. 26)

Comparing (4. 18) and (4. 19) with (4. 24) and (4. 25) we see that the system of GIORGI, especially in the second rationalized form is the most recommendable for all practical purposes.

§ 5. *The relative dimensions independent of the choice of the electromagnetic units.*

The table (4. 14) or (4. 15) is the base of all dimension formulas. The constants a, h, h', k, k', p and p' can be chosen in such a way that all or a part of them depend on some natural unit (cf. § 2). In fact this is done in the four practical systems mentioned above by relating the constants with the velocity of light (cf. (4. 16)). There would be no objection to the use of other natural units as the mass of the electron, the charge of the electron or the elementary quantum.

In order to deduce from (4. 14) or (4. 15) the dimensions in m, l, t or m, l, t, q of the units of a definitely given system it is not allowed to drop only the factors α', $\varkappa$, $\varkappa'$, χ, χ', ω and ω' because among the constants a, k, k', h, h', p and p', fixing the system, there may be some having a dimension. This dimension has to be taken into account. Then we get the

following *general dimension formulas*, valid for vectors only as far as *orthogonal components* are concerned ($[\varphi] = [\psi] = [1]$):

$$\left.\begin{array}{ll}
\mathbf{Q}: & [m^{1/2}\, l^{3/2}\, t^{-1}]\, [h'^{1/2}] = [q] \\
\mathbf{F}: & [m^{1/2}\, l^{-1/2}\, t^{-1}]\, [h'^{-1/2}\, a^{-1}] = [m\, l\, t^{-2}\, q^{-1}]\, [a^{-1}] \\
\mathbf{D}: & [m^{1/2}\, l^{-1/2}\, t^{-1}]\, h\, h'^{1/2}] = [l^{-2}\, q]\, [h] \\
\mathbf{H}: & [m^{1/2}\, l^{1/2}\, t^{-2}]\, [k^{-1}\, h'^{1/2}] = [l^{-1}\, t^{-1}\, q]\, [k^{-1}] \\
\mathbf{B}: & [m^{1/2}\, l^{-3/2}]\, [k'\, h'^{-1/2}\, a^{-1}] = [m\, t^{-1}\, q^{-1}]\, [k'\, a^{-1}] \\
\varepsilon: & [p] = [h\, h'\, a] = [m^{-1}\, l^{-3}\, t^{2}\, q^{2}]\, [h\, a] \\
\mu: & [p'] = [l^{-2}\, t^{2}]\, [k\, k'\, h'^{-1}\, a^{-1} = [m\, l\, q^{-2}]\, [k\, k'\, a^{-1}] \\
E: & [m^{1/2}\, l^{1/2}\, t^{-1}]\, [h'^{-1/2}\, a^{-1}] = [m\, l^{2}\, t^{-2}\, q^{-1}]\, [a^{-1}] \\
I: & [m^{1/2}\, l^{3/2}\, t^{-2}]\, [h'^{1/2}] = [t^{-1}\, q] \\
R: & [l^{-1}\, t]\, [h'^{-1}\, a^{-1}] = [m\, l^{2}\, t^{-1}\, q^{-2}]\, [a^{-1}]
\end{array}\right\} \quad . \quad . \quad (5.1)$$

Mathematics. — *On the theory of linear integral equations.* III. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 23, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the preceding two papers bearing the same title, to which we shall refer with I and II [1]). In many textbooks on integral equations kernels of the form $K(x, y) = A(x, y)\, h(y)$ are considered, where $A(x, y)$ is Hermitean and the positive function $h(y)$ is bounded. It is proved then, that every theorem for integral equations with Hermitean kernel has its analogue for integral equations with a kernel of this kind. This is done by writing

$$K(x, y) = \frac{h^{1/2}(y)}{h^{1/2}(x)}\, h^{1/2}(x)\, A(x, y)\, h^{1/2}(y) = \frac{h^{1/2}(y)}{h^{1/2}(x)}\, \widetilde{K}(x, y), \quad . \quad . \quad (1)$$

where $\widetilde{K}(x, y)$ is evidently Hermitean, and proving then in the first place that $K(x, y)$ and $\widetilde{K}(x, y)$ have the same sequence of characteristic values $\neq 0$ (compare I, Theorem 18). To be able to write (1), it is however necessary to suppose that $h^{-1}(x)$ is also bounded, hence in particular that $h(x) \neq 0$ for every x. We shall show now that this restriction is not necessary or, more precisely, that every theorem proved in II for Hermitean kernels, belonging to $L_2^{(2m)}$, has its analogue for kernels of the form $A(x, y)\, h(y)$, where $A(x, y) \in L_2^{(2m)}$ is Hermitean and $h(y)$ is measurable, bounded and non-negative. For this purpose we prove the following

Theorem 1. *If, for $x \in \triangle$ ($\triangle$ is the same m-dimensional interval as in* II*), $h(x)$ is a measurable, bounded, non-negative function, the linear transformation H in $L_2^{(m)}(\triangle)$, defined by*

$$Hf = h(x)\, f(x),$$

is bounded, self-adjoint and positive. Necessary and sufficient that H should not be identical with the nulltransformation O is that $h(x) \neq 0$ on a set of positive measure. The uniquely determined transformation $H^{1/2}$ is given by

$$H^{1/2} f = h^{1/2}(x)\, f(x).$$

Proof. If $M =$ upper bound $h(x)$ in $\triangle$, we have

$$\| Hf \|^2 = \int_{\triangle} h^2 |f|^2\, dx \leqq M^2 \int_{\triangle} |f|^2\, dx \leqq M^2 \| f \|^2$$

or $\| Hf \| \leq M \| f \|$, so that H is bounded. Further

$$(Hf, g) = \int_{\triangle} hf \cdot \bar{g}\, dx = \int_{\triangle} f \cdot \overline{hg}\, dx = (f, Hg)$$

and

$$(Hf, f) = \int_{\Delta} h\,|f|^2\,dx \geqq 0,$$

so that H is self-adjoint and positive.

If $H \neq O$, it is clear that $h(x) \neq 0$ in a set of positive measure. Conversely, if this is the case, we have for $f(x) = 1$, $Hf = h(x) \neq 0$, hence $H \neq O$.

The proof of the last statement is trivial.

To terminate this paragraph we mention that the condition of H-orthogonality for two functions $f(x)$ and $g(x)$, belonging to L_2, takes the form

$$\int_{\Delta} h(x)\,f(x)\,\overline{g(x)}\,dx = 0,$$

and that $f(x)$ is H-normal if

$$\int_{\Delta} h(x)\,|f(x)|^2\,dx = 1.$$

§ 2. *Integral equations with kernel $K(x, y) = A(x, y)h(y)$, where $A(x, y) \in L_2^{(2m)}$ is Hermitean, and $h(y)$ is measurable, bounded and non-negative.*

If $A(x, y) \in L_2^{(2m)}$ is Hermitean, the linear transformation A in the HILBERT space $L_2(\Delta)$, defined by

$$Af = \int_{\Delta} A(x, y)\,f(y)\,dy,$$

is completely continuous and self-adjoint; if $h(x)$, for $x \in \Delta$, is measurable, bounded and non-negative, the transformation H, defined by

$$Hf = h(x)\,f(x),$$

is bounded, self-adjoint and positive by Theorem 1; the transformation $K = AH$, given by

$$Kf = A\,Hf = \int_{\Delta} K(x, y)\,f(y)\,dy = \int_{\Delta} A(x, y)\,h(y)\,f(y)\,dy,$$

is therefore completely continuous and, by I, Theorem 14, symmetrisable relative to the transformation H. Supposing that $h(x) \neq 0$ on a set of positive measure, so that $H \neq O$, all theorems proved in I, and in particular those in I, § 5, may therefore be applied to the integral equation $Kf - \lambda f = g$ or

$$\int_{\Delta} K(x, y)\,f(y)\,dy - \lambda\,f(x) = g(x), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $f(x)$ and $g(x)$ belong to L_2. Here we may mention that they are even applicable to a wider class of equations, since it is not difficult to prove that every kernel of the form $k_1(x)A(x, y)k_2(y)$, where $A(x, y)$ is Hermitean, $k_1(x)$, $k_2(x)$, $k_1^{-1}(x)$ are bounded, and $k_1(x)\,k_2(x)$ is non-negative or non-positive, is expressible in the form $\widetilde{A}(x, y)h(y)$, where $\widetilde{A}(x, y)$ is Hermitean, and $h(y)$ is bounded and non-negative.

Theorem 2. *The characteristic values of* (2) *are real and characteristic functions, belonging to different characteristic values, are H-orthogonal.*

Proof. Follows from I, Theorem 5.

Theorem 3. *If $\lambda \neq 0$ is a characteristic value of* (2), *this equation has, for a given function $g(x) \in L_2$, a solution $f(x) \in L_2$ for those and only those functions $g(x)$ that are H-orthogonal to all characteristic functions, belonging to the characteristic value λ. If $\lambda \neq 0$ is no characteristic value, so that it is a regular value, the equation has a uniquely determined solution for every $g(x) \in L_2$.*

Proof. Follows from I, Theorem 12 and Theorem 3.

Theorem 4. *If*

$$\int_{\Delta} K_2(x, x)\,dx = \int_{\Delta\times\Delta} K(x, y)\,K(y, x)\,dx\,dy \neq 0,$$

the equation (2) *has a characteristic value $\neq 0$.*

Proof. Since, by I, Theorem 18, the transformation $K = AH$ has the same characteristic values $\neq 0$ as the self-adjoint transformation $\widetilde{K} = H^{1/2}AH^{1/2}$ with Hermitean kernel $\widetilde{K}(x, y) = h^{1/2}(x)A(x, y)h^{1/2}(y)$, the equation (2) has a characteristic value $\neq 0$ if only

$$\|\widetilde{K}(x, y)\|_{2m}^2 = \int_{\Delta\times\Delta} |\widetilde{K}(x, y)|^2\,dx\,dy \neq 0$$

(cf. II, Theorem 4). Since

$$|\widetilde{K}(x,y)|^2 = h(x)\,|A(x,y)|^2\,h(y) = A(x,y)\,h(y)\,A(y,x)\,h(x) = K(x,y)\,K(y,x)$$

almost everywhere in $\Delta \times \Delta$, this relation is equivalent with

$$\int_{\Delta\times\Delta} K(x, y)\,K(y, x)\,dx\,dy = \int_{\Delta} K_2(x, x)\,dx \neq 0.$$

Let now λ_i ($|\lambda_1| \geq |\lambda_2| \geq \ldots$) be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\psi_i(x)$ a corresponding H-orthonormal sequence of characteristic functions. These functions satisfy therefore the relations

$$\int_{\Delta} h(x)\,|\psi_i(x)|^2\,dx = 1,$$

$$\int_{\Delta} h(x)\,\psi_i(x)\,\overline{\psi_j(x)}\,dx = 0 \quad \text{for } i \neq j,$$

or, writing $h(x)\,\psi_i(x) = \chi_i(x)$,

$$\int \psi_i(x)\,\overline{\chi_j(x)}\,dx = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

so that the sequences $\psi_i(x)$ and $\chi_i(x)$ are biorthogonal.

Theorem 5. *(Expansion Theorem). If* $a_i = (f, \chi_i) = \int_{\Delta} f(x)\overline{\chi_i(x)}\,dx$, *then*

$$\int_{\Delta} K(x, y)\,f(y)\,dy \backsim \Sigma \lambda_i\, a_i\, \psi_i(x) + p(x)$$

for any $f(x) \in L_2$, *where the function* $p(x)$ *satisfies the relation*

$$Hp = h(x)p(x) = 0.$$

Furthermore

$$\int_{\Delta\times\Delta} h(x)\,K(x, y)\,\overline{f(x)}\,f(y)\,dx\,dy = \Sigma\ \lambda_i\,|a_i|^2.$$

Proof. Follows from I, Theorems 15 and 9.

Theorem 6. *Let* $\lambda_{n_i} (i = 1, 2, \ldots)$ *be the subsequence of all positive characteristic values where* $\lambda_{n_1} \geq \lambda_{n_2} \geq \ldots$, *let the functions* $p_1(x), \ldots, p_{i-1}(x) \in L_2$ *be arbitrarily given, and let*

$$\mu_i = \text{upper bound} \int_{\Delta\times\Delta} h(x)\,K(x, y)\,\overline{f(x)}\,f(y)\,dx\,dy \Big/ \int_{\Delta} h(x)\,|f(x)|^2\,dx$$

for all $f(x) \in L_2$ *satisfying* $\int_{\Delta} h(x)\,|f(x)|^2 dx \neq 0$ *and*

$$\int_{\Delta} h f\,\overline{p_1}\,dx = \ldots = \int_{\Delta} h f\,\overline{p_{i-1}}\,dx = 0.$$

The number μ_i *depends on* $p_1(x), \ldots, p_{i-1}(x)$. *Letting now these functions run through the whole space* L_2, *we have* $\lambda_{n_i} = \min \mu_i$.

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 7. *Let* $\lambda \neq 0$, *and let* $g(x) \in L_2$ *be H-orthogonal to all char-*

acteristic functions of (2), belonging to the characteristic value λ. (If λ is no characteristic value, $g(x)$ is therefore arbitrary). Then every solution $f(x)$ of (2) satisfies a relation of the form

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)} a_i \psi_i(x) + q(x),$$

where $a_i = \int_\Delta g(x)\overline{\chi_i(x)}dx$ *for* $\lambda_i \neq \lambda$, $h(x)q(x) = 0$, *and where* Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* $\psi_i(x)$ *has the value* $\int_\Delta f(x)\overline{\chi_i(x)}dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of (2).*

Proof. Follows from I, Theorem 17.

Theorem 8. *We have*

$$K_2(x, x) = \int_\Delta K(x,y)K(y,x)dy = \Sigma\lambda_i^2 \psi_i(x)\overline{\chi_i(x)} \text{ for almost every } x \in \Delta, \quad (3)$$

$$\int_\Delta K_2(x, x)\, dx = \Sigma\lambda_i^2, \quad \ldots\ldots\ldots \quad (4)$$

$$K(x, y) - p(x, y) \backsim \Sigma\lambda_i \psi_i(x)\overline{\chi_i(y)} \quad (\textit{Expansion Theorem}), \quad (5)$$

where $p(x, y) \in L_2^{(2m)}$ *satisfies* $h(x)p(x, y) = 0$.

Proof.

$$K_2(x, x) = \int_\Delta K(x, y)\, K(y, x)\, dy = \int_\Delta h(x)\, |A(x, y)|^2\, h(y)\, dy =$$

$$= \int_\Delta |h^{1/2}(x)\, A(x, y)\, h^{1/2}(y)|^2\, dy$$

for almost every $x \in \Delta$, and, since the Hermitean kernel $h^{1/2}(x)A(x, y)h^{1/2}(y)$ has, by I, Theorem 18, the sequence λ_i of characteristic values $\neq 0$ with the corresponding orthonormal characteristic functions $h^{1/2}(x)\psi_i(x)$, we have by II, Theorem 8

$$\int_\Delta |h^{1/2}(x)\, A(x, y)\, h^{1/2}(y)|^2\, dy = \Sigma\lambda_i^2\, h(x)\, |\psi_i(x)|^2 = \Sigma\lambda_i^2\, \psi_i(x)\, \overline{\chi_i(x)}$$

for almost every $x \in \Delta$, hence the relation (3). Integrating over x, we find (4).

To obtain (5), we observe first that the expansion theorem for $h^{1/2}(x)A(x,y)h^{1/2}(y)$ gives

$$h^{1/2}(x)\,A(x,y)\,h^{1/2}(y) \backsim \Sigma\,\lambda_i\,h^{1/2}(x)\,\psi_i(x)\,h^{1/2}(y)\,\overline{\psi_i(y)};$$

hence, since $h(y)$ is bounded, by multiplication with $h^{1/2}(y)$,

$$h^{1/2}(x)\,K(x,y) \backsim \Sigma\,\lambda_i\,h^{1/2}(x)\,\psi_i(x)\,\overline{\chi_i(y)} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is evidently not permitted to multiplicate now with $h^{-\frac{1}{2}}(x)$, since $h^{-\frac{1}{2}}(x)$ may be infinite on a set of positive measure. We must proceed, therefore, in a different way. Now, since $A(x,y) \in L_2^{(2m)}$ *and* $h(y)$ is bounded, the function $k(y) = A(x,y)h^{1/2}(y)$ belongs to L_2 for almost every $x \in \triangle$. Hence, by BESSEL's inequality (the system $h^{1/2}(y)\overline{\psi_i(y)}$ is orthonormal),

$$\Sigma\,|(k, H^{1/2}\overline{\psi_i})|^2 \leqq \|k\|^2$$

or, since

$$(k, H^{1/2}\overline{\psi_i}) = \int_{\triangle} A(x,y)\,h(y)\,\psi_i(y)\,dy = \lambda_i\,\psi_i(x),$$

$$\Sigma\,\lambda_i^2\,|\psi_i(x)|^2 \leqq \int_{\triangle} |A(x,y)|^2\,h(y)\,dy \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for almost every $x \in \triangle$.

Then we observe that the functions $f_i(x,y) = \lambda_i\psi_i(x)h^{1/2}(y)\overline{\psi_i(y)}$ are orthogonal in the space $L_2^{(2m)}$ on account of the orthogonality of the system $h^{1/2}(y)\overline{\psi_i(y)}$ in the space $L_2^{(m)}$. Furthermore, by (7).

$$\Sigma\,\|f_i(x,y)\|_{2m}^2 = \Sigma \int_{\triangle\times\triangle} \lambda_1^2\,|\psi_i(x)|^2\,h(y)\,|\psi_i(y)|^2\,dx\,dy =$$

$$\Sigma \int_{\triangle} \lambda_i^2\,|\psi_i(x)|^2\,dx \leqq \int_{\triangle\times\triangle} |A(x,y)|^2\,h(y)\,dx\,dy < \infty,$$

which shows, since

$$\left\| \sum_{i=k}^{l} f_i(x,y) \right\|_{2m}^2 = \sum_{i=k}^{l} \|f_i(x,y)\|_{2m}^2$$

by the orthogonality, that $\Sigma f_i(x,y)$ converges in mean. Denoting the sum-function by $f(x,y)$, we have therefore

$$f(x,y) \backsim \Sigma\,\lambda_i\,\psi_i(x)\,h^{1/2}(y)\,\overline{\psi_i(y)},$$

so that, writing $f(x,y)h^{1/2}(y) = K(x,y) - p(x,y)$,

$$K(x,y) - p(x,y) \backsim \Sigma\,\lambda_i\,\psi_i(x)\,\overline{\chi_i(y)} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The only thing that remains to be proved is $h(x)p(x, y) = 0$. From (5) we deduce

$$h^{1/2}(x)\, K(x, y) - h^{1/2}(x)\, p(x, y) \backsim \Sigma \lambda_i\, h^{1/2}(x)\, \psi_i(x)\, \overline{\chi_i(y)},$$

hence, comparing this with (6),

$$h^{1/2}(x)\, p(x, y) = 0 \text{ or } h(x)\, p(x, y) = 0.$$

This completes the proof.

Defining the Hermitean kernels $B_n(x, y) \in L_2^{2(m)}$ by

$$B_1(x, y) = A(x, y),$$

$$B_n(x, y) = \int_{\Delta} A(x, z)\, h(z)\, B_{n-1}(z, y)\, dz \quad (n = 2, 3, \ldots),$$

it is not difficult to see that the iterated kernels

$$K_n(x,y) \text{ and } \widetilde{K}_n(x, y)$$

of

$$K(x, y) = A(x, y)\, h(y) \text{ and } \widetilde{K}(x, y) = h^{1/2}(x)\, A(x, y)\, h^{1/2}(y)$$

are given by

$$K_n(x, y) = B_n(x, y)\, h(y) \text{ and } \widetilde{K}_n(x, y) = h^{1/2}(x)\, B_n(x, y)\, h^{1/2}(y).$$

These kernels stand therefore in the same relation to each other as the original kernels $K(x, y)$ and $\widetilde{K}(x, y)$. Since $\widetilde{K}_n(x, y)$ has the sequence $\lambda_i{}^n$ of characteristic values $\neq 0$ with the orthonormal sequence $h^{1/2}(x)\psi_i(x)$ of characteristic functions, it follows that $K_n(x, y)$ has also the sequence $\lambda_i{}^n$ of characteristic values $\neq 0$ with the H-orthonormal sequence $\psi_i(x)$ of characteristic functions. The Theorems 2—8 hold therefore for the integral equation with kernel $K_n(x, y)$, replacing everywhere λ_i by λ_i^n. in particular

$$\int_{\Delta} K_n(x, y)\, f(y)\, dy \backsim \Sigma \lambda_i^n\, a_i\, \psi_i(x) + p_n(x), \text{ where } h(x)\, p_n(x) = 0,$$

$$K_n(x, y) - p_n(x, y) \backsim \Sigma \lambda_i^n\, \psi_i(x)\, \overline{\chi_i(y)}, \quad \text{where } h(x)\, p_n(x, y) = 0.$$

We shall show now that, for $n \geq 2$, the functions $p_n(x)$ and $p_n(x, y)$ vanish. That this is not necessarily true for $n = 1$, will be shown in § 3.

Theorem 9. *For $n \geq 2$ we have*

$$\int_{\Delta} K_n(x, y)\, f(y)\, dy \backsim \Sigma \lambda_i^n\, a_i\, \psi_i(x),$$

$$K_n(x, y) \backsim \Sigma \lambda_i^n\, \psi_i(x)\, \overline{\chi_i(y)}.$$

Proof. The first part of the theorem follows from I, Theorem 15. To prove the second part, we observe that, by SCHWARZ's inequality,

$$K(z, y) - p(z, y) \backsim \Sigma \lambda_i \psi_i(z) \overline{\chi_i(y)}$$

implies

$$\int_{\triangle} K(x, z) K(z, y)\, dz - \int_{\triangle} K(x, z) p(z, y)\, dz \backsim \Sigma \lambda_i \overline{\chi_i(y)} \int_{\triangle} K(x, z) \psi_i(z)\, dz$$

or, since

$$K(x, z) p(z, y) = A(x, z) h(z) p(z, y) = 0,$$

$$K_2(x, y) \backsim \Sigma \lambda_i^2 \psi_i(x) \overline{\chi_i(y)}.$$

The proof for $n > 2$ follows easily by induction.

Theorem 10. *For the traces of $K(x, y)$ we have*

$$\int_{\triangle} K_n(x, x)\, dx = \Sigma \lambda_i^n \text{ for } n \geqq 2.$$

Proof. Since, for $n \geq 2$,

$$K_n(x, x) = B_n(x, x) h(x) = h^{1/2}(x) B_n(x, x) h^{1/2}(x) = \widetilde{K}_n(x, x)$$

for almost every $x \in \triangle$, we have by II, Theorem 9,

$$\int_{\triangle} K_n(x, x)\, dx = \int_{\triangle} \widetilde{K}_n(x, x)\, dx = \Sigma \lambda_i^n.$$

Remark. It is not difficult to prove that, if $\int_{\triangle} |A(x, y)|^2 dy$ is finite for every $x \in \triangle$ and

$$\lim_{x_2 \to x_1} \int_{\triangle} |A(x_2, y) - A(x_1, y)|^2\, dy = 0,$$

while moreover $h(y)$ is continuous on $\triangle$, several of the results in this paragraph can be improved. The convergence in mean in the Theorems 5 and 7, and also the convergence almost everywhere in Theorem 8, (3), may be replaced by uniform convergence, and the function $p(x)$ in Theorem 5 is now continuous. Moreover, for $n \geq 2$, the series $\Sigma \lambda_i^n \psi_i(x) \overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$ to $K_n(x, y)$.

§ 3. *An example.*

We shall illustrate the theorems, proved in the preceding paragraph, by an example, showing that the functions $p(x)$ and $p(x, y)$, occurring in the Theorems 5 and 8, need not vanish identically.

Let $\triangle$ be the linear interval $[0, 3]$ and $\triangle_1, \triangle_2, \triangle_3$ the subintervals $[0, 1]$, $(1, 2)$, $[2, 3]$. The orthonormal system of functions

$$\varphi_1(x), \varphi_2(x), \varphi_3(x)$$

is defined by

$$\varphi_i(x) = \begin{cases} 1 \text{ in } \triangle_i, \\ 0 \text{ in } \triangle - \triangle_i \end{cases} \qquad (i = 1, 2, 3).$$

Denoting by V the 3-dimensional subspace of $L_2(\triangle)$, determined by φ_1, φ_2 and φ_3, we shall suppose that the system χ_n is orthonormal and complete in the orthogonal subspace of V.

Let now

$$h(x) = 1 \text{ in } \triangle_1 \text{ and } \triangle_3, \; h(x) = 0 \text{ in } \triangle_2.$$

The bounded, linear transformation H in the space L_2, defined by $Hf = h(x)f(x)$, is then self-adjoint and positive. We observe that

$$H\varphi_1 = \varphi_1, \qquad H\varphi_2 = 0, \qquad H\varphi_3 = \varphi_3.$$

The bounded, self-adjoint transformation A is now defined by

$$A\varphi_1 = \varphi_2, \qquad A\varphi_2 = \varphi_1, \qquad A\varphi_3 = \varphi_3, \qquad A\chi_n = 0.$$

Evidently A has only two characteristic values $\neq 0$:
$\lambda = 1$ with the orthonormal characteristic functions $2^{-1/2}\{\varphi_1(x) + \varphi_2(x)\}$ and $\varphi_3(x)$,
$\lambda = -1$ with the normal characteristic function $2^{-1/2}\{\varphi_1(x) - \varphi_2(x)\}$; this transformation is therefore given by

$$Af = \int_{\triangle} A(x, y) f(y)\, dy,$$

where

$$A(x, y) = \tfrac{1}{2}\{\varphi_1(x) + \varphi_2(x)\}\{\varphi_1(y) + \varphi_2(y)\} + \varphi_3(x)\varphi_3(y)$$
$$-\tfrac{1}{2}\{(\varphi_1(x) - \varphi_2(x)\}\{\varphi_1(y) - \varphi_2(y)\} = \varphi_1(x)\varphi_2(y) + \varphi_2(x)\varphi_1(y) + \varphi_3(x)\varphi_3(y).$$

The symmetrisable kernel $K(x, y) = A(x, y)\, h(y)$ and the Hermitean kernel $\widetilde{K}(x, y) = h^{1/2}(x) A(x, y) h^{1/2}(y)$, corresponding with the transformations $K = AH$ and $\widetilde{K} = H^{1/2} AH^{1/2}$, are given by

$$K(x, y) = \varphi_2(x)\varphi_1(y) + \varphi_3(x)\varphi_3(y),$$
$$\widetilde{K}(x, y) = \varphi_3(x)\varphi_3(y).$$

As we know, $K(x, y)$ and $\widetilde{K}(x, y)$ have the same characteristic values $\neq 0$; hence, since $\widetilde{K}(x, y)$ has evidently only the characteristic value $\lambda = 1$, different from 0, with the characteristic function $\varphi_3(x)$, the kernel $K(x, y)$ has also $\lambda_1 = 1$ as the only characteristic value $\neq 0$ with the H-normal characteristic function $\psi_1(x) = \varphi_3(x)$. Observing that

$\overline{\chi_1(y)} = h(y)\overline{\psi_1(y)} = h(y)\varphi_3(y) = \varphi_3(y)$, so that $\lambda_1 \psi_1(x)\overline{\chi_1(y)} = \varphi_3(x)\varphi_3(y)$,

we have therefore

$$K(x, y) - p(x, y) = \lambda_1 \psi_1(x) \overline{\chi_1(y)},$$

where $p(x, y) = \varphi_2(x)\varphi_1(y) \neq 0$ for $(x, y) \in \triangle_2 \times \triangle_1$, and $h(x)p(x, y) = 0$, as required by Theorem 8.

Furthermore, by Theorem 5,

$$\int_{\triangle} K(x, y) f(y)\, dy = \lambda_1 a_1 \psi_1(x) + p(x),$$

where

$$a_1 = \int_{\triangle} f(x) \overline{\chi_1(x)}\, dx \text{ and } h(x) p(x) = 0.$$

Taking $f(x) = \varphi_1(x)$, we have

$$\int_{\triangle} K(x,y) f(y)\, dy = K\varphi_1 = A H\varphi_1 = \varphi_2(x) \text{ and } a_1 = \int_{\triangle} \varphi_1(x)\varphi_3(x)\, dx = 0;$$

hence

$$\varphi_2(x) = p(x),$$

which shows that $p(x) \neq 0$ for $x \in \triangle_2$. Evidently $h(x)p(x) = 0$, as required.

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 194 and 205 (1946).

Mathematics. — *Over het bestaan der oplossing van* $\triangle^k u = 0$, *die met haar* $k-1$ *eerste normale afgeleiden gegeven waarden aanneemt in de punten van een gegeven gesloten kromme.* II. By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

§ 3. Wij gaan nu over tot het bewijs, dat we een functie u kunnen construeeren, die binnen C voldoet aan $\triangle^3 u = 0$, terwijl in de punten van C $u = f_1$, $\frac{\partial u}{\partial n} = f_2$, $\frac{\partial^2 u}{\partial n^2} = f_3$ gegeven analytische functies van de plaats op de kromme zijn, wat weer is op te vatten, zooals in het begin van § 1 is uiteengezet.

Wij stellen daarbij

$$u = r^4 u_2 + r^2 u_1 + u_0, \tag{14}$$

waar $\triangle u_2 = \triangle u_1 = \triangle u_0 = 0$, dan is, zooals bekend, aan $\triangle^3 u = 0$ voldaan en wij zullen nu u_0, u_1, u_2 zoo bepalen, dat aan C

$$r^4 u_{2,c} + r^2 u_{1,c} + u_{0,c} = f_1, \quad \ldots\ldots\ldots \tag{15}$$

$$\frac{\partial}{\partial n}(r^4 u_2 + r^2 u_1 + u_0)_c = f_2, \quad \ldots\ldots\ldots \tag{16}$$

$$\frac{\partial^2}{\partial n^2}(r^4 u_2 + r^2 u_1 + u_0)_c = f_3. \quad \ldots\ldots\ldots \tag{17}$$

Met behulp van deze vergelijkingen trachten we $u_{0,c}$, $u_{1,c}$, $u_{2,c}$ te vinden. Wij stellen $r^2 u_2 + u_1 = v$ dan is dus $\triangle^2 v = 0$, dus volgens (13)

$$v(P) = \frac{1}{8\pi}\int_C \left\{ v_c(Q)\,\frac{\partial \triangle H_2(P,Q)}{\partial n_Q} + \left(\frac{\partial v(Q)}{\partial n}\right)_C \triangle H_2(P,Q) \right\} ds.$$

Uit (16) lossen wij op

$$\left(\frac{\partial v(Q)}{\partial n_Q}\right)_C = -\frac{2}{r_Q}\frac{\partial r_Q}{\partial n_Q}\, v_c(Q) - \frac{1}{r_Q^2}\left(\frac{\partial u_0}{\partial n_Q}\right)_C + \frac{f_2(Q)}{r_Q^2},$$

dus wordt

$$\left.\begin{aligned} v(P) = \frac{1}{8\pi}\int_C v_c(Q)\left\{\frac{\partial \triangle H_2(P,Q)}{\partial n_Q} + \frac{2}{r_Q}\frac{\partial r_Q}{\partial n_Q}\triangle H_2(P,Q)\right\} ds + \\ + \frac{1}{8\pi}\int_C \frac{1}{r_Q^2}\left(\frac{\partial u_0}{\partial n_Q}\right)_c \triangle H_2(P,Q)\, ds - \frac{1}{8\pi}\int_C \frac{f_2(Q)}{r_Q^2}\triangle H_2(P,Q)\, ds. \end{aligned}\right\} \tag{18}$$

Verder is

$$u_0(Q) = -\frac{1}{2\pi}\int_C u_{0,c}(Q')\,\frac{\partial G(Q,Q')}{\partial n_{Q'}}\,ds' =$$

$$-\frac{1}{2\pi}\int_C f_1(Q')\,\frac{\partial G(Q,Q')}{\partial n_{Q'}}\,ds' + \frac{1}{2\pi}\int_C r_{Q'}^2\, v_c(Q')\,\frac{\partial G(Q,Q')}{\partial n_{Q'}}\,ds'$$

en dus

$$\frac{\partial u_0}{\partial n_Q} = -\frac{1}{2\pi}\frac{\partial}{\partial n_Q}\int_C f_1(Q')\,\frac{\partial G(Q,Q'}{\partial n_{Q'}}\,ds' + \frac{1}{2\pi}\int_C r_{Q'}^2\, v_c(Q')\,\frac{\partial^2 G(Q,Q')}{\partial n_Q\,\partial n_{Q'}}\,ds.$$

Hierin is

$$\frac{1}{2\pi}\int_C f_1(Q')\,\frac{\partial G(Q,Q')}{\partial n_{Q'}}\,ds' = F_1(Q)$$

een bekende functie, dus ook

$$\frac{1}{8\pi}\int_C \frac{1}{r_Q^2}\,\frac{\partial F_1(Q)}{\partial n_Q}\,\triangle H_2(P,Q)\,ds = \Phi_1(P).$$

Evenzoo

$$\frac{1}{8\pi}\int_C \frac{f_2(Q)}{r_Q^2}\,\triangle H_2(P,Q)\,ds = F_2(P).$$

Door substitutie in (18) vinden wij nu

$$\left.\begin{aligned} v(P) = \frac{1}{8\pi}\int_C v_c(Q)\left\{\frac{\partial \triangle H_2(P,Q)}{\partial n_Q} + \frac{2}{r_Q}\,\frac{\partial r_Q}{\partial n_Q}\,\triangle H_2(P,Q)\right\}ds + \\ + \frac{1}{16\pi^2}\int_C \frac{1}{r_Q^2}\,\triangle H_2(P,Q)\,ds \int_C r_{Q'}^2\, v_c(Q')\,\frac{\partial^2 G(Q,Q')}{\partial n_Q\,\partial n_{Q'}}\,ds' - \Phi_1(P) - F_2(P). \end{aligned}\right\} \quad (19)$$

Door nu P tot een punt P_1 van C te laten naderen, vinden we een integraalvergelijking voor v_c. De daarbij optredende integralen zijn op te vatten op dezelfde wijze als die in (6). Wij zullen in de eerste plaats de tweede integraal nader beschouwen. Daarbij is overeenkomstig het vroeger gezegde onder

$$\int_C r_{Q'}^2\, v_c(Q')\,\frac{\partial^2 G(Q_1,Q')}{\partial n_{Q_1}\,\partial n_{Q'}}\,ds'$$

te verstaan

$$\lim_{Q_1 \to Q}\int_C r_{Q'}^2\, v_c(Q')\,\frac{\partial^2 G(Q_1,Q')}{\partial n_{Q_1}\,\partial n_{Q'}}\,ds'.$$

Noemen wij nu

$$\frac{1}{r_{Q'}^2} \Delta H_2(P, Q_1) \int_C r_{Q'}^2 v_c(Q') \frac{\partial^2 G(Q_1, Q')}{\partial n_{Q_1} \partial n_{Q'}} ds' = f(P, Q_1),$$

dan wordt de herhaalde integraal in (19)

$$\int_C \lim_{Q_1 \to Q} f(P, Q_1) ds.$$

Wij denken ons nu in de omgeving der kromme C de in de inleiding genoemde veranderlijken ξ, η ingevoerd, zoodat de kromme C wordt voorgesteld door $\eta = 0$ en op die kromme ξ loopt van 0 tot a, als men de geheele kromme doorloopt. De krommen $\xi =$ const zijn dan de orthogonale trajectorien van het stelsel krommen $\eta =$ const. Het punt P zij bepaald door (ξ_0, η_0). Wij zullen ons nu voorstellen, dat Q_1, langs een kromme $\xi =$ const. naar Q gaat, zijn dan de coördinaten van $Q(\xi, 0)$, dan kunnen wij die van Q_1 stellen (ξ, η_1), wij zullen ons voorstellen, dat binnen de kromme η negatief is. Wij nemen verder aan, dat op de kromme $\eta = h$, als $|h|$ beneden zekere grens ligt, $\frac{\partial s}{\partial \xi} = \varphi(\xi, \eta)$ voor alle waarden van ξ tusschen 0 en a doorloopend is, dan wordt onze integraal

$$\int_0^a \lim_{\eta_1 \to 0} f(\xi_0, \eta_0; \xi, \eta_1) \varphi(\xi, 0) d\xi$$

en daarvoor kunnen we achtereenvolgens schrijven

$$\int_0^a \lim_{\eta_1 \to 0} f(\xi_0, \eta_0; \xi, \eta_1) \varphi(\xi, \eta_1) d\xi = \lim_{\eta_1 \to 0} \int_0^a f(\xi_0, \eta_0; \xi, \eta_1) \varphi(\xi, \eta_1) d\xi,$$

wat wij ook kunnen voorstellen door de notatie

$$\lim_{C_1 \to C} \int_{C_1} f(P, Q_1) ds_1.$$

Wij kunnen dus voor de beschouwde herhaalde integraal schrijven

$$\lim_{C_1 \to C} \int_{C_1} \frac{1}{r_{Q_1}^2} \Delta H_2(P, Q_1) ds_1 \int_C r_{Q'}^2 v_c(Q') \frac{\partial^2 G(Q_1, Q')}{\partial n_{Q_1} \partial n_{Q'}} ds'.$$

Daar wij nu met eigenlijke integralen te doen hebben, kunnen we de volgorde der integraties verwisselen, waardoor we vinden

$$\lim_{\eta_1 \to 0} \int_C \frac{1}{r_{Q_1}^2} r_{Q'}^2 v_c(Q') ds' \int_{C_1} \frac{1}{r_{Q_1}^2} \Delta H_2(P, Q_1) \frac{\partial^2 G(Q_1, Q')}{\partial n_{Q_1} \partial n_{Q'}} ds_1.$$

Hierin is de binnenintegraal een bekende functie van P, Q' en η_1. Wij willen nu vooreerst bewijzen, dat die oneindig wordt van de eerste orde in

de andere integratie-veranderlijke, als $\eta_1 \to 0$ en $P \to Q'$. Wij kunnen voor die integraal schrijven

$$\frac{\partial}{\partial n_{Q'}} \int_{C_1} \frac{1}{r_{Q_1}^2} \Delta H_2 (P, Q_1) \frac{\partial G (Q_1, Q')}{\partial n_{Q_1}} ds_1.$$

waarbij we kunnen opmerken, dat differentieëren naar n_{Q_1} op hetzelfde neerkomt als differentiëeren naar η.

Wij hebben alleen te letten op de termen in H_2 en G, wier afgeleiden oneindig worden als P en Q' naar Q_1 gaan, dat is in H_2 $PQ_1^2 \ln PQ_1$ en in G $\ln Q_1 Q$. De termen, die we nader te bestudeeren hebben, zijn dus van den vorm

$$\int_{C_1} \frac{1}{r_{Q_1}^2} \ln (P, Q_1) \frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') ds_1 \text{ en}$$

$$\int_{C_1} \frac{1}{r_{Q_1}^2} \frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') ds_1.$$

Wij merken vooreerst op, dat, daar het punt Q' buiten de kromme C_1 ligt, zooals welbekend

$$\int \frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') ds_1 = 0.$$

Om onze integralen verder te onderzoeken, brengen we het coördinatenstelsel aan met de raaklijn in Q' aan C als x-as en de normaal als y-as De coördinaten van Q noemen wij (x, y), die van Q_1 (x_1, y_1), die van P (x_0, y_0) en wij ontwikkelen de te behandelen functies in machtreeksen naar x, dan is weer

$$y = cx^2 + \ldots$$

verder $x_1 = a_1 x + a_2 x^2 + \ldots$, $y_1 = b_0 + b_1 x + b_2 x^2 + \ldots$, waarin de coëfficienten functies zijn van η_1 (den parameter der kromme C_1). Als η_1 naar 0 gaat, gaan x_1 en y_1 naar x en y, waaruit volgt, dat dan a_1 naar 1 gaat, de overige a's naar nul, b_0 en b_1 naar nul, b_2 naar c enz.

Wij hebben dan

$$\frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') = \frac{y_1}{x_1^2 + y_1^2} = \frac{b_0 + b_1 x + b_2 x^2 + \ldots}{b_0^2 + 2 b_0 b_1 x + (a_1^2 + b_1^2 + 2 b_0 b_2) x^2 + \ldots}$$

Voor $\frac{1}{r_{Q_1}^2} ds_1$ vinden we een ontwikkeling $(a_0 + a_1 x + \ldots) dx$ en wij vinden

$$\int \frac{1}{r_{Q_1}^2} \frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') ds_1 = \int \left\{ \frac{a_0 (b_0 + b_1 x + \ldots)}{b_0^2 + 2 b_0 b_1 x + (a_1^2 + b_1^2 + 2 b_0 b_2) x^2 + \ldots} + \right.$$

$$\left. + \frac{a_1 x (b_0 + b_1 x + \ldots)}{b_0^2 + 2 b_0 b_1 x + (a_1^2 + b_1^2 + 2 b_0 b_2) x^2 + \ldots} \right\} dx.$$

Nu is

$$\int \frac{b_0 + b_1 x}{b_0^2 + 2 b_0 b_1 x + (a_1^2 + b_1^2 + 2 b_0 b_2) x^2} dx =$$

$$\frac{b_1}{2(a_1^2 + b_1^2 + 2 b_0 b_2)} ln \{ b_0^2 + 2 b_0 b_1 x + (a_1^2 + b_1^2 + 2 b_0 b_2) x^2 \} -$$

$$\frac{\sqrt{a_1^2 + 2 b_0 b_2}}{a_1^2 + b_1^2 + 2 b_0 b_2} \text{arc tg} \frac{b_0 + b_1 x}{x \sqrt{a_1^2 + 2 b_0 b_2}}.$$

Als η_1 klein is, ligt a_1 dicht bij 1 en b_0 en b_1 zijn zeer klein, de laatste term is de hoofdterm, deze stelt bij verwaarloozing van hoogere machten van x, de verandering voor van arc $tg \frac{x_1}{y_1}$, dat is dus van den hoek, waaronder van uit Q' gezien wordt de boog van C_1 van de y-as tot Q_1. (Wanneer we over de geheele kromme integreeren, is die verandering nul, in overeenstemming met onze voorafgaande opmerking, hetzelfde geldt trouwens van den eersten term, die, bij de zelfde verwaarloozing, de verandering voorstelt van $\frac{1}{2} b_1 ln\, Q_1 Q'$). Hieruit volgt, dat de beschouwde integraal eindig blijft, hetzelfde geldt voor de afgeleide naar η.

Om $$\int \frac{1}{r_{Q_1}^2} ln\, PQ_1 \frac{1}{Q_1 Q'} \cos (n_{Q''}, Q_1 Q') ds_1$$

nader te beschouwen, schrijven we

$$ln\, PQ_1 = ln\, PQ_1' + \beta_1 x + \beta_2 x^2 + \ldots,$$

waar Q'_1 het punt is, waar de normaal in Q' de kromme C_1 snijdt. Het is uit het voorgaande duidelijk, dat wij alleen zullen hebben te letten op een term van den vorm

$$A\, ln\, PQ_1' \int \frac{1}{Q_1 Q'} \cos (n_{Q'}, Q_1 Q') ds_1,$$

die een uitkomst oplevert van den vorm $B\, ln\, PQ'_1$, waarbij B met zijn afgeleide naar η eindig blijft. In de afgeleide van deze integraal naar η komt dus een term voor, die $\frac{1}{PQ_1'}$ als factor bevat en die dus oneindig wordt van de eerste orde, als P en Q'_1 naar Q_1 gaan.

Voor de som der integralen in (19) kunnen wij dus schrijven

$$\int_C v_c (Q)\, \Phi (P,Q)\, ds,$$

waarbij $\Phi(P, Q)$, in het algemeen, een enkelvoudige pool heeft voor Q in P (men merke op, dat ook $\frac{\partial \triangle H_2(P,Q)}{\partial n_Q}$ een enkelvoudige pool heeft, als P en Q samenvallen).

Wij kunnen nu evenals bij de integraal in (6) bewijzen, dat het op het zelfde neerkomt, als we P op de kromme nemen en onder de integraal de hoofdwaarde verstaan. Zoo vinden we voor v_c de integraalvergelijking

$$v_c(P) = \int_C' v_c(Q)\, \Phi(P,Q)\, ds - \Phi_1(P) - F_2(P) \tag{20}$$

Wanneer nu de bijbehoorende homogene integraalvergelijking geen oplossing heeft behalve de nuloplossing — en wij zullen ons aan het eind van deze § er van overtuigen, dat dat zoo is — dan kunnen we uit (20) v_c bepalen en (15) geeft dan $u_{0,c}$.

De vergelijkingen (15), (16) en (17) leveren nu

$$r^4 u_{2,c} + r^2 u_{1,c} = \varphi_1, \tag{21}$$

$$4r^3 \frac{\partial r}{\partial n} u_{2,c} + r^4 \left(\frac{\partial u_2}{\partial n}\right)_c + 2r \frac{\partial r}{\partial n} u_{1,c} + r^2 \left(\frac{\partial u_1}{\partial n}\right)_c = \varphi_2, \tag{22}$$

$$\left\{12 r^2 \left(\frac{\partial r}{\partial n}\right)^2 + 4r^3 \frac{\partial^2 r}{\partial n^2}\right\} u_{2,c} + 8r^3 \frac{\partial r}{\partial n}\left(\frac{\partial u_2}{\partial n}\right)_c + r^4 \left(\frac{\partial^2 u_1}{\partial n^2}\right)_c +$$
$$+ \left\{2\left(\frac{\partial r}{\partial n}\right)^2 + 2r \frac{\partial^2 r}{\partial n^2}\right\} u_{1,c} + 4r \frac{\partial r}{\partial n}\left(\frac{\partial u_1}{\partial n}\right)_c + r^2 \left(\frac{\partial^2 u}{\partial n^2}\right)_c = \varphi_3, \tag{23}$$

waarin φ_1, φ_2, φ_3 bekende functies zijn, φ_1 en φ_2 zijn, zoo bepaald, dat de beide eerste vergelijkingen met behulp van $\Delta u_1 = 0$ en $\Delta u_2 = 0$ voor $r^2 u_2 + u_1$ dezelfde functie opleveren. Voor het bepalen dezer functies zijn die vergelijkingen gelijkwaardig, een van beide is voldoende. Wij zullen ze echter beide gebruiken om $u_{1,c}$ en haar afgeleiden te elimineeren.

Noemen we de richting der raaklijn t dan is

$$\frac{\partial^2 u_2}{\partial n^2} + \frac{\partial^2 u_2}{\partial t^2} = 0 \text{ en } \frac{\partial u_2}{\partial t} = \frac{\dfrac{\partial u_2}{\partial s}}{\dfrac{\partial t}{\partial s}},$$

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{\dfrac{\partial^2 u_2}{\partial s^2}}{\left(\dfrac{\partial t}{\partial s}\right)^2} - \frac{\dfrac{\partial u_2}{\partial s}\dfrac{\partial^2 t}{\partial s^2}}{\left(\dfrac{\partial t}{\partial s}\right)^3}, \text{ dus } \frac{\partial^2 u_2}{\partial n^2} = -\frac{\dfrac{\partial^2 u_2}{\partial s^2}}{\left(\dfrac{\partial t}{\partial s}\right)^2} + \frac{\dfrac{\partial u_2}{\partial s}\dfrac{\partial^2 t}{\partial s^2}}{\left(\dfrac{\partial t}{\partial s}\right)^3}$$

en evenzoo

$$\frac{\partial^2 u_1}{\partial n^2} = -\frac{\dfrac{\partial^2 u_1}{ds^2}}{\left(\dfrac{\partial t}{\partial s}\right)^2} + \frac{\dfrac{\partial u_1}{\partial s}\dfrac{\partial^2 t}{\partial s^2}}{\left(\dfrac{\partial t}{\partial s}\right)^3}.$$

Voeren wij dat in (23) in en leiden we uit (21) door tweemaal differentiëeren af

$$\left\{12r^2\left(\frac{\partial r}{\partial s}\right)^2+4r^3\frac{\partial^2 r}{\partial s^2}\right\}u_{2,c}+8r^3\frac{\partial r}{\partial s}\left(\frac{\partial u_2}{\partial s}\right)_c+r^4\left(\frac{\partial^2 u_2}{\partial s^2}\right)_c+$$

$$+\left\{2\left(\frac{\partial r}{\partial s}\right)^2+2r\frac{\partial^2 r}{\partial s^2}\right\}u_{1,c}+4r\frac{\partial r}{\partial s}\left(\frac{\partial u_1}{\partial s}\right)_c+r^2\left(\frac{\partial^2 u_1}{\partial s^2}\right)_c=\varphi_1''$$

dan kunnen wij $r^4\frac{\partial^2 u_2}{\partial s^2}+r^2\frac{\partial^2 u_1}{\partial s^2}$ elimineeren.

Verder leiden we uit (21) af $u_{1,c}=\frac{\varphi_1}{r^2}-r^2u_{2,c}$

en dus

$$\left(\frac{\partial u_1}{\partial s}\right)_c=-\frac{2\varphi_0}{r^3}\frac{\partial r}{\partial s}+\frac{\varphi_0'}{r^2}-2r\frac{\partial r}{\partial s}u_{2,c}-r^2\left(\frac{\partial u_2}{\partial s}\right)_c,$$

waardoor we voor $u_{2,c}$ een vergelijking vinden van de gedaante

$$Au_{2,c}+B\left(\frac{\partial u_2}{\partial n}\right)_c+C\left(\frac{\partial u_2}{\partial s}\right)_c=F,$$

dat is juist die, welke door POINCARÉ is behandeld (l.c. zie pag. 252). Hij bepaalt daartoe de lading μ_2 op de kromme, die aanleiding geeft tot den potentiaal u_2. Voor μ_2 vindt men dan de integraalvergelijking

$$B\pi\mu_2(P)+\int_C'\mu_2(Q)\,ds\,\left(A\ln\frac{1}{\varrho}+B\frac{\cos\psi}{\varrho}+C\frac{\sin\psi}{\varrho}\right)=F, \quad (24)$$

waarbij ϱ weer den afstand aanwijst van het punt P, waar we μ_2 willen bepalen, naar het punt Q van het lijnelement ds en Ψ den hoek tusschen QP en de normaal in Q. Ook hier wordt wegens den term $\frac{\sin\Psi}{\varrho}$ de kern oneindig van de eerste orde als $Q\to P$. Daarom wordt ook hier de hoofdwaarde genomen. Volgens de theorie van POINCARÉ kunnen we uit (24) μ_2 bepalen tenzij de bijbehoorende homogene integraalvergelijking andere oplossingen heeft dan de nuloplossing. Door μ_2 wordt ook u_2 bekend en dus $u_{2,c}$, daardoor ook $u_{1,c}$ en volgens het principe van DIRICHLET u_1. Men kan trouwens ook even goed u_1, eerst bepalen door uit (21) u_2 en haar afgeleiden te elimineeren.

Dat het geval van onoplosbaarheid zich noch bij (20) noch bij (24) kan voordoen, blijkt weer daaruit, dat wij dan voor de gegeven functies zoodanige keus zouden kunnen doen, dat het vraagstuk oneindig veel oplossingen zou toelaten en wij weten, dat dat onmogelijk is.

§ 4. Op deze wijze kunnen wij nu stap voor stap voortgaan. Om het bewijs af te maken, zullen wij nu nog doen zien, hoe we de stelling kunnen bewijzen voor de vergelijking $\triangle^{\nu+1}u = 0$, aannemende, dat zij voor $\triangle^{\nu}u = 0$ geldt. Wij zullen daartoe vooreerst de oplossing van $\triangle^{\nu}u = 0$, waarbij aan C

$$u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2}, \ldots \frac{\partial^{\nu-1} u}{\partial n^{\nu-1}}$$

gegeven waarden aannemen, voorstellen door een integraal over die kromme met behulp van een functie van GREEN $H_\nu(P, Q)$, bepaald door de volgende eischen.

1°. Als P en Q willekeurige punten binnen C zijn, geldt voor $P \neq Q$, $\triangle^{\nu} H_\nu(P, Q) = 0$, (de differentiaties hebben betrekking op de coördinaten van P).

2°. Als $P \to Q$ heeft $H_\nu(P, Q)$ een singulariteit als $\varrho^{2(\nu-1)} ln\, \varrho$ waarbij ϱ den afstand PQ voorstelt, d.w.z. als men stelt $H_\nu(P, Q) = \varrho^{2(\nu-1)}\, ln\, \varrho + h_\nu(P, Q)$, heeft $h_\nu(P, Q)$ doorloopende partiëele afgeleiden tot die van de orde 2ν inclusief, h_ν voldoet dus in het geheele gebied binnen C aan $\triangle^{\nu} h_\nu = 0$.

3°. $$H_\nu(P, Q), \frac{\partial H_\nu(P, Q)}{\partial n}, \frac{\partial^2 H_\nu(P, Q)}{\partial n^2}, \ldots \frac{\partial^{\nu-1} H_\nu(P, Q)}{\partial n^{\nu-1}}$$

gaan naar nul, als P naar eenig punt van C gaat.

Men ziet, dat deze definitie geheel analoog is aan die van H_2, waarin zij trouwens overgaat voor $\nu = 2$, en dat de aanname, dat onze stelling bewezen is voor $\triangle^{\nu} u = 0$, meebrengt dat wij weten, dat we h_ν en dus ook H_ν kunnen bepalen.

Om nu de gevraagde functie u voor te stellen door een integraal, leiden we uit het theorema van GREEN een hulpstelling af. Wij passen de formule (11) achtereenvolgens toe door voor u en v te nemen

$$U \text{ en } \triangle^{\nu-1} U', \triangle U \text{ en } \triangle^{\nu-2} U', \ldots \triangle^{\nu-1} U \text{ en } U'.$$

Dat geeft

$$\int (U \triangle^{\nu} U' - \triangle U \triangle^{\nu-1} U')\, d\sigma = \int \left(U \frac{\partial \triangle^{\nu-1} U'}{\partial n} - \frac{\partial U}{\partial n} \triangle^{\nu-1} U' \right) ds,$$

$$\int (\triangle U \triangle^{\nu-1} U' - \triangle^2 U \triangle^{\nu-2} U')\, d\sigma = \int \left(\triangle U \frac{\partial \triangle^{\nu-2} U'}{dn} - \frac{\partial \triangle U}{\partial n} \triangle^{\nu-2} U' \right) ds,$$

. . . . . .

$$\int (\triangle^{\nu-1} U \triangle U' - \triangle^{\nu} U U')\, d\sigma = \int \left(\triangle^{\nu-1} U \frac{\partial U'}{\partial n} - \frac{\partial \triangle^{\nu-1} U}{\partial n} U' \right) ds,$$

waaruit door optelling

$$\int (U \triangle^{\nu} U' - U' \triangle^{\nu} U)\, \partial\sigma = \int \left(U \frac{\partial \triangle^{\nu-1} U'}{\partial n} - \frac{\partial U}{\partial n} \triangle^{\nu-1} U' + \right.$$

$$\left. + \triangle U \frac{\partial \triangle^{\nu-2} U'}{\partial n} - \ldots + \triangle^{\nu-1} U \frac{\partial U'}{\partial n} - \frac{\partial \triangle^{\nu-1} u}{\partial n} U' \right) ds.$$

Hierin nemen we nu voor U de gevraagde functie u, voor U' de functie H_ν en voor het integratiegebied het gebied tusschen de gegeven kromme C en een cirkel om Q, met straal δ, zoo klein, dat de cirkel geheel binnen C ligt. Dan wordt het eerste lid nul. In het tweede lid moet de integraal over den cirkel van teeken omgekeerd worden, als we de normaal in de richting naar buiten nemen, waardoor de differentiatie naar n op het zelfde neerkomt, als differentiatie naar ϱ. De integraal over den cirkel blijkt onafhankelijk van δ en wij zullen die berekenen door δ naar nul te doen naderen. Wij hebben dan aan den cirkel

$$\frac{\partial H_\nu}{\partial \varrho} = \delta^{2\nu-3}\{(2\nu-2)\ln\delta + 1\} + \ldots.$$

$$\triangle H_\nu = 2^2\,\delta^{2(\nu-2)}\{(\nu-1)^2\ln\delta + \nu - 1\} + \ldots.$$

$$\triangle^2 H_\nu = 2^4\,\delta^{2(\nu-3)}\{(\nu-1)^2(\nu-2)^2\ln\delta + \ldots.\} + \ldots.$$

. . . . . .

$$\triangle^{\nu-1} H_\nu = 2^{2(\nu-1)}\{(\nu-1)!^2\ln\delta + \ldots.\} + \ldots.,$$

$$\frac{\partial \triangle^{\nu-1} H_\nu}{\partial \varrho} = 2^{2(\nu-1)}\left\{(\nu-1)!^2\,\frac{1}{\delta} + \ldots.\right\} + \ldots.$$

waarbij telkens de niet opgeschreven termen met onbepaald afnemende δ sterker nul worden, of minder sterk toenemen dan de opgeschrevene. Het blijkt dus, dat daarbij voor de integraal over den cirkel alleen de eerste term van de integrand iets oplevert en wel $-2^{2\nu-1}\,(\nu-1)!^2\,\pi\,u(Q)$. Voor de integraal over C leveren wegens de definitie van H_ν de ν laatste termen in de integrand niets op en wij vinden voor ν even, $\nu = 2\mu$,

$$2^{\nu-1}(\nu-1)!^2\,\pi\,u(Q) = \int_C \left(U\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n} - \frac{\partial u}{\partial n}\triangle^{\nu-1} H_\nu + \right.$$

$$\left. + \triangle u\frac{\partial \triangle^{\nu-2} H_\nu}{\partial n} - \ldots. - \frac{\partial \triangle^{\mu-1} u}{\partial n}\triangle^{\mu} H_\nu\right) ds. \quad . \quad . \quad (25)$$

en voor ν oneven, $\nu = 2\mu + 1$,

$$2^{\nu-1}(\nu-1)!^2\,\pi\,u(Q) = \int_C \left(u\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n} - \frac{\partial u}{\partial n}\triangle^{\nu-1} H_\nu + \right.$$

$$\left. + \triangle U\frac{\partial \triangle^{\nu-2} H_\nu}{\partial n} - \ldots. - \frac{\partial \triangle^{\mu-1} u}{\partial n}\triangle^{\mu} H_\nu + \triangle^{\mu}_{(u)}\frac{\partial \triangle^{\mu} H_\nu}{\partial n}\right) ds. \quad (26)$$

Uit de gegeven waarden van u, $\frac{\partial u}{\partial n}, \ldots \frac{\partial^{\nu-1} u}{\partial n^{\nu-1}}$ aan C kunnen wij in die punten alle partiëele afgeleiden van u tot de $(\nu-1)^{\text{de}}$ orde inclusief berekenen. Doordat u bekend is in alle punten van C, is ook $\frac{\partial u}{\partial s}$ bekend en door $\frac{\partial u}{\partial s}$ en $\frac{\partial u}{\partial n}$ in $\frac{\partial u}{\partial x}$ en $\frac{\partial u}{\partial y}$ uit te drukken vinden we twee vergelijkingen,

waaruit in alle punten van C deze laatste partiële afgeleiden kunnen worden opgelost. Daardoor worden ook $\frac{\partial^2 u}{\partial x\,\partial s}$ en $\frac{\partial^2 u}{\partial y\,\partial s}$ bekend; door die en $\frac{\partial^2 u}{\partial n^2}$ in

$$\frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial x \partial y} \text{ en } \frac{\partial^2 u}{\partial y^2}$$

uit te drukken vinden we drie vergelijkingen, waaruit deze drie afgeleiden kunnen worden opgelost en zoo kunnen wij voortgaan. Het blijkt daarbij, dat wij telkens stelsels vergelijkingen krijgen, waarvoor de determinant ± 1 is, zoodat inderdaad alle genoemde partiële afgeleiden in de gegevens kunnen worden uitgedrukt en dus ook als $\nu = 2\mu$

$$\triangle u,\ \frac{\partial \triangle u}{\partial n},\ \triangle^2 u, \ldots \triangle_u^{\mu-1},\ \frac{\partial \triangle_u^{\mu-1}}{\partial n},$$

en als $\nu = 2\mu + 1$

$$\triangle u,\ \frac{\partial \triangle u}{\partial n},\ \triangle^2 u \ldots \triangle^{\mu} u.$$

Door de formules (25 of (26) wordt dus voor de beide gevallen de waarde van u in een willekeurig punt Q binnen C in de gegevens uitgedrukt en men kan bewijzen, dat de zoo gevonden waarden van u en haar afgeleiden in de door n bepaalde richting, wanneer Q naar eenig punt van C gaat, naderen tot de voorgeschreven waarden.

Mathematics. — *Over het bestaan der oplossing van* $\Delta^k u = 0$, *die met haar* $k-1$ *eerste normale afgeleiden gegeven waarden aanneemt in de punten van een gegeven gesloten kromme.* III. By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of January 26, 1946.)

§ 5. Wij gaan nu over tot het bewijs van het bestaan der oplossing van $\Delta^{\nu+1} u = 0$ waarbij aan C

$$u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2} \ldots, \frac{\partial^\nu u}{\partial n^\nu}$$

gegeven analytische functies zijn. Wij zullen dat bewijs geven voor ν even $\nu = 2\mu$, het geval $\nu = 2\mu + 1$ eischt slechts onbeduidende veranderingen. Omdat $\Delta^{\nu+1} u = 0$, kunnen wij stellen

$$u = r^2 v_0 + u_0,$$

waarbij $\Delta^\nu v_0 = 0$ en $\Delta u_0 = 0$.

Volgens de vorige § is dus (we nemen nu aan, dat wij onze stelling reeds voor de vergelijking $\Delta^\nu u = 0$ bewezen hebben)

$$2^{2\nu-1}(\nu-1)!^2 \pi v_0(Q) = \int_C \Big\{ V_0(P) \frac{\partial \Delta^{\nu-1} H_\nu(P,Q)}{\partial n_P} - \frac{\partial v(P)}{\partial n} \Delta^{\nu-1} H_\nu(P,Q) +$$

$$+ \Delta v_0(P) \frac{\partial \Delta^{\nu-2} H(P,Q)}{\partial n_P} \ldots - \frac{\partial \Delta^{\mu-1} v_0(P)}{\partial n_P} \Delta^\mu H_\nu(P,Q) \Big\} ds.$$

Uit de gegevens aan C leiden we nu af

$$u = r^2 v_0 + u_0 = f_1, \frac{\partial u}{\partial n} = 2r \frac{\partial r}{\partial n} v_0 + r^2 \frac{\partial v_0}{\partial n} + \frac{\partial u_0}{\partial n} = f_2, \ldots$$

$$\frac{\partial^\nu u}{\partial n^\nu} = \frac{\partial^\nu (r^2 v_0 + u_0)}{\partial n^\nu} = f_{\nu+1},$$

en dus ook

$$\Delta u = 4 v_0 + 4 \alpha r \frac{\partial v_0}{\partial n} + r^2 \Delta v_0 = \varphi_2,$$

$$\frac{\partial \Delta u}{\partial n} = 8 \frac{\partial v_0}{\partial n} + 4 \alpha r \frac{\partial^2 v_0}{\partial n^2} + 2 \alpha r \Delta v_0 + r^2 \frac{\partial \Delta v_0}{\partial n} = \varphi_3,$$

waarbij $\alpha = \cos(n, r)$ een in elk punt der kromme bekende functie is,

$$\triangle^2 u = 16\,\triangle v_0 + 8\,\alpha r\,\frac{\partial \triangle v_0}{\partial n} + r^2\,\triangle^2 v_0 = \varphi_4,$$

$$\frac{\partial \triangle^2 u}{\partial n} = 24\,\frac{\partial \triangle v_0}{\partial n} + 8\,\alpha r\,\frac{\partial^2 \triangle v_0}{\partial n^2} + 2\,\alpha r\,\triangle^2 v_0 + r^2\,\frac{\partial \triangle^2 v_0}{\partial n} = \varphi_5,$$

. . . . . .

$$\triangle^k u = 4k^2\,\triangle^{k-1} v_0 + 4k\,\alpha r\,\frac{\partial \triangle^{k-1} v_0}{\partial n} + r^2\,\triangle^k v_0 = \varphi_{2k},$$

$$\frac{\partial \triangle^k u}{\partial n} = 4k(k+1)\,\frac{\partial \triangle^{k-1} v_0}{\partial n} + 4k\,\alpha r\,\frac{\partial^2 \triangle^{k-1} v_0}{\partial n^2} + 2\,\alpha r\,\triangle^{k-1} v_0 + r^2\,\frac{\partial \triangle^k v_0}{\partial n} = \varphi_{2k+1},$$

. . . . . .

$$\triangle^\mu u = 4\mu^2\,\triangle^{\mu-1} v_0 + 4\,\mu\,\alpha r\,\frac{\partial \triangle^{\mu-1} v_0}{\partial n} + r^2\,\triangle^\mu v_0 = \varphi_\nu,$$

waarbij f_1, f_2 ... $f_{\nu+1}$ en ook φ_2, φ_3 ... φ_ν in elk punt der kromme bekende functies zijn. Men toont de juistheid van die formules gemakkelijk aan met behulp van een Bernoulliaansch bewijs. Voegen wij bij de vergelijkingen, waarvan het tweede lid φ_2 en φ_3 is nog $\triangle v_0 = \frac{\partial^2 v_0}{\partial n^2} + \beta\frac{\partial^2 v_0}{\partial s^2} + \gamma\frac{\partial v_0}{\partial s}$, waarbij ook β en γ weer in elk punt der kromme bekende functies zijn, dan kunnen wij uit deze drie vergelijkingen $\triangle v_0$, $\frac{\partial \triangle v_0}{\partial n}$ en $\frac{\partial^2 v_0}{\partial n^2}$ lineair uitdrukken in v_0, $\frac{\partial v_0}{\partial n}$, $\frac{\partial v_0}{\partial s}$ en $\frac{\partial^2 v_0}{\partial s^2}$.

Bij de twee volgende vergelijkingen voegen wij

$$\triangle^2 v_0 = \frac{\partial^2 \triangle v_0}{\partial n^2} + \beta\,\frac{\partial^2 \triangle v_0}{\partial s^2} + \gamma\,\frac{\partial \triangle v_0}{\partial s},$$

waardoor we drie vergelijkingen krijgen, waaruit we $\triangle^2 v_0$, $\frac{\partial \triangle^2 v_0}{\partial n}$ en $\frac{\partial^2 \triangle v_0}{\partial n^2}$ kunnen oplossen in $\triangle v_0$ en $\frac{\partial \triangle v_0}{\partial n}$ dus ook in v_0, $\frac{\partial v_0}{\partial n}$, $\frac{\partial v_0}{\partial s}$ en $\frac{\partial^2 v_0}{\partial s^2}$. Uit de daarop volgende vergelijkingen vinden we evenzoo $\triangle^2 v$ en $\frac{\partial \triangle^2 v_0}{\partial n}$, zoo gaan we door tot het voorlaatste tweetal, dat $\triangle^{\mu-1} v_0$, en $\frac{\partial \triangle^{\mu-1} v_0}{\partial n}$ oplevert. Door dat alles in de integraalvoorstelling voor $v(Q)$ te substitueeren vinden we

$$v_0(Q) = \int_C \left\{ A\,v_0(P) + B\,\frac{\partial v_0(P)}{\partial n} + C\,\frac{\partial v_0(P)}{\partial s} + D\,\frac{\partial^2 v_0(P)}{\partial s^2} \right\} ds, \quad (27)$$

waarin A, B, C, D bekende functies zijn, opgebouwd uit α, β, γ en H_ν en

haar afgeleiden, waarbij wij moeten opmerken, dat in A een term voorkomt met $\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n}$, die, wanneer P en Q tot een zelfde punt van de kromme naderen, een enkelvoudige oneindigheid krijgt. In (27) substitueeren wij nu

$$\frac{\partial v_0}{\partial n} = \frac{f_2}{r^2} - \frac{2}{r}\frac{\partial r}{\partial n}\left(f_1 - \frac{u_0}{r^2}\right) - \frac{\partial u_0}{\partial n} = g_1 + \lambda u_0 - \frac{\partial u_0}{\partial n},$$

$$\frac{\partial v_1}{\partial s} = \frac{\partial}{\partial s}\left(\frac{f_1}{r^2}\right) + \frac{2}{r^3}\frac{\partial r}{\partial s}\cdot u_0 - \frac{1}{r^2}\frac{\partial u_0}{ds} = \psi + \lambda_1 u_0 - \frac{1}{r^2}\frac{\partial u_0}{\partial s},$$

$$\frac{\partial^2 v_0}{\partial s^2} = \psi + \lambda_2 u_0 + \varkappa_2 \frac{\partial u_0}{\partial s} - \frac{1}{r^2}\frac{\partial^2 u_0}{\partial s^2},$$

waarin g_1, ψ_1, ψ_2, λ, λ_1, λ_2 en $\varkappa_2$ bekende functies zijn. Daar $u_0 = f_1 - r^2 v_0$ en $\triangle u_0 = 0$, hebben we

$$u_0(P) = -\frac{1}{2\pi}\int_C f_0(Q')\frac{\partial G(P,Q')}{\partial n_{Q'}}ds' + \frac{1}{2\pi}\int_C r_{Q'}^2 v_0(Q')\frac{\partial G(P,Q')}{\partial n_{Q'}}ds'.$$

De vergelijking (27) wordt dus

$$\begin{aligned} v_0(Q) = \int_C \Bigg[& (Bg_1 + C\psi_1 + D\psi_2) - \frac{1}{2\pi}(\lambda B + \lambda_1 C + \lambda_2 D)\int_C f_1(Q')\frac{\partial G(P,Q')}{\partial n_{Q'}}ds' + \\ & + \frac{1}{2\pi}(\lambda B + \lambda_1 C + \lambda_2 D)\int_C r_{Q'}^2 v_0(Q')\frac{\partial G(P,Q')}{\partial n_{Q'}}ds' + \\ & + \Bigg\{ A v_0(P) - \frac{B}{2\pi}\int_C f_1(Q')\frac{\partial^2 G(P,Q')}{\partial n_P\,\partial n_{Q'}}ds' + \frac{B}{2\pi}\int r_{Q'}^2 v_0(Q')\frac{\partial^2 G(P,Q')}{\partial n_P\,\partial n_{Q'}}ds' + \\ & + \frac{1}{2\pi}\left(-\frac{C}{r_P^2} + \varkappa_2 D\right)\int f_0(Q')\frac{\partial^2 G(P,Q')}{\partial s_P\,\partial n_Q'}ds' + \\ & + \frac{1}{2\pi}\left(\frac{C}{r_P^2} - \varkappa_2 D\right)\int r_{Q'}^2 v_0(Q')\frac{\partial^2 G(P,Q')}{\partial s_P\,\partial n_Q'}ds' - \frac{1}{2\pi}\frac{D}{r_P^2}\int f_1(Q')\frac{\partial^3 G(P,Q')}{\partial^2 s_P\,\partial n_{Q'}}ds - \\ & - \frac{1}{2\pi}\frac{D}{r_P^2}\int r_{Q'}^2 v_0(Q')\frac{\partial^3 G(P,Q')}{\partial^2 s_P\,\partial n_{Q'}}ds' \Bigg\} \Bigg] ds. \end{aligned}$$

Deze vergelijking is het analogon van (20). Wij zullen ze ook op de zelfde manier behandelen, zoo vinden we b.v.

$$\begin{aligned} & \int \left\{ (\lambda B + \lambda_1 C + \lambda_2 D)\int r_{Q'}^2 v_0(Q')\frac{\partial G(P,Q')}{\partial n_{Q'}}ds' \right\} ds = \\ & = \int v_0(Q') r_{Q'}^2 \left\{ \int (\lambda B + \lambda_1 C + \lambda_2 D)\frac{\partial G(P,Q')}{\partial n_{Q'}}ds \right\} ds'. \end{aligned}$$

enz., waardoor we geraken tot een integraalvergelijking

$$v_0(Q) = F(Q) + \int_C v_0(Q')\, K(Q, Q')\, ds', \tag{28}$$

waarbij

$$F(Q) = \int_C (B g_1 + C\psi_1 + D\psi_2)\, ds -$$

$$-\frac{1}{2\pi}\int_C \left\{(\lambda B + \lambda_1 C + \lambda_2 D)\int_C f_1(Q')\frac{\partial G(Q,Q')}{\partial n_{Q'}}\, ds'\right\} ds$$

$$-\frac{\lambda}{2\pi}\int_C B\, ds \int_C f_1(Q')\frac{\partial^2 G(Q,Q')}{\partial n_P\, \partial n_{Q'}} ds' +$$

$$+\frac{1}{2\pi}\int_C \left(-\frac{C}{r^2} + \varkappa_2 D\right) ds \int_C f_1(Q')\frac{\partial^2 G(Q,Q')}{\partial s_Q\, \partial n_{Q'}} ds' -$$

$$-\frac{1}{2\pi}\int_C \frac{D}{r_Q^2}\, ds \int f_1(Q')\frac{\partial^3 G(Q,Q')}{\partial^2 s_Q\, \partial n_{Q'}} ds',$$

en

$$K(Q, Q') = \frac{1}{2\pi} r_{Q'}^2 \left\{\int (\lambda B + \lambda_1 C + \lambda_2 D)\frac{\partial G(Q,Q')}{\partial n_{Q'}} + B\frac{\partial^2 G(Q,Q')}{\partial n_{Q'}\, \partial n_Q} -\right.$$

$$\left. -\frac{\partial}{\partial s}\left(\frac{C}{r_Q^2} - \varkappa_2 D\right)\frac{\partial G(Q,Q')}{\partial n_{Q'}} - \frac{\partial^2}{\partial s^2}\left(\frac{D}{r_Q^2}\right)\frac{\partial G(Q,Q')}{\partial n_{Q'}}\right\} ds.$$

Het bewijs van deze formules komt geheel overeen met het behandelde in § 3. Wij merken nog op, dat bij de afleiding van de laatste twee termen in de integrand, na omkeering der integratievolgorde, de binnenintegraal door partieele integratie herleid is.

Uit de vergelijking (28) lossen wij nu op v_{oc} en dan wordt door $r^2 v_{oc} + u_{oc} = f_1\ u_{oc}$ bekend en daardoor u_0, dus ook

$$\left(\frac{\partial u_0}{\partial n}\right)_C, \left(\frac{\partial^2 u_0}{\partial n^2}\right)_C \cdots \left(\frac{\partial^{\nu-1}}{\partial n^{\nu-1}}\right)_C$$

en daardoor

$$\frac{\partial}{\partial n}(r^2 v_0), \frac{\partial^2}{\partial n^2}(r^2 v_0)_C \cdots \frac{\partial^{\nu-1}}{\partial n^{\nu-1}}(r^2 v_0)_C$$

en dus ook

$$\left(\frac{\partial v_0}{\partial n}\right)_C, \left(\frac{\partial^2 v_0}{\partial n^2}\right)_C \cdots \left(\frac{\partial^{\nu-1} v_0}{\partial n^{\nu-1}}\right)_C.$$

Stellen wij nu $v_0 = r^2 v_1 + u_1$, waarbij $\Delta^{\nu-1} v_1 = 0$ en $\Delta u_1 = 0$, dan wordt aan C bekend

$$r^2 v_{1C} + u_{1C} = \psi_0, \left(\frac{\partial}{\partial n}\right)_C (r^2 v_1)_C + \left(\frac{\partial u_1}{\partial n}\right)_C = \psi_1,$$

$$\ldots\ldots \left(\frac{\partial^{\nu-1}}{\partial n^{\nu-1}}\right)_C (r^2 v_1)_C + \left(\frac{\partial^{\nu-1} u_1}{\partial n^{\nu-1}}\right)_C = \psi_{\nu-1}$$

Van deze ν vergelijkingen kan de laatste vervallen, zoodat er nog $\nu - 1$ vergelijkingen overblijven. De functies $\psi, \ldots \psi_{\nu-1}$ zijn namelijk zoo bepaald, dat wanneer wij met behulp van de eerste $\nu - 1$ vergelijkingen v_1 en u_1 zoo bepalen, dat $\Delta^{\nu-1} v_1 = 0$ en $\Delta u_1 = 0$, aan de laatste vanzelf voldaan is. Het vraagstuk u_1 en v_1 uit deze gegevens te bepalen komt overeen met het vraagstuk u_0 en v_0 te bepalen uit de gegevens over u met vervanging van ν door $\nu - 1$. Vervolgens stellen wij $v_1 = r^2 v_2 + u_2$ waarbij $\Delta^{\nu-2} v_2 = 0$ en $\Delta u_2 = 0$, de bepaling van v_2 en u_2 geschiedt dan weer op de zelfde wijze, waarbij de reeds gebruikte gegevens aan den rand ons de noodige $\nu - 2$ vergelijkingen leveren. Zoo gaan we door. Ten laatste komt er $v_{\nu-2} = r^2 v_{\nu-1} + u_{\nu-1}$ met $\Delta v_{\nu-1}$ en $\Delta u_{\nu-1} = 0$.

Het stelsel vergelijkingen, dat wij uit de reeds gebruikte gegevens aan C afleiden (analoog aan de vergelijkingen 21 en 22 in § 3) is dan gelijkwaardig met één vergelijking, daar voegen wij nu de nog niet gebruikte laatste van het stelsel vergelijkingen, dat de gegevens aan C uitdrukt, het stelsel van het begin van § 5, bij ,analoog aan vergelijking 23). Het zoo verkregen stelsel vergelijkingen behandelen we geheel als het stelsel, gevormd door 21, 22 en 23, waardoor we weer geraken tot een vergelijking van POINCARÉ, waaruit we $v_{\nu-1,c}$ oplossen, zoodat $v_{\nu-1}$ en $u_{\nu-1}$ en daardoor ten slotte u bekend worden.

Het geval, dat een der integraalvergelijkingen, die wij bij de behandeling ontmoeten onoplosbaar blijkt, kan zich niet voordoen, anders zouden wij gegevens kunnen construeeren, waarbij het vraagstuk oneindig veel oplossingen toelaat, wat zooals wij weten onmogelijk is.

§ 6. Wanneer we Δ opvatten als den operator van LAPLACE in drie veranderlijken, heeft de vergelijking $\Delta^k u = 0$ een binnen S geldende (ondubbelzinnig bepaalde *) oplossing, waarbij $u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2} \cdots \frac{\partial^{k-1} u}{\partial n^{k-1}}$ in de punten van het gegeven gesloten oppervlak S, dat in ieder punt een bepaald raakvlak heeft, gegeven waarden aannemen. Wanneer we, analoog aan het voorgaande, daarbij aannemen, dat, wanneer we in een willekeurig punt van het oppervlak een rechthoekig coördinatenstelsel aanbrengen met de normaal als z-as, in de omgeving van den oorsprong de z van een punt van het oppervlak een analytische functie is van x en y, zoodat we door

*) Cf. H. BREMEKAMP, Over de bepaaldheid der oplossingen van $\Delta^k u = 0$, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 48, 222 (1945).

een draaiing van het coördinatenstelsel om de z-as kunnen krijgen

$$z = ax^2 + by^2 + \ldots,$$

dan hebben we in het voorgaande bijna niets te veranderen. Alleen zullen we den aard der singulariteit der te gebruiken functies van GREEN opnieuw moeten vastleggen. Wij zullen het in § 4 behandelde voor de nu beschouwde vergelijking moeten wijzigen.

Wij bepalen nu een functie $H_\nu\,(P, Q)$ door de volgende eischen:

1°. Als P en Q willekeurige punten binnen het oppervlak S zijn, geldt voor $P \neq Q$, $\triangle^\nu H_\nu(P, Q) = 0$.

2°. Als $P \to Q$ kan men stellen $H_\nu\,(P, Q) = \varrho^{2\nu-3} + h_\nu\,(P, Q)$, waarbij ϱ den afstand PQ aanwijst en $h_\nu\,(P, Q)$ doorloopende partiëele afgeleiden tot die van de orde 2ν inclusief heeft, h_ν voldoet dus in het geheele gebied binnen S aan $\triangle^\nu h_\nu = 0$.

3°. $H_\nu\,(P, Q), \frac{\partial H_\nu\,(P, Q)}{\partial n}, \frac{\partial^2 H_\nu\,(P, Q)}{\partial n^2} \ldots \frac{\partial^{\nu-1} H_\nu\,(P, Q)}{\partial n^{\nu-1}}$ gaan naar nul, als P naar eenig punt van S gaat.

De hulpstelling

$$\int (U \triangle^\nu U' - U' \triangle^\nu U)\, d\tau = \int \Big(U \frac{\partial \triangle^{\nu-1} U'}{\partial n} - \frac{\partial U}{\partial n} \triangle^{\nu-1} U' + \\ + \triangle U \frac{\partial \triangle^{\nu-1} U'}{\partial n} - \ldots . \triangle^{\nu-1} U \frac{\partial U'}{\partial n} - \frac{\partial \triangle^{\nu-1} U}{\partial n} U' \Big) d\sigma$$

analoog aan de hulpstelling van § 4 wordt op geheel de zelfde manier bewezen. Hierin nemen wij nu voor U de gevraagde functie u, voor U' de functie H_ν en voor het integratiegebied het gebied tusschen het gegeven oppervlak S en een bol om Q met straal δ zoo klein, dat de bol geheel binnen S ligt. Het eerste lid wordt dan nul. In het tweede lid moet de integraal over den bol van teeken omgekeerd worden als we de normaal in de richting naar buiten nemen, waardoor differentiatie naar n op het zelfde neerkomt als differentiatie naar ϱ. De integraal over den bol blijkt weer onafhankelijk van δ en wij berekenen die door δ tot nul te laten naderen. Wij hebben dan aan den bol, als we alleen de grootste termen opschrijven

$$\frac{\partial H_\nu}{\partial \varrho} = (2\nu-3)\, \varrho^{2\nu-4} + \ldots .$$

$$\triangle H_\nu = (2\nu-3)(2\nu-2)\, \varrho^{2\nu-5} + \ldots .$$

$$\triangle^2 H_\nu = (2\nu-5)(2\nu-4)(2\nu-3)(2\nu-2)\, \varrho^{2\nu-7} + \ldots .$$

. . . . . .

$$\triangle^{\nu-1} H_\nu = (2\nu-2)! \frac{1}{\varrho} + \ldots .$$

$$\frac{\partial \triangle^{\nu-1} H^\nu}{\partial \varrho} = -(2\nu-2)! \frac{1}{\varrho^2} + \ldots .$$

Voor de integraal over den bol in het tweede lid van onze hulpstelling levert dus alleen de eerste term van de integrand iets op en wel $-(2\nu-2)!\,4\pi U(Q)$. Wij vinden dus voor $\nu = 2\mu$

$$4(2\nu-2)!\,\pi u(Q) = -\int_S \left(u\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n} - \frac{\partial u}{\partial n}\triangle^{\nu-1} H_\nu + \right.$$

$$\left. + \triangle u \frac{\partial \triangle^{\nu-2} H_\nu}{\partial n} - \ldots . - \frac{\partial \triangle^{\mu-1} u}{\partial n}\triangle^{\mu} H_\nu \right) d\sigma$$

en voor $\nu = 2\mu + 1$.

$$4(2\nu-2)!\,\pi u(Q) = -\int_S \left(u\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n} - \frac{\partial u}{\partial n}\triangle^{\nu-1} H_\nu + \right.$$

$$\left. + \triangle u \frac{\partial \triangle^{\nu-2} H_\nu}{\partial n} - \ldots . - \frac{\partial \triangle^{\mu-1} u}{\partial n}\triangle^{\mu} H_\nu + \triangle^{\mu} u \frac{\partial \triangle^{\mu} H_\nu}{\partial n} \right) d\sigma.$$

Zijn nu aan S de waarden van u, $\frac{\partial u}{\partial n}, \ldots \frac{\partial^{\nu-1} u}{\partial n^{\nu-1}}$ gegeven, dan kunnen wij daaruit de waarden vinden van alle partiëele afgeleiden tot inclusief, die van de orde $\nu-1$. Daardoor worden voor $\nu = 2\mu$ ook $\frac{\partial u}{\partial n}$, $\triangle u$, $\frac{\partial \triangle u}{\partial n} \ldots \frac{\partial \triangle^{\mu-1} u}{\partial n}$ bekend en voor $\nu = 2\mu + 1$, $\frac{\partial u}{\partial n}$, $\triangle u$, ... $\triangle^{\mu} u$. Door de laatste formules, die het analogon zijn van (25) en (26) wordt de waarde van u in een willekeurig punt binnen S uitgedrukt in de gegevens aan S.

Overigens loopt het bewijs geheel zooals voor het in de vorige §§ behandelde geval.

Mathematics. — *Over de oplossingen der vergelijking* $\triangle\triangle u = 0$, *die aan zekere randvoorwaarden voldoen.* By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 23, 1946.)

§ 1. Het is bekend, dat de functie u binnen een gesloten kromme zonder dubbelpunten, die in ieder punt één bepaalde raaklijn heeft, ondubbelzinnig bepaald is, als zij in dat gebied voldoet aan de vergelijking $\triangle\triangle u = 0$, en aan de randkromme u en $\frac{\partial u}{\partial n}$ gegeven waarden aannemen.

Analoog aan de wijze, waarop men de oplossing der vergelijking van LAPLACE voor het gebied binnen een dergelijke kromme in de randwaarden kan uitdrukken met behulp der, voor de gegeven kromme te bepalen, functie van GREEN, kan men de hier bedoelde functie u voorstellen door een bepaalde integraal langs de grenskromme, als men een op dit probleem betrekking hebbende functie van GREEN invoert. Dat geschiedt door de volgende definitie:

De functie $H(P, Q)$ waarbij P en Q punten zijn binnen de gegeven kromme C, voldoet, beschouwd als functie van de coördinaten van $P(x, y)$ als $P \neq Q$ aan $\triangle\triangle H = 0$ en als $P \to Q$ is $H = PQ^2 \ln PQ + h(P, Q)$, waarbij $h(P, Q)$ doorloopende partiëele afgeleiden heeft tot die van de vierde orde inclusief, zoodat dus in het geheele gebied binnen C $\triangle\triangle h = 0$. Voor een willekeurig punt Q binnen de kromme gaan $H(P, Q)$ en $\frac{\partial}{\partial n} H(P, Q)$ naar nul, als P tot eenig punt der randkromme nadert.

Het construeeren der functie H komt klaarblijkelijk neer op het construeeren der functie h, dus op een bijzonder geval van het boven genoemde algemeene vraagstuk. Daarmee zullen wij ons hier niet bezig houden. Om de gevraagde functie u uit te drukken met behulp der functie H leiden we vooreerst uit het theorema van GREEN

$$\int (V \triangle U - U \triangle V)\, d\sigma = \int \left(V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n} \right) ds,$$

waarbij de integraal in het eerste lid genomen wordt over het gebied binnen de kromme C en die in het tweede lid over den rand, een hulpstelling af. Wij nemen $V = \triangle U'$; daar voor de toepassing van het theorema van GREEN van de functies U en V verlangd moet worden, dat ze doorloopende partiëele afgeleiden van de tweede orde hebben, zullen wij van U' eischen, dat de partiëele afgeleiden tot die van de vierde orde inclusief doorloopend zijn.

Wij vinden:

$$\int (\triangle U \triangle U' - U \triangle \triangle U')\, d\sigma = \int \left(\triangle U' \frac{\partial U}{\partial n} - U \frac{\partial \triangle U'}{\partial n} \right) ds.$$

Onderstellen wij nu, dat ook U doorloopende partiëele afgeleiden inclusief die van de vierde orde heeft, dan kunnen wij hierin U en U' verwisselen, hetgeen geeft

$$\int (\triangle U \triangle U' - U' \triangle \triangle U)\, d\sigma = \int \left(\triangle U \frac{\partial U'}{\partial n} - U' \frac{\partial \triangle U}{\partial n} \right) ds.$$

Door aftrekking der beide laatste formules vinden wij nu de bedoelde hulpstelling

$$\int (U \triangle \triangle U' - U' \triangle \triangle U) d\sigma = \int \left(U \frac{\partial \triangle U'}{\partial n} - \frac{\partial U}{\partial n} \triangle U' + \triangle U \frac{\partial U'}{\partial n} - \right.$$
$$\left. - \frac{\partial \triangle U}{\partial n} U' \right) ds.$$

Nemen wij nu hierin voor U de gevraagde functie u, voor U' de functie $H(P, Q)$ en voor het integratiegebied het gebied tusschen C en een cirkel om Q waarvan de straal δ zoo klein is, dat de cirkel geheel binnen C valt, dan wordt het linkerlid nul; de integraal rechts moet genomen worden langs C in positieven, langs den cirkel in negatieven zin. Voor de integraal langs C leveren de twee laatste termen in de integrand niets op wegens de definitie van H. Om de integraal over den cirkel te vinden, merken we vooreerst op, dat deze onafhankelijk is van δ. Stellen we weer $H = \varrho^2 \ln \varrho + h$, waarbij ϱ den afstand QP beteekent, dan zien we door δ onbepaald te laten afnemen, dat de termen, die van h afkomstig zijn niets opleveren. Verder is

$$\frac{\partial \varrho^2 \ln \varrho}{\partial n} = \varrho (2 \ln \varrho + 1),\ \triangle \varrho^2 \ln \varrho = 4 \ln \varrho + 4,\ \frac{\partial}{\partial n} \triangle \varrho^2 \ln \varrho = \frac{4}{\varrho},$$

de bijdrage van den eersten term in de integrand tot de integraal over den cirkel nadert dus, als $\delta \to 0$, tot $-8\pi u(Q)$, die van de overige termen tot nul. Wij vinden dus

$$8\pi u(Q) = \int_C \left(u \frac{\partial \triangle H}{\partial n} - \frac{\partial n}{\partial n} \triangle H \right) ds. \quad . \quad . \quad . \quad . \quad (1)$$

§ 2. Voor het geval, dat de gegeven kromme een cirkel is, kunnen wij de functie H vrij gemakkelijk min of meer probeerenderwijs construeeren. Wij maken daarbij gebruik er van, dat iedere functie u, die binnen den cirkel voldoet aan $\triangle\triangle u = 0$ geschreven kan worden in de gedaante

$$u = v + l^2 w \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

waarbij v en w binnen den cirkel voldoen aan $\triangle v = 0$, $\triangle w = 0$, en l den afstand tot een willekeurig punt beteekent. Wij hebben nu

$$H(P, Q) = \varrho^2 \ln \varrho + h(P, Q)$$

waarbij ϱ den afstand QP beteekent en h in het geheele gebied binnen den cirkel voldoet aan $\triangle\triangle h = 0$. Wij noemen het middelpunt van den cirkel O

en leggen onze x-as door O. De straal van den cirkel zij R en $OQ = a$. Wij bepalen op de x-as het punt Q_1 zoo, dat $OQ_1 = a_1 = \frac{R^2}{a}$. Den afstand van een willekeurig punt P tot Q_1 noemen we ϱ_1, zoodat volgens een bekende meetkundestelling, als P op den cirkel ligt, $\frac{\varrho_1}{\varrho} = \frac{R}{a}$. Wij stellen nu

$$H = \varrho^2 \ln \varrho - \frac{a^2}{R^2}\varrho_1^2 \ln \varrho_1 + \varrho^2 \ln \frac{R}{a} + (R^2 - r^2) F.$$

$$= \varrho^2 \ln \varrho \frac{R}{a} - \frac{a^2}{R^2}\varrho_1^2 \ln \varrho_1 + (R^2 - r^2) F.$$

Hierdoor is voldaan aan de voorwaarde $H = 0$ aan den cirkel. Aan $\triangle^2 H = 0$ is voldaan, als $\triangle F = 0$. De voorwaarde $\frac{\partial H}{\partial n} = 0$ aan den cirkel zal ons nu de waarden van F aan den cirkel geven en daardoor, tezamen met $\triangle F = 0$, is F in het geheele gebied binnen den cirkel bepaald. Voor de voorwaarden, $\frac{\partial H}{\partial n} = 0$ aan den cirkel, kunnen wij schrijven $\frac{\partial H}{\partial r} = 0$ voor $r = R$. Nu is $\varrho \frac{\partial \varrho}{\partial r} = r - a \cos \vartheta$, dus aan den cirkel $\varrho \frac{\partial \varrho}{\partial r} = R - a \cos \vartheta$ en $\varrho_1 \frac{\partial \varrho_1}{\partial r} = r - a_1 \cos \vartheta$, dus aan den cirkel $\varrho_1 \frac{\partial \varrho_1}{\partial r} = \frac{R}{a}(a - R \cos \vartheta)$. Dus moet aan den cirkel

$$0 = \left(2 \ln \varrho \frac{R}{a} + 1\right)(R - a \cos \vartheta) - \frac{a}{R}(2 \ln \varrho_1 + 1)(a - R \cos \vartheta) - 2RF$$

dus

$$F = \frac{R^2 - a^2}{2R^2}(2 \ln \varrho_1 + 1)$$

en daar de uitdrukking in het rechterlid binnen den cirkel voldoet aan $\triangle F = 0$, geldt zij in dat geheele gebied. Wij vinden dus

$$H = \varrho^2 \ln \varrho \frac{R}{a} - \frac{a^2}{R^2}\varrho_1^2 \ln \varrho_1 + \frac{(R^2 - a^2)(R^2 - r^2)}{2R^2}(2 \ln \varrho_1 + 1) \quad . \qquad (3)$$

§ 3. Deze uitdrukking willen wij in de formule (1) substitueeren, waartoe we nog de waarden van $\triangle H$ en $\frac{\partial \triangle H}{\partial n}$ in de punten van den cirkel moeten berekenen. Wij vinden

$$\triangle \varrho^2 \ln \varrho = 4(\ln \varrho + 1),\ \triangle \varrho^2 = 4,\ \triangle \varrho_1^2 \ln \varrho_1 = 4(\ln \varrho_1 + 1),\ \triangle \varrho_1^2 = 4,$$

$$\triangle r^2 \ln \varrho_1 = 4 \ln \varrho_1 + \frac{4(r^2 - ra_1 \cos \vartheta)}{\varrho_1^2}$$

en dus

$$\triangle H = 4 \ln \varrho - 4 \ln \varrho_1 - \frac{4(R^2 - a^2)}{R^2} \frac{r^2 - r a_1 \cos \vartheta}{\varrho_1^2} + 2 \frac{R^2 - a^2}{R^2} + 4 \ln \frac{R}{a}.$$

en

$$\frac{\partial \triangle H}{\partial n} = \frac{4(R - a \cos \vartheta)}{\varrho^2} - \frac{4(R - a_1 \cos \vartheta)}{\varrho_1^2} - 4 \frac{R^2 - a^2}{R^2} \frac{2R - a_1 \cos \vartheta}{\varrho_1^2} + \\ + 8 \frac{R^2 - a^2}{R} \frac{(R - a_1 \cos \vartheta)^2}{\varrho_1^4}.$$

en aan den cirkel

$$\triangle H = 2 \frac{(R^2 - a^2)^2}{R^2} \frac{1}{\varrho^2} \quad \text{en} \quad \frac{\partial \triangle H}{\partial n} = \frac{4(R^2 - a^2)^2}{R^2} \frac{R - a \cos \vartheta}{\varrho^4}.$$

Denken wij ons nu de gegeven waarden aan den cirkel voorgesteld door $u = f(\vartheta), \frac{\partial u}{\partial n} = g(\vartheta)$; dan geeft formule (1)

$$u(Q) = \frac{(R^2 - a^2)^2}{2\pi R} \int_0^{2\pi} \frac{(R - a \cos \vartheta) f(\vartheta)}{(R^2 + a^2 - 2Ra \cos \vartheta)^2} d\vartheta - \\ - \frac{(R^2 - a^2)^2}{4\pi R} \int_0^{2\pi} \frac{g(\vartheta)}{R^2 + a^2 - 2Ra \cos \vartheta} d\vartheta. \tag{4}$$

Men kan in deze formule een analogon zien van de bekende integraal van POISSON uit de potentiaaltheorie.

Daar voor $g = 0$, $f = 1$ klaarblijkelijk $u = 1$, vindt men

$$\int_0^{2\pi} \frac{R - a \cos \vartheta}{(R^2 + a^2 - 2Ra \cos \vartheta)^2} d\vartheta = \frac{2\pi R}{(R^2 - a^2)^2}, \tag{5}$$

wat zich door directe berekening laat bevestigen.

§ 4. Wij kunnen de formule (4) ook direct afleiden, zonder gebruik te maken van de formule (1) en de functie H. Daartoe stellen wij

$$u = v + r^2 w,$$

waarbij v en w harmonische functies zijn. Wij kunnen dus stellen

$$u = a_0 r^2 + b_0 R^2 + \sum_1^\infty \{(a_{n,1} r^2 + b_{n,1} R^2) r^n \cos n\vartheta + (a_{n,2} r^2 + b_{n,2} R^2) r^n \sin n\vartheta\}$$

Om aan de gegeven grensvoorwaarden te voldoen, moet dan

$$f(\vartheta) = (a_0 + b_0) R^2 + \sum_1^\infty \{(a_{n,1} + b_{n,1}) R^{n+2} \cos n\vartheta + (a_{n,2} + b_{n,2}) R^{n+2} \sin n\vartheta\}$$

en

$$g(\vartheta) = 2a_0 R + \sum_1^\infty \{((n+2) a_{n,1} + n b_{n,1}) R^{n+1} \cos n\vartheta + \\ + ((n+2) a_{n,2} + n b_{n,2}) R^{n+1} \sin n\vartheta\}$$

Aannemende, dat de functies $f(\vartheta)$ en $g(\vartheta)$ een FOURIER-ontwikkeling toelaten, vinden wij dus

$$(a_0 + b_0)\, R^2 = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi \quad , (a_{n,1} + b_{n,1})\, R^{n+2} = \frac{1}{\pi}\int_0^{2\pi} f(\varphi) \cos n\varphi\, d\varphi \ ,$$

$$(a_{n,2} + b_{n,2})\, R^{n+2} = \frac{1}{\pi}\int_0^{2\pi} f(\varphi) \sin n\varphi\, d\varphi,$$

$$2a_0\, R = \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\, d\varphi, \ ((n+2)\, a_{n,1} + n b_{n,1})\, R^{n+1} = \frac{1}{\pi}\int_0^{2\pi} g(\varphi) \cos n\varphi\, d\varphi,$$

$$((n+2)\, a_{n,2} + n\, b_{n,2})\, R^{n+1} = \frac{1}{\pi}\int_0^{2\pi} g(\varphi) \sin n\varphi\, d\varphi.$$

en dus

$$a_0 = \frac{1}{4\pi R^2}\int_0^{2\pi} R\, g(\varphi)\, d\varphi, \ b_0 = \frac{1}{4\pi R^2}\int_0^{2\pi} \{2 f(\varphi) - R\, g(\varphi)\}\, d\varphi,$$

$$a_{n,1} = \frac{1}{2\pi R^{n+2}}\int_0^{2\pi} \{R\, g(\varphi) - n\, f(\varphi)\} \cos n\varphi\, d\varphi,$$

$$b_{n,1} = \frac{1}{2\pi R^{n+2}}\int_0^{2\pi} \{-R\, g(\varphi) + (n+2)\, f(\varphi)\} \cos n\varphi\, d\varphi,$$

$$a_{n,2} = \frac{1}{2\pi R^{n+2}}\int_0^{2\pi} \{Rg(\varphi) - n\, f(\varphi)\} \sin n\varphi\, d\varphi,$$

$$b_{n,2} = \frac{1}{2\pi R^{n+2}}\int_0^{2\pi} \{Rg(\varphi) + (n+2)\, f(\varphi)\} \sin n\varphi\, d\varphi.$$

Dus ten slotte

$$u = \frac{r^2 - R^2}{4\pi R^2}\int_0^{2\pi} Rg(\varphi)\, d\varphi + \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi +$$

$$+ \frac{1}{2\pi R^2}\sum_1^{\infty}\left[(r^2 - R^2)\int_0^{2\pi} \{Rg(\varphi) - n\, f(\varphi)\} \cos n\varphi\, d\varphi + 2R^2\int_0^{2\pi} f(\varphi) \cos n\varphi\, d\varphi\right]\frac{r^n}{R^n}\cos n\vartheta +$$

$$+ \frac{1}{2\pi R^2}\sum_1^{\infty}\left[(r^2 - R^2)\int_0^{2\pi} \{Rg(\varphi) - nf(\varphi)\} \sin n\varphi\, d\varphi + \right.$$

$$\left. + 2R^2\int_0^{2\pi} f(\varphi) \sin n\varphi\, d\varphi\right]\frac{r^n}{R^n}\sin n\vartheta.$$

Zijn nu de functies f en φ van dien aard, dat hierin sommatie en integratie mogen worden verwisseld, dan herleiden wij dit tot

$$u = \frac{1}{2\pi R^2} \int_0^{2\pi} [\{(r^2 - R^2) R g(\varphi) + 2R^2 f(\varphi)\} (\tfrac{1}{2} + \sum_1^\infty \frac{r^n}{R^n} \cos n \quad - \vartheta -$$

$$- (r^2 - R^2) f(\varphi) \sum_1^\infty n \frac{r^n}{R^n} \cos n (\varphi -$$

Nu is

$$\frac{1}{2} + \sum_1^\infty \frac{r^n}{R^n} \cos n (\varphi - \vartheta) = \frac{R^2 - r^2}{2\{R^2 + r^2 - 2Rr \cos(\varphi - \vartheta}$$

$$\sum_1^\infty n \frac{r^n}{R^n} \cos n (\varphi - \vartheta) = \frac{Rr\{(R^2 + r^2) \cos(\varphi - \vartheta) - 2Rr}{\{R^2 + r^2 - 2Rr \cos(\varphi - \vartheta)}.$$

dus

$$\left.\begin{aligned} u = \frac{(R^2 - r^2)^2}{2\pi R} \int_0^{2\pi} \frac{\{R - r\cos(\varphi - \vartheta)\} f(\varphi)}{\{R^2 + r^2 - 2Rr\cos(\varphi - \vartheta)\}^2} d\varphi - \\ - \frac{(R^2 - r^2)^2}{4\pi R} \int_0^{2\pi} \frac{g(\varphi)}{R^2 + r^2 - 2Rr\cos(\varphi - \vartheta)} d\varphi. \end{aligned}\right\} \quad \ldots \quad (6)$$

Om tot de notatie van § 3 terug te komen, moeten we r door a en ϑ door 0 vervangen, waardoor we de vergelijking (4) terugvinden.

Heeft men op deze wijze de formule (4) bewezen, dan kan men ook het omgekeerde doen van wat in §§ 2 en 3 gedaan is, namelijk uit de laatste formule weer de formule (3) afleiden. Door (4) en (1) te vergelijken vinden we, dat aan den cirkel

$$\triangle H = 2 \frac{(R^2 - r^2)^2}{R^2} \frac{1}{\varrho^2}, \text{ dus } \triangle h = 2 \frac{(R^2 - r^2)^2}{R^2} \frac{1}{\varrho^2} - 4(ln\, \varrho + 1).$$

Daar $\triangle h$ binnen den cirkel een harmonische functie is, kan die dus b.v. met behulp van de integraal van POISSON berekend worden. Daarna kan h gevonden worden doordat binnen den cirkel $\triangle h$ en op den cirkel $h = -\varrho^2 ln\, \varrho$ bekend zijn. Daardoor wordt ook H bekend.

§ 5. Wij hebben in het vorige slechts bewezen, dat, aannemende dat er een oplossing is van $\triangle\triangle u = 0$, waarbij aan den cirkel

$$u = f(\varphi), \frac{\partial u}{\partial n} = g(\varphi),$$

deze door de formule (6) wordt voorgesteld. Ons rest nog, te bewijzen, dat de door (6) gegeven functie inderdaad aan alle eischen van het vraagstuk voldoet.

Om te bewijzen, dat elk der termen van (6) en dus ook hun verschil aan $\triangle\triangle = 0$ voldoet, hebben we, opmerkend, dat we met eigenlijke integralen

te doen hebben, zoodat differentiaties onder het integraalteeken kunnen worden uitgevoerd, slechts te doen zien, dat

$$\frac{(R^2-r^2)^2\,(R-r\cos\vartheta)}{\varrho^4} \text{ en } \frac{(R^2-r^2)^2}{\varrho^2}$$

aan die vergelijking voldoen. Dat vereischt slechts eenig, nogal omvangrijk, rekenwerk, dat nog eenigszins vereenvoudigd kan worden door toepassing van de stelling, dat $\triangle\triangle r^2 v = 0$, als $\triangle v = 0$.

In de tweede plaats moeten wij aantoonen, dat als het punt Q, waar u door de formule (6) bepaald is, op willekeurige wijze tot een punt van den cirkel nadert, u nadert tot de in dat punt gegeven waarde $f(\vartheta)$ en $\frac{\partial u}{\partial r}$ tot de daar gegeven waarde $g(\vartheta)$. Wij zullen dat bewijs geven voor het geval, dat f en g in dat punt continue functies voor ϑ zijn. Daar we in het vorige de x-as willekeurig kunnen kiezen, kunnen wij die leggen door het beschouwde punt, zoodat voor dat punt $\vartheta = 0$ wordt en wij met de formule (4) te doen krijgen. Wij hebben voor dit bewijs slechts een uit de theorie der vergelijking van LAPLACE bekende redeneering na te volgen.

Wij beschouwen vooreerst den eersten term in (4) en schrijven

$$\frac{(R^2-r^2)^2}{2\pi R}\int_0^{2\pi}\frac{R-r\cos\varphi}{\varrho^4}f(\varphi)\,d\varphi = \frac{(R^2-r^2)^2}{2\pi R}f(0)\int_0^{2\pi}\frac{R-r\cos\varphi}{\varrho^4}d\varphi +$$

$$+\frac{(R^2-r^2)^2}{2\pi R}\int_0^{2\pi}\{f(\varphi)-f(0)\}\,\frac{R-r\cos\varphi}{\varrho^4}d\varphi =$$

$$= f(0) + \frac{(R^2-r^2)^2}{2\pi R}\int_0^{2\pi}\{f(\varphi)-f(0)\}\,\frac{R-r\cos\varphi}{\varrho^4}d\varphi,$$

de laatste herleiding door toepassing der formule (5). Is nu ε een willekeurig positief getal, dan kunnen wij wegens de continuïteit van f, δ zoo bepalen, dat voor $|\varphi| < \delta$, $|f(\varphi) - f(0)| < \varepsilon$. Wij bepalen nu op den cirkel het punt A met poolhoek δ en het punt B met poolhoek $-\delta$, zoodat de koorde AB loodrecht op de x-as staat, haar snijpunt met de x-as noemen we C, het middelpunt van den cirkel O. Wij verdeelen nu de integraal in de laatste formule in twee stukken, de integraal van $+\varepsilon$ tot $2\pi - \varepsilon$, die dus te nemen is over den boog van A over het snijpunt met de negatieve x-as naar B en de integraal over het overblijvende deel van den cirkel. Het laatste deel is in absolute waarde kleiner dan

$$\varepsilon\,\frac{(R^2-r^2)^2}{2\pi R}\int_{-\varepsilon}^{\varepsilon}\frac{R-r\cos\varphi}{\varrho^4}d\varphi.$$

Deze uitdrukking vergrooten wij nog, als wij de integraal over den geheelen cirkel nemen, dus φ van 0 tot 2π laten gaan, dan komt er volgens

het voorgaande ε. Wij gaan nu over tot de beschouwing van het eerste deel. Daarbij trekken wij nog een tweede koorde A_1B_1 loodrecht op de x-as, die de x-as snijdt in C_1 zoo gelegen, dat $R - OC_1 = h$. Wij denken ons Q gelegen in het kleinste door deze koorde afgesneden segment en bewijzen, dat dan dat deel met h tot nul nadert, dat is dus als Q op willekeurige wijze tot het snijpunt van den cirkel met de x-as nadert.

Wij hebben

$$o < R^2 - r^2 < R^2 - OC_1^2 < 2Rh, \varrho > AA_1 > CC_1 = R(1 - \cos\delta) - h,$$
$$o < R - r\cos\varphi < h.$$

Is dus M de grootste waarde van $f(\varphi) - f(0)$, dan is

$$\left| \frac{(R^2 - r^2)^2}{2\pi R} \int_{\varepsilon}^{2\pi-\varepsilon} \{ f(\varphi) - f(o) \} \frac{R - r\cos\varphi}{\varrho^4} d\varphi \right| < \frac{RMh^3}{(2R\sin^2 \tfrac{1}{2}\delta - h)^4},$$

waaruit het beweerde volgt.

De tweede term in (4) kan op de zelfde manier behandeld worden, waarbij blijkt, dat die tot nul nadert als Q naar P gaat. Trouwens deze term is $\frac{R^2 - r^2}{2R}$ maal een uitdrukking, waarvan uit de theorie van de integraal van POISSON bekend is, dat die onder de beschouwde omstandigheden tot $g(0)$ nadert.

Wij gaan nu over tot de beschouwing van $\frac{\partial u}{\partial r}$, waarvoor (4) geeft

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \int_0^{2\pi} \frac{\partial \psi}{\partial r} f(\varphi)\, d\varphi - \int_0^{2\pi} \frac{\partial}{\partial r} g(\varphi) d\varphi, \\ \text{waarin } \psi &= \frac{(R^2 - r^2)^2 (R - r\cos\varphi)}{2\pi R \varrho^4}, \quad \chi = \frac{(R^2 - r^2)^2}{4\pi R \varrho^2} \end{aligned}\right\} \qquad (7)$$

Door te beschouwen $u = 1$, waarbij $f = 1$ en $g = 0$ vinden we $\int \frac{\partial \psi}{\partial r} d\varphi = 0$

Door te beschouwen $u = \frac{r^2 - R^2}{2R}$, waarbij $f = 0$ en $g = 1$,

$$\frac{r}{R} = -\int_0^{2\pi} \frac{\partial \chi}{\partial r} d\varphi.$$

Wij beschouwen nu weer vooreerst den eersten term in (7) en schrijven

$$\int_0^{2\pi} \frac{\partial \psi}{\partial r} f(\varphi)\, d\varphi = f(o) \int_0^{2\pi} \frac{\partial \psi}{\partial r} d\varphi + \int_0^{2\pi} \{ f(\varphi) - f(0) \} \frac{\partial \psi}{\partial r} \partial\varphi =$$

$$= \int_0^{2\pi} \{ f(\varphi) - f(o) \} \frac{\partial \psi}{\partial r} d\varphi.$$

De laatste integraal splitsen wij op de zelfde manier als bij de voorgaande beschouwing is uiteengezet in twee deelen. Het deel dat betrekking heeft op den kleinsten boog BA is weer in absolute waarde kleiner dan ε. Voor de beschouwing van het andere deel hebben we

$$\frac{\partial \psi}{\partial r} = -\frac{2r(R^2-r^2)(R-r\cos\varphi)}{\pi R \varrho^4} - \frac{(R^2-r^2)^2\cos\varphi}{2\pi R\varrho^4} - \\ - \frac{2(R^2-r^2)^2(R-r\cos\varphi)(r-R\cos\varphi)}{\pi R\varrho^6}.$$

Wij zien in de figuur direct dat $R-r\cos\varphi < h$ en $r-R\cos\varphi < h$ en $R\cos\varphi < R-h$, dus

$$\left|\frac{\partial\psi}{\partial r}\right| < \frac{4Rh^2}{\pi(2R\sin^2\frac{1}{2}\delta - h)^4} + \frac{2(R-h)h^2}{\pi(2R\sin^2\frac{1}{2}\delta - h)^4} + \frac{8Rh^4}{\pi(2R\sin^2\frac{1}{2}\delta - h)^6},$$

waaruit blijkt, dat dat deel van de integraal met h tot nul nadert. De eerste term in (7) nadert dus tot nul, als Q naar P gaat. Voor den tweeden term schrijven we

$$-\int_0^{2\pi}\frac{\partial\chi}{\partial r}g(\varphi)\,d\varphi = -g(o)\int_0^{2\pi}\frac{\partial\chi}{\partial r}d\varphi - \int_0^{2\pi}\{g(\varphi)-g(o)\}\frac{\partial\chi}{\partial r}d\varphi.$$

De eerste term hiervan nadert tot $g(0)$, als Q naar P gaat. Wij hebben dus nog te bewijzen, dat de tweede daarbij naar nul gaat. Daarbij verdeelen we de integraal weer in twee stukken, door nu het getal δ, en daarmee de punten A en B, zoo te bepalen, dat voor $|\varphi| < \delta$, $|g(\varphi)-g(0)| < \varepsilon$. De integraal over den kleinsten boog BA is dan weer in absolute waarde kleiner dan ε. Bij het overige deel hebben we

$$\frac{\partial\chi}{\partial r} = -\frac{r(R^2-r^2)}{\pi R\varrho^2} - \frac{(R^2-r^2)^2(r-R\cos\varphi)}{2\pi R\varrho^4},$$

dus

$$\left|\frac{\partial\chi}{\partial r}\right| < \frac{2Rh}{\pi(2R\sin^2\frac{1}{2}\delta - h)^2} + \frac{2Rh^3}{\pi(2R\sin^2\frac{1}{2}\delta - h)^4},$$

waaruit we weer de verlangde conclusie kunnen trekken.

Men kan nu trachten, het existentiebewijs voor andere krommen te geven door die, waar dat mogelijk is, door een analytische substitutie $x = g(\xi,\eta)$, $y = h(\xi,\eta)$ op een cirkel af te beelden. Daarbij gaat echter wel de vergelijking $\triangle u = 0$ over in $\triangle_{\xi\eta}u = 0$ maar de vergelijking $\triangle\triangle u = 0$ geeft $\triangle_{\xi,\eta}(\alpha\triangle_{\xi,\eta}u) = 0$, waarbij $\alpha = \left(\frac{\partial g}{\partial\xi}\right)^2 + \left(\frac{\partial g}{\partial\eta}\right)^2$. Men vindt dus een vergelijking, waarin de termen van de vierde orde de gedaante $\triangle\triangle u$ houden. Van deze vergelijking kan men oplossingen construeeren door successieve benadering en het komt nog aan op het bewijs van de convergentie van dat proces. Op deze zaak hoop ik nog terug te komen [1]).

[1]) Vgl. intusschen H. BREMEKAMP, On the Partial Differential Equations occurring in the Theory of the Elastic Plate. Nieuw Archief voor Wiskunde, Bnd. XXII.

§ 6. Wanneer het symbool $\triangle$ den operator van LAPLACE in drie afmetingen aanwijst, is de oplossing der vergelijking $\triangle\triangle u = 0$ ondubbelzinnig bepaald in de ruimte binnen een gesloten oppervlak, dat in elk van zijn punten een volkomen bepaald raakvlak heeft, als de waarden van u en $\frac{\partial u}{\partial n}$ in de punten van dat oppervlak zijn voorgeschreven. Ook hier kan men die functie door een integraal, nu over het oppervlak genomen, voorstellen, als men een functie van GREEN invoert. Deze functie, die wij weer $H(P, Q)$ zullen noemen, wordt nu als volgt gedefinieerd.

Als P en Q punten zijn binnen het gegeven oppervlak en $P \neq Q$ voldoet $H(P, Q)$ als functie van de coördinaten van $P(x, y, z)$ aan $\triangle\triangle H = 0$ en als $P \to Q$ is $H(P, Q) = PQ + h(P, Q)$, waarbij $h(P, Q)$ doorloopende partieele afgeleiden tot die van de vierde orde inclusief heeft, zoodat dus overal binnen het oppervlak $\triangle\triangle h = 0$. Voor een willekeurig punt Q binnen het oppervlak gaan $H(P, Q)$ en $\frac{\partial H(P,Q)}{\partial n}$ naar nul, als P naar eenig punt van het oppervlak gaat. Den aard van de singulariteit als $P \to Q$ vindt men door een oplossing van $\triangle\triangle H = 0$ te zoeken, die alleen afhangt van $PQ = \varrho$.

Om de functie u met behulp van H uit te drukken, leiden we geheel analoog aan hetgeen in § 1 geschied is uit het theorema van GREEN de volgende hulpstelling af

$$\int (U \triangle\triangle U' - U' \triangle\triangle U)\, d\tau = \int \left(U \frac{\partial \triangle U'}{\partial n} - \frac{\partial U}{\partial n} \triangle U' + \right.$$
$$\left. + \triangle U \frac{\partial U'}{\partial n} - \frac{\partial \triangle U}{\partial n} U' \right) d\sigma$$

waarbij de integraal in het eerste lid genomen wordt over de ruimte binnen een gesloten oppervlak, dat in ieder punt een ondubbelzinnig bepaald raakvlak heeft en die in het tweede lid over het begrenzend oppervlak. Van de functies U en U' wordt voorondersteld, dat alle hier voorkomende partiëele afgeleiden doorloopend zijn.

In deze formule nemen we voor U de gevraagde functie u en voor U' H. Voor het integratiegebied nemen we de ruimte tusschen het gegeven oppervlak en een bol om Q met straal δ. De integraal in het eerste lid is dan nul. Voor de integraal over het buitenoppervlak leveren wegens de definitie van H de laatste twee termen in de integrand niets op. Om de integraal over den bol te vinden stellen wij weer $H = \varrho + h$, waarbij als $\delta \to 0$ de bijdrage van de termen, die van h afkomstig zijn ook onbepaald afnemen en wij van te voren kunnen opmerken, dat de geheele integraal over den bol onafhankelijk is van δ. Verder hebben we $\triangle \varrho = \frac{2}{\varrho}$ dus

$$\frac{\partial}{\partial n} (\triangle \varrho) = -\frac{2}{\varrho^2}.$$

Tenslotte levert dus alleen de eerste term van de integrand iets op en wel $-8\pi u(Q)$. Wij vinden dus tenslotte

$$8\pi u(Q) = -\int \left(u \frac{\partial \triangle H}{\partial n} - \frac{\partial u}{\partial n} \triangle H \right) d\sigma. \quad . \quad . \quad . \quad (7)$$

§ 7. Voor het geval, dat het gegeven oppervlak een bol is, kunnen we de functie H op een dergelijke manier vinden als voor den cirkel. Ook hier kan iedere oplossing van $\triangle\triangle u = 0$ geschreven worden in de gedaante

$$u = v + l^2 w,$$

waarbij $\triangle v = \triangle w = 0$ en l den afstand tot een willekeurig vast punt aanwijst. Wij noemen het middelpunt van den bol O, den straal R, den afstand $OQ = a$, ϱ den afstand van een willekeurig punt tot Q, Q_1 het punt op OQ gelegen zóó, dat $OQ_1 = a_1 = \frac{R^2}{a}$ en ϱ_1 den afstand van een willekeurig punt tot Q_1.

Analoog aan hetgeen we in § 2 deden, stellen wij nu

$$H = \varrho - \frac{a}{R}\varrho_1 + (R^2 - r^2) F_1$$

waardoor aan den bol $H = 0$ wordt en aan $\triangle\triangle H = 0$ voldaan is, als $\triangle F = 0$. De voorwaarde $\frac{\partial H}{\partial n} = 0$ aan den bol, waarvoor we weer kunnen schrijven $\frac{\partial H}{\partial r} = 0$, geeft weer de waarden van F aan den bol, waardoor tezamen met $\triangle F = 0$, F in het geheele gebied binnen den bol bepaald is. Wij vinden aan den bol

$$0 = \frac{1}{\varrho}(R - a\cos\vartheta) - \frac{1}{\varrho_1}(a - R\cos\vartheta) - 2RF,$$

dus

$$F = \frac{R^2 - a^2}{2aR}\frac{1}{\varrho_1}$$

en daar dit binnen den bol voldoet aan $\triangle F = 0$, geldt deze uitdrukking ook voor dat geheele gebied. Wij vinden dus

$$H = \varrho - \frac{a}{R}\varrho_1 + \frac{(R^2 - a^2)(R^2 - r^2)}{2aR}\frac{1}{\varrho_1}. \quad (8)$$

Voor de substitutie in (7) berekenen we

$$\triangle\varrho = \frac{2}{\varrho}, \triangle\varrho_1 = \frac{2}{\varrho_1}, \triangle\frac{1}{\varrho_1} = 0, \triangle\frac{r^2}{\varrho_1} = \frac{6}{\varrho_1} - \frac{4r^2}{\varrho_1^3} + \frac{4R^2 r\cos\vartheta}{a\varrho_1^3}$$

dus

$$\triangle H = \frac{2}{\varrho} - \frac{2a}{R}\frac{1}{\varrho_1} - \frac{R^2 - a^2}{aR}\left(\frac{3}{\varrho_1} - \frac{2r^2}{\varrho_1^3} + \frac{2a_1 r\cos\vartheta}{\varrho_1^3}\right)$$

waarbij evenals in het voorgaande $a_1 = \frac{R^2}{a}$,

$$\frac{\partial \triangle H}{\partial r} = -\frac{2(r - a\cos\vartheta)}{\varrho^3} + \frac{2a(r - a_1\cos\vartheta)}{R\,\varrho_1^3} + \frac{3(R^2 - a^2)(r - a_1\cos\vartheta)}{a\,R\,\varrho_1^3} +$$

$$+ \frac{4(R^2 - a^2)\,r}{a\,R\,\varrho_1^3} - \frac{6(R^2 - a^2)\,r^2\,(r - a_1\cos\vartheta)}{a\,R\,\varrho_1^5} - \frac{2(R^2 - a^2)\,a_1\cos\vartheta}{a\,R\,\varrho_1^3} +$$

$$+ \frac{6(R^2 - a^2)\,a_1\,r\cos\vartheta\,(r - a_1\cos\vartheta)}{a\,R\,\varrho_1^5}$$

Dus aan het boloppervlak

$$\triangle H = -\frac{(R^2 - a^2)^2}{R^2}\frac{1}{\varrho^3}$$

$$\frac{\partial \triangle H}{\partial n} = \frac{R^2 - a^2}{R^3\,\varrho^3}\left\{-2R^2 + 5a^2 - 3aR\cos\vartheta + \frac{-6a^4 + 12a^3R\cos\vartheta - 6a^2R^2\cos^2\vartheta}{\varrho^2}\right\} =$$

$$= -\frac{(R^2 - a^2)^2\,(2R^2 - a^2 - aR\cos\vartheta)}{R^3\,\varrho^5}$$

De formule (7) geeft dus

$$u = \frac{(R^2 - a^2)^2}{8\pi R^3}\int\left\{u\,\frac{(2R^2 - a^2 - aR\cos\vartheta)}{\varrho^5} - \frac{\partial u}{\partial n}\frac{R}{\varrho^3}\right\} d\sigma. \qquad (9)$$

Het bewijs, dat de op deze wijze geconstrueerde functies aan alle eischen voldoen, kan op geheel overeenkomstige wijze als dat in § 5 gegeven worden.

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems).* By K. MAHLER. (First communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 23, 1946.)

The star body K is called *reducible* if there exists a star body H contained in, but different from, K such that $\Delta(H) = \Delta(K)$, and is otherwise called *irreducible*. We say further that K is *boundedly reducible* if a bounded star H with $\Delta(H) = \Delta(K)$ exists. In this Part II, I prove conditions for K to be reducible (irreducible) or boundedly reducible. The irreducible star bodies, as well as the boundedly reducible star bodies, seem to have many interesting properties, and I believe that a further study might lead to results of importance in themselves and for applications to other problems.

As in Part I, I have stated a number of problems which seem to deserve further study *).

In order to make this paper intelligible to a reader who has not seen the first part, I repeat here the main definitions and theorems.

Let R_n be the n-dimensional space of all points $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, etc., with real coordinates; $O = (o, \ldots, o)$ is the origin of R_n. We put $|X| = (x_1^2 + \ldots + x_n^2)^{1/2}$, and use the notation $X + Y$ and tX for the points $(x_1 + y_1, \ldots, x_n + y_n)$ and $(tx_1, \ldots tx_n)$. The determinant $\| x_{hk} \|$ of n points $X_h = (x_{h2}, \ldots, x_{hn})$ is denoted by $\{X_1, \ldots, X_n\}$. If this determinant does not vanish, then the set of all points

$$X = u_1 X_1 + \ldots + u_n X_n \qquad (u_1, \ldots, u_n = 0, \mp 1, \mp 2, \ldots)$$

forms a lattice Λ of basis $X_1, \ldots, X_n$ and determinant

$$d(\Lambda) = |\{X_1, \ldots, X_n\}|.$$

There always exist reduced bases, i.e. such that the quadratic form

$$\Phi(u_1, \ldots, u_n) = \sum_{k=1}^{n} (x_{1k} u_1 + \ldots + x_{nk} u_n)^2$$

is reduced in the sense of MINKOWSKI; for such reduced bases,

$$|X_1| \, |X_2| \ldots |X_n| \leqq \gamma_n \, d(\Lambda),$$

where $\gamma_n > 0$ depends only on n.

An infinite sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is called *bounded* if there are two constants $c > 0$ and $c' > 0$ such that

$$d(\Lambda_r) \leq c, \text{ and } |X| \geq c' \text{ for every point } X \neq O \text{ of } \Lambda_r \quad (r = 1, 2, \ldots).$$

The following theorem is fundamental for the whole paper:

*) Part I of this paper is to appear in the Proceedings of the Royal Society.

Theorem 2: *Every bounded infinite sequence S of lattices contains a convergent infinite subsequence S'* (i.e., reduced bases of the elements of S' tend to a basis of the limiting lattice Λ, and so Λ consists just of the limit points of the lattices in S').

A function $F(X) = F(x_1, \ldots, x_n)$ of the variable point X in R_n is called a distance function if

(i) $F(X) > 0$ for all $X \neq O$ in R_n, and $F(O) = 0$.
(ii) $F(tX) = |t| F(X)$ for all X and all real t.
(iii) $F(X)$ is a continuous function of X.

We then say that the point set K: $F(X) \leq 1$ is a *star body*. Such a star body contains, and is symmetrical in, the origin; and with every point X the whole line segment OX belongs to it. The boundary C of K may extend to infinity, but is continuous in the finite part.

A lattice Λ is *K-admissible* if no point of Λ except O is an *inner point* of K. According as to whether K-admissible lattices do, or do not, exist, we say that K is of the finite or the infinite type; in the first case, $\Delta(K)$ denotes the lower bound of $d(\Lambda)$ extended over all admissible lattices, and we put $\Delta(K) = \infty$ in the second case. It is nearly trivial that bounded star bodies are of the finite type, and that unbounded star bodies of both types exist. One proves easily:

Theorem 7: *If the star body H is contained in the star body K, then* $\Delta(H) \leq \Delta(K)$.

From Theorem 2, the following existence theorem is derived:

Theorem 8: *Every star body of the finite type possesses at least one critical lattice,* i.e. a lattice Λ such that (i) $d(\Lambda) = \Delta(K)$, and (ii) Λ is K-admissible.

Part 1 deals mainly with the properties of critical lattices, and discusses, in particular, their points on, or near to, the boundary of the star body. I quote the following theorems, since I refer to them in the present paper:

Theorem 9: *Let $K, K_1, K_2, \ldots$ be an infinity of star bodies of the finite type such that for every $\varepsilon > 0$ and every $t > 0$,* (i) *K_r is contained in $(1 + \varepsilon)K$ if $r \geq r_0(\varepsilon)$, and* (ii) *the subset $|X| \leq t$, $F(X) \leq 1$ of K is contained in $(1 + \varepsilon)K_r$ if $r \geq r_1(\varepsilon, t)$. Then* $\lim_{r \to \infty} \Delta(K_r) = \Delta(K)$.

Theorem 11: *Every critical lattice of a bounded star body has n independent points on its boundary.*

Theorem 15: *There exists an unbounded star body of the finite type with a critical lattice which has no points on its boundary.*

Theorem 16: *Let K: $F(X) \leq 1$ be of the finite type, Ω a linear transformation of R_n into itself of determinant $\omega \neq 0$, $F'(X) = F(\Omega X)$, and let K' be the star body $F'(X) \leq 1$. Then also K' is of the finite type, and* $\Delta(K') = |\omega|^{-1} \Delta(K)$.

Theorem 17: *If the star body K of the finite type admits the automorphism Ω* (i.e., $F(\Omega X) = F(X)$ identically in X), *then Ω is of determinant $\omega = \pm 1$.*

Theorem 19: *Let $\Lambda_1, \Lambda_2, \ldots$ be a convergent sequence of lattices of limit Λ, and let $\varphi = \lim_{r \to \infty} F(\Lambda_r) > 0$ exist* ($F(\Lambda_r)$ denotes the lower bound of $F(X)$ extended over all points $X \neq O$ of Λ_r). *Let there be a constant $c > 0$ such that in each Λ_r there is a point $P_r \neq 0$ satisfying $|P_r| \leq c$ and $\lim_{r \to \infty} F(P_r) = \varphi$. Then $\lim_{r \to \infty} F(\Lambda_r) = F(\Lambda)$, and there is a point $P \neq 0$ of Λ such that $F(P) = F(\Lambda)$.*

Theorem 23: *Let K: $F(X) \leq 1$ be an unbounded star body of the finite type, and let Γ be the group of its automorphisms. Let there be a constant $c > 0$ such that, if P is any point of K, then $|\Omega P| \leq c$ for at least one element Ω of Γ; and if $t > 0$ is arbitrary and $P \neq 0$ is any point of K, let $|\Omega P| > t$ for at least one element Ω of Γ. Then, however small $\varepsilon > 0$, every critical lattice of K contains an infinity of different points P satisfying $1 \leq F(P) < 1 + \varepsilon$.*

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§ 1. *Preliminary remark.*

The theorems and definitions in Part I of this paper were numbered with Arabic numerals, and the problems with Latin capitals. In order to simplify references, theorems and definitions in the present Part II shall be numbered with Latin capitals, and problems with Arabic numerals.

§ 2. *The relations $H < K$ and $H \prec K$.*

Let H: $G(X) \leq 1$ and K: $F(X) \leq 1$ be two star bodies of the finite type in R_n. If H is a proper subset of K, then we write

$$H < K \quad \text{or} \quad K > H.$$

By Theorem 7, this relation implies that

$$\triangle(H) \leq \triangle(K),$$

where the equality sign may, or may not, hold. We therefore use the further symbol

$$H \prec K \text{ or } K \succ H$$

to denote that both

$$H < K \text{ and } \triangle(H) = \triangle(K).$$

From this definition,

$$\textit{if } H \prec K \textit{ and } K \prec L, \textit{ then } H \prec L,$$

and

$$\textit{if } H < K < L \textit{ and } H \not\prec L, \textit{ then } H \prec K \prec L.$$

It is further clear that

$H < K$ implies $H \prec K$, if and only if at least one critical lattice of H is K-admissible.

Theorem A: *To every star body H of the finite type, there exist star bodies K of the finite type such that $H \prec K$.*

Proof: Choose for Λ any critical lattice of H, and for K any star body such that $K > H$ and that Λ is K-admissible.

On the other hand, when K is given, then it is not always possible to find a star body H such that $H \prec K$. We therefore define:

Definition A: *The star body K is called reducible if there exists a star body H such that $H \prec K$, and it is called irreducible if no star body H with $H \prec K$ exists.*

Theorem A shows that there are reducible star bodies. We shall show later that also irreducible star bodies exist, and give some examples of such bodies. But we shall first obtain necessary, respectively sufficient, conditions of irreducibility. So far, I have not yet succeeded in finding conditions which are both necessary and sufficient.

§ 3. *A necessary condition for irreducibility.*

We need the following lemma which is closely related to Theorem 19 in Part 1:

Theorem B: *Let K: $F(X) \leq 1$ be a star body of the finite type, and $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ an infinite sequence of K-admissible lattices with the following properties:*

(a): $$\lim_{r \to \infty} d(\Lambda_r) = \triangle(K);$$

(b): *Every lattice Λ_r contains a point P_r on the boundary C: $F(X) = 1$ of K;*

(c): *The points $P_1, P_2, P_3, \ldots$ tend to a limit point P.*

Then P lies on C, and there exists a critical lattice Λ of K containing P.

Proof: The first part of the assertion follows at once from

$$F(P) = \lim_{r \to \infty} F(P_r) = \lim_{r \to \infty} 1 = 1.$$

For the second part, we first remark that the sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is bounded. Hence, by Theorem 2, there is an infinite subsequence

$$\Lambda^{(1)} = \Lambda_{k_1}; \Lambda^{(2)} = \Lambda_{k_2}; \Lambda^{(3)} = \Lambda_{k_3}; \ldots, \qquad (k_1 < k_2 < k_3 < \ldots)$$

which tends to a limit, the lattice Λ say. We write $P^{(r)} = P_{k_r}$. Choose in every lattice $\Lambda^{(r)}$ a reduced basis

$$Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$$

and in Λ a basis

$$Y_1, Y_2, \ldots, Y_n$$

such that

$$\lim_{r \to \infty} | Y_g^{(r)} - Y_g | = 0 \qquad (g = 1, 2, \ldots, n).$$

Then there are integers $u_1^{(r)}, u_2^{(r)}, \ldots, u_n^{(r)}$ not all zero such that

$$P^{(r)} = u_1^{(r)} Y_1^{(r)} + u_2^{(r)} Y_2^{(r)} + \ldots + u_n^{(r)} Y_n^{(r)} \qquad (r = 1, 2, 3, \ldots),$$

and real numbers $u_1, u_2, \ldots, u_n$ not all zero such that

$$P = u_1 Y_1 + u_2 Y_2 + \ldots + u_n Y_n.$$

From the hypothesis, the points

$$P^{(r)}, Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \qquad (r = 1, 2, 3, \ldots)$$

are bounded; further

$$| \{ Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \} | = d(\Lambda^{(r)}) \geqq \Delta(K).$$

Hence, by the equations

$$u_g^{(r)} \{ Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \} = \{ Y_1^{(r)}, \ldots, Y_{g-1}^{(r)}, P^{(r)}, Y_{g+1}^{(r)}, \ldots, Y_n^{(r)} \}$$
$$(g = 1, 2, \ldots, n).$$

all coefficients $u_1^{(r)}, \ldots, u_n^{(r)}$ are bounded. Hence the left-hand side of the identity

$$(P - P^{(r)}) + u_1^{(r)} (Y_1^{(r)} - Y_1) + \ldots + u_n^{(r)} (Y_n^{(r)} - Y_n) = (u_1 - u_1^{(r)}) Y_1 +$$
$$+ \ldots + (u_n - u_n^{(r)}) Y_n$$

and so also the right-hand side tends to O when r tends to infinity. Since the points $Y_1, Y_2, \ldots, Y_n$ are independent, this implies that

$$\lim_{r \to \infty} u_g^{(r)} = u_g \qquad (g = 1, 2, \ldots, n).$$

Now the coefficients $u_g^{(r)}$ are integers, hence also their limits u_g, and so P

is a point of Λ. As in earlier proofs, it is easy to prove that Λ is K-admissible and therefore critical. This completes the proof.

Theorem C: *Let K: $F(X) \leq 1$ be an irreducible star body of the finite type, and let P be any finite point on the boundary C of K. Then there exists at least one critical lattice of K containing P.*

Proof: Let us assume that on the contrary no critical lattice of K passes through a certain finite point P on C; we shall show that K is then reducible.

By the continuity of $F(X)$, every point X on C sufficiently near to P lies at a bounded distance from O. We assert, firstly, that a positive number σ exists such that no critical lattice of K passes through a point of the set S consisting of all points X on C for which

$$| \, |X|^{-1} X - |P|^{-1} P \, | \geqq \sigma.$$

For let this assertion be false. Then there exists an infinite sequence of critical lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

of K with a point P_r in each lattice Λ_r such that

$$\lim_{r \to \infty} | P_r - P | = 0.$$

But then, by Theorem B, at least one critical lattice of K passes through P, contrary to hypothesis.

We assert, secondly, that there is a positive number δ such that

$$d(\Lambda) \geqq (1 + \delta)^n \triangle (K)$$

for every K-admissible lattice which contains a point of S. For let this assertion be false. Then an infinite sequence of K-admissible lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

exists such that

$$\lim_{r \to \infty} d(\Lambda_r) = \triangle(K),$$

and that further each lattice Λ_r contains a point P_r of S. Since S is bounded and closed, it is then possible to find an infinite sequence of indices

$$k_1, k_2, k_3, \ldots \qquad (k_1 < k_2 < k_3 < \ldots)$$

such that the corresponding points

$$P_{k_1}, P_{k_2}, P_{k_3}, \ldots$$

tend to a point of S, the point P^* say. Theorem B, if applied to the sequence of lattices

$$\Lambda_{k_1}, \Lambda_{k_2}, \Lambda_{k_3}, \ldots,$$

gives now the existence of a critical lattice of K passing through P^*, contrary to what had just been proved.

Next put

$$G(X) = F(X)\left\{1 + \frac{\delta}{\sigma} \max\,(0, \sigma - |\,|X|^{-1}X - |P|^{-1}P\,|)\right\} \quad \text{if } X \neq O,$$

and

$$G(X) = 0 \text{ if } X = O,$$

so that $G(X)$ is a distance function. Evidently

$$F(X) \leq G(X) \leq (1+\delta)F(X) \qquad \text{for all points } X,$$

and

$$G(X) > F(X) \quad \text{if and only if} \quad X \neq 0, \quad |\,|X|^{-1}X - |P|^{-1}P\,| < \sigma.$$

Hence the star body H: $G(X) \leq 1$ satisfies the relation

$$H < K.$$

The theorem is then proved if we can show that even

$$H \prec K.$$

Let this be untrue. Then there exists a critical lattice Λ of H which is not K-admissible, and so the set Σ of all points

$$Q_s \neq 0 \qquad (s = 1, 2, 3, \ldots)$$

of Λ which are *inner* points of K, is not empty. Every point Q_s satisfies the inequalities

$$F(Q_s) < 1 \leq G(Q_s) \leq (1+\delta)\,F(Q_s),$$

whence

$$\frac{1}{1+\delta} \leqq F(Q_s) < 1,$$

and it also satisfies the inequality

$$|\,|Q_s|^{-1}Q_s - |P|^{-1}P| < \sigma.$$

By the definition of S, these formulae imply that Q_s belongs to a bounded part of R_n. Hence Σ has only a finite number of elements, the lattice points

$$Q_1, Q_2, \ldots, Q_r$$

say. Put

$$\min_{s=1,2,\ldots,r} F(Q_s) = \frac{1}{\lambda},$$

so that

$$1 < \lambda \leq 1 + \delta.$$

The lattice

$$\Lambda^* = \lambda\Lambda$$

is then K-admissible, and at least one of the points

$$\lambda Q_1, \lambda Q_2, \ldots, \lambda Q_r$$

lies in the subset S on C. Hence, as we showed above,

$$d(\Lambda^*) \geq (1+\delta)^n \triangle(K).$$

On the other hand,

$$d(\Lambda^*) = \lambda^n d(\Lambda) = \lambda^n \triangle(H),$$

since Λ is a critical lattice of H. Hence

$$\triangle(H) = \frac{1}{\lambda^n} d(\Lambda^*) \geqq \left(\frac{1+\delta}{\lambda}\right)^n \triangle(K) \geqq \triangle(K),$$

contrary to the assumption that $\triangle(H) < \triangle(K)$.

The so proved Theorem C applies to all irreducible star bodies irrespective of whether these are bounded or not. The following problem arises now:

Problem 1: *Do there exist unbounded irreducible star bodies?*

I have reasons to believe that the answer is in the negative. However, I cannot prove this generally, but only for a special class of star bodies to be considered later (Theorem H).

§ 4. *A sufficient condition for irreducibility.*

For this reason, the sufficient condition for irreducibility given in this paragraph will apply only to bounded star bodies.

Let K: $F(X) \leq 1$ be a bounded star body, and Λ any critical lattice of K. This lattice has only a finite number of points on the boundary C of K, say the points

$$\mp P_1, \mp P_2, \ldots, \mp P_m;$$

here $m \geq n$ by Theorem 11. Denote by $Y_1, \ldots, Y_n$ a reduced basis of Λ, and write the points P_k in the form

$$P_k = u_1^{(k)} Y_1 + \ldots + u_n^{(k)} Y_n \qquad (k = 1, 2, \ldots, m),$$

so that the coefficients $u_g^{(k)}$ are integers. Let $\varepsilon > 0$. Any n points $Y_1^*, \ldots, Y_n^*$ satisfying

$$|Y_g^* - Y_g| < \varepsilon \qquad (g = 1, 2, \ldots, n)$$

generate a neighbouring lattice Λ^*; we say that Λ^* lies in a ε-neighbourhood of Λ. The points

$$P_k^* = u_1^{(k)} Y_1^* + \ldots + u_n^{(k)} Y_n^* \qquad (k = 1, 2, \ldots, m)$$

are then the only points of Λ^* which lie on, or near to, the boundary C of K, if ε sufficiently small.

Definition B: *The critical lattice Λ of K is called a free lattice if to every index $k = 1, 2, \ldots, m$ and to every $\varepsilon > 0$, there exists a lattice Λ^* with the following properties:*

(a): *Λ^* lies in an ε-neighbourhood of Λ.*

(b): $$d(\Lambda^*) < d(\Lambda).$$

(c): *Λ^* contains no inner points of K except O and $\mp P_k$.*

For instance, every singular lattice Λ of K (i.e. a critical lattice with $m = n$) is free. For let $\mp P_1, \ldots, \mp P_n$ be the points of Λ on C. There exists then a neighbouring lattice Λ^* such that the points P_g^* of Λ^* corresponding to $P_1, \ldots, P_n$ are given by

$$P_g^* = \begin{cases} P_g & \text{if } g = 1, \ldots, k-1, k+1, \ldots, n, \\ (1-\varepsilon) P_k & \text{if } g = k. \end{cases}$$

Since

$$d(\Lambda^*) = (1-\varepsilon)\, d(\Lambda) < d(\Lambda),$$

the conditions (a), (b), (c) are satisfied.

Other examples of free lattices are the critical lattices of the unit circle

$$x_1^2 + x_2^2 \leqq 1$$

in R_2 ($m = 3$, $n = 2$), and the critical lattices of the unit sphere

$$x_1^2 + x_2^2 + x_3^2 \leqq 1$$

in R_3 ($m = 6$, $n = 3$). See the next paragraph.

Not every critical lattice is free, as is seen by the following example in R_2 which is easily extended to more dimensions:

Denote by K the non-convex hexagon in R_2 of vertices

$$(1, 0),\ (2, 1),\ (-2, 1),\ (-1, 0),\ (-2, -1),\ (2, -1).$$

This hexagon contains the square Q,

$$|x_1| \leq 1,\ |x_2| \leq 1$$

as a subset; moreover, the critical lattice

Λ: $$x_1 = u_1,\quad x_2 = u_2 \qquad (u_1, u_2 = 0, \mp 1, \mp 2, \ldots)$$

of Q is K-admissible. Hence $K \succ Q$, and Λ is also a critical lattice of K. Denote by

Λ^*: $x_1 = (1+\alpha) u_1 + \beta u_2,\ x_2 = \gamma u_1 + (1+\delta) u_2$ $(u_1, u_2, = 0, \mp 1, \mp 2, \ldots)$

a neighbouring lattice; thus

$$|\alpha| + |\beta| + |\gamma| + |\delta|$$

is very small. Then it is impossible that the point

$$(\beta, 1+\delta)$$

of Λ^* is an inner point of K, while the two other points

$$(1+\alpha+\beta,\ 1+\gamma+\delta),\ (-1-\alpha+\beta,\ 1-\gamma+\delta)$$

of Λ^* lie on the boundary or outside of K, since the three inequalities

$$1+\delta < 1,\ 1+\gamma+\delta \geq 1,\ 1-\gamma+\delta \geq 1$$

are mutually contradictory. Hence Λ is not free.

In this example, K is reducible. We may then ask:

Problem 2: *Does there exist a bounded irreducible star body with at least one critical lattice which is not free?*

I have so far not been able to solve this question.

Theorem D: *Let K be a bounded star body, and Σ a set of points on its boundary C which is everywhere dense. If further P is any point of Σ, let there be a free lattice of K containing P. Then K is irreducible.*

Proof: Assume there is a star body H such that $H \prec K$. There exists then a point Q on C which lies outside H; hence $\varepsilon > 0$ can be chosen so small that all points X with

$$|X - Q| \leq \varepsilon$$

lie also outside H. By the hypothesis, we can find a point P of Σ such that

$$|P - Q| \leqq \frac{\varepsilon}{2};$$

let Λ be the free lattice through P. Then, by Definition B, there is a neighbouring lattice Λ^* of determinant

$$d(\Lambda^*) < d(\Lambda)$$

and containing no inner points of K except O and two symmetrical points $\mp P^*$ such that

$$|P^* - P| \leqq \frac{\varepsilon}{2} \qquad \text{that is,} \qquad |P^* - Q| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence Λ^* is H-admissible, and so

$$\triangle(H) \leq d(\Lambda^*) < d(\Lambda) = \triangle(K),$$

contrary to hypothesis.

Remark: The same proof shows that if the hypothesis of Theorem D is satisfied, then at least one free lattice of K passes through every point on C.

§ 5. *Examples of irreducible convex star bodies.*

By means of Theorem D, any number of irreducible star bodies can be constructed. We first show that some of the two- and three-dimensional convex regions considered already by MINKOWSKI [1]) are irreducible. To this end, we use the following results of MINKOWSKI:

(A) Every critical lattice Λ of a convex star domain K in R_2 contains at least *six* points on the boundary C of K. If Λ contains only six such points, then a system of parallel coordinates ξ_1, ξ_2 can be chosen in which these six points are of coordinates [2])

$$(1, 0), \quad (0, 1), \quad (-1, 1).$$

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[1]) Diophantische Approximationen, 24—28, 54—55, 67—75, 75—77, 105—111. For the case of the cylinder, see my note: "On lattice points in a cylinder", which is to appear in the Quarterly Journal.

[2]) Diophantische Approximationen, 51—54.

(B) Every critical lattice Λ of a convex star body K in R_3 contains at least *twelve* points on the boundary C of K. If Λ contains only twelve such points, then a system of parallel coordinates ξ_1, ξ_2, ξ_3 can be chosen in which these twelve points are either of coordinates

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (0,1,-1),\quad (-1,0,1),\quad (1,-1,0),$$

or of coordinates [1])

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (0,1,1),\quad (1,0,1),\quad (1,1,0).$$

Theorem E: *Let K be a convex star domain in R_2, and Λ a critical lattice of K with just six points on C. Then Λ is a free lattice.*

Proof: It is clear from (A) that no three of the lattice points on C are collinear. Hence there exists a line L which separates any chosen point among these six from the five other ones. Let $-L$ be the line symmetrical to L in O, and let K^* be the set of all points of K which are not separated from O by either L or $-L$; then K^* is also a convex star domain. Since only four points of Λ lie on the boundary of K^*, there exists a neighbouring lattice Λ^* which is K^*-admissible and of determinant $d(\Lambda^*) < d(\Lambda)$; this lattice satisfies the conditions of definition B.

Theorem F: *Let K be a convex star body in R_3, and Λ a critical lattice of K with just twelve points on C. Then Λ is a free lattice.*

Proof: We first show that each one of the twelve lattice points on C can be separated from the eleven others by a plane L. For if the twelve lattice points are

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (0,1,-1),\quad (-1,0,1),\quad (1,-1,0),$$

then $(1,0,0)$ is separated from the other points by the plane

$$4\xi_1 + 2\xi_2 + 2\xi_3 - 3 = 0,$$

and $(0,1,-1)$ is separated by the plane

$$2\xi_2 - 2\xi_3 - 3 = 0.$$

If, however, the twelve lattice points are

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad (0,1,1),\quad (1,0,1),\quad (1,1,0),$$

then $(1,0,0)$ is separated from the other points by the plane

$$6\xi_1 - 2\xi_2 - 2\xi_3 - 5 = 0,$$

and $(0,1,1)$ is separated by the plane

$$2\xi_1 - 2\xi_2 - 2\xi_3 + 3 = 0.$$

By a cyclical permutation of the coordinates ξ_1, ξ_2, ξ_3 and by changing ξ_1, ξ_2, ξ_3 into $-\xi_1, -\xi_2, -\xi_3$, we obtain the equations of separating planes belonging to the other points. Having thus fixed L, let $-L$ be the plane symmetrical to L in O, and K^* the set of all points of K which are

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[1]) Gesammelte Abhandlungen, II.

not separated from O by either L or $-L$; then K^* is also a convex star body. Since Λ has only ten points on the boundary of K^*, there exists a neighbouring lattice Λ^* which is K^*-admissible and of determinant $d(\Lambda^*) < d(\Lambda)$, hence satisfies the conditions of Definition B.

From Theorems E and F, we obtain now the following examples of irreducible convex star bodies:

(a) The square K_1,

$$\max(|x_1|, |x_2|) \leq 1$$

in R_2 of determinant $\triangle(K_1) = 1$. For if $0 < |\xi| < 1$, then the critical lattice of basis

$$(1, 0),\ (\xi, 1)$$

passes through the boundary point $(\xi, 1)$ and the critical lattice of basis

$$(1, \xi),\ (0, 1)$$

passes through the boundary point $(1, \xi)$; further both types of lattice have just six points on the boundary of K_1.

(b) The circle K_2,

$$+\sqrt{(x_1^2 + x_2^2)} \leqslant 1$$

in R_2 of determinant $\triangle(K_2) = \frac{1}{2}\sqrt{3}$. For if $(\cos\theta, \sin\theta)$ is any point on C, then the critical lattice of basis

$$(\cos\theta, \sin\theta),\quad \left(\cos\left\{\theta + \frac{\pi}{3}\right\}, \sin\left\{\theta + \frac{\pi}{3}\right\}\right)$$

passes through this point and has just six points on the boundary.

(c) The cube K_3,

$$\max(|x_1|, |x_2|, |x_3|) \leq 1,$$

in R_3 of determinant $\triangle(K_3) = 1$. For if $0 < |\xi_1| < 1$, $0 < |\xi_2| < 1$, then the critical lattice of basis

$$(1, 0, 0),\ (\tfrac{1}{2}, 1, 0),\ (\xi_1, \xi_2, 1)$$

passes through the boundary point $(\xi_1, \xi_2, 1)$, and similar lattices pass through the points $(\xi_1, 1, \xi_2)$ and $(1, \xi_1, \xi_2)$. Moreover, these three types of critical lattices have just twelve points on C.

(d) The sphere K_4,

$$+\sqrt{(x_1^2 + x_2^2 + x_3^2)} \leqslant 1,$$

in R_3 of determinant $\triangle(K_4) = \sqrt{\tfrac{1}{2}}$. For all critical lattices are obtained from the lattice of basis

$$(1, 0, 0), (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, 0), (0, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{2}{3}})$$

by a rotation about O. They each contain twelve points on C, and they pass through every point on this boundary.

(e) The cylinder K_5,

$$\max(+\sqrt{(x_1^2 + x_2^2)}, |x_3|) \leqslant 1,$$

in R_3 of determinant $\triangle(K_5) = \sqrt{\frac{3}{4}}$. For one type of critical lattice is obtained from the lattice of basis

$$(1, 0, 0), (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, 0), (\xi_1, \xi_2\ 1)$$

by all rotations about the x_3-axis; if

$$0 < \xi_1^2 + \xi_2^2 < 1, \quad 0 < (\xi_1 - 1)^2 + \xi_2^2 < 1, \quad 0 < (\xi_1 - \tfrac{1}{2})^2 + (\xi_2 - \sqrt{\tfrac{3}{4}})^2 < 1,$$

then this type of lattice has just twelve points on C, and it passes through every point (x_1, x_2, x_3) on C for which

$$0 < x_1^2 + x_2^2 < 1, \qquad x_3 = \mp 1.$$

Further a second type of critical lattice is obtained from the lattice of basis

$$(1, 0, \xi_1), \ (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, \xi_2), \ (0, 0, 1)$$

by all rotations about the x_3-axis; if

$$0 < |\xi_1| < \tfrac{1}{2}, \ 0 < |\xi_2| < \tfrac{1}{2},$$

then also this type of lattice contains just twelve points on C, and it passes through every point on C for which

$$x_1^2 + x_2^2 = 1, \qquad 0 < |x_3| < 1.$$

Mathematics. — *On the G-function.* II. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 23, 1946.)

§ 4. **The fundamental systems of the differential equation satisfied by** $G_{p,q}^{m,n}(z)$

The function $y = G_{p,q}^{m,n}(z)$ defined by (2) satisfies the homogeneous linear differential equation of the q-th order

$$\left\{(-1)^{p-m-n} z \prod_{j=1}^{p} (\theta - a_j + 1) - \prod_{j=1}^{q} (\theta - b_j)\right\} y = 0, \quad . \quad . \quad (34)$$

where θ denotes the operator $z \frac{d}{dz}$.

This may be established in a very simple way. For the particular case with $m = 1$, $n = p = q = 2$ and $b_1 = 0$, that is if the function $G_{p,q}^{m,n}(z)$ reduces to the ordinary hypergeometric function ${}_2F_1$, it has been proved by BARNES [21]). The proof of the general case is almost similar to that of BARNES' special case, so that it may be omitted here.

Replacing z by $ze^{(p-m-n+1)\pi i}$ we find that the q functions [22])

$$\left.\begin{aligned} & e^{(m+n-p-1)\pi i b_h} G_{p,q}^{1,p}\left(z e^{(p-m-n+1)\pi i} \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_h, b_1, \ldots, b_{h-1}, b_{h+1}, \ldots, b_q \end{matrix}\right) \\ = & \frac{\prod_{j=1}^{p} \Gamma(1+b_h-a_j)}{\prod_{\substack{j=1 \\ j\neq h}}^{q} \Gamma(1+b_h-b_j)} z^{b_h} {}_pF_{q-1}\left(\begin{matrix} 1+b_h-a_1, \ldots, 1+b_h-a_p; \\ 1+b_h-b_1, \ldots^*\ldots, 1+b_h-b_q \end{matrix}; (-1)^{p-m-n} z\right), \end{aligned}\right\} \quad (35)$$

where $h = 1, \ldots, q$ satisfy too the differential equation (34). When

$$b_h - b_j \neq 0, \pm 1, \pm 2, \ldots \qquad (h = 1, \ldots, q; j = 1, \ldots, q; h \neq j),$$

the functions (35) are obviously linearly independent and form therefore a fundamental system of solutions valid in the vicinity of the origin [23]). Hence the function $G_{p,q}^{m,n}(z)$ must be linearly expressible in terms of the functions (35). The actual expression is given by (7).

We now proceed to determine a fundamental system of solutions of (34) valid near infinity; for this purpose we distinguish two cases:

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[21]) BARNES, [5], 145.

[22]) Equation (35) follows from (7).

[23]) If one or more of the differences $b_h - b_j$ $(h \neq j)$ is equal to $0, \pm 1, \pm 2, \ldots$, some of the fundamental solutions (35) must be replaced by expressions involving logarithmic terms.

First case: $q > p$. To every value of $\arg z$ it is possible to determine an integer λ such that

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n-2\lambda+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi \ . \quad (36)$$

It is also possible to determine an integer ω such that [24])

$$-(q-p+\varepsilon)\pi < \arg z+(q-m-n-2\psi)\pi < (q-p+\varepsilon)\pi. \ . \ . \ . \ . \quad (37)$$

for $\psi = \omega,\ \omega+1, \ldots, \omega+q-p-1$.

We now suppose

$$a_j - a_h \neq 0, \pm 1, \pm 2, \ldots (j = 1, \ldots, p; h = 1, \ldots, p; j \neq h) \ . \quad (38)$$

and consider the functions

$$G_{p,q}^{q,1}(z\, e^{(q-m-n-2\lambda+1)\pi i} \,||\, a_t) \qquad (t = 1, \ldots, p). \ . \ . \ . \quad (39)$$

They obviously satisfy the differential equation (34). On account of (36) and (13) it follows from theorem A that these functions tend algebraically to zero or to infinity for $|z| \to \infty$; it is further clear in virtue of (38) and (13) that they have a mutually different algebraic behaviour for large values of $|z|$. Hence they form a system of p linearly independent solutions of the differential equation (34). The fundamental system is therefore not yet complete.

In order to determine the failing solutions we consider the $q-p$ functions

$$G_{p,q}^{q,0}(z\, e^{(q-m-n-2\psi)\pi i}) \qquad (\psi = \omega, \omega+1, \ldots, \omega+q-p-1) \ . \quad (40)$$

It is easily seen that these functions satisfy too the differential equation (34). Because of (37) and (25) it follows from (26) that they tend exponentially to zero or to infinity as $|z| \to \infty$; besides it appears from (26) and (25) that they are mutually linearly independent. Hence they supply the $q-p$ failing fundamental solutions of (34).

We have therefore proved: *When $q > p$ and the conditions* (36), (37) *and* (38) *are satisfied, a fundamental system of* (34) *valid in the vicinity of $z = \infty$ is formed by the p functions* (39) *and the $q-p$ functions* (40).

Second case: $q = p$. We shall assume that the numbers $a_1 \ldots, a_p$ satisfy the condition (38). Besides we suppose

$$\arg z + (p-m-n)\pi \neq 0, \pm 2\pi, \pm 4\pi, \ldots \ . \ . \ . \ . \quad (41)$$

Then it is possible to determine an integer λ such that

$$-\pi < \arg z + (p-m-n-2\lambda+1)\pi < \pi. \ . \ . \ . \ . \quad (42)$$

We now consider the functions

$$G_{p,p}^{p,1}(z\, e^{(p-m-n-2\lambda+1)\pi i} \,||\, a_t) \qquad (t = 1, \ldots, p) \ . \ . \ . \ . \quad (43)$$

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[24]) The number ε is defined by (24).

These functions satisfy the differential equation (34) with $q = p$. Because of (42), (31) and (38) it follows from theorem F that they have outside the unit-circle a mutually different behaviour for $|z|$ tending to infinity. The functions (43) form therefore a system of p linearly independent solutions of the differential equation (34) with $q = p$.

Hence, if $q = p$ and the conditions (41), (42) *and* (38) *are satisfied, the p functions* (43) *form a fundamental system adapted for the region outside the circle $|z| = 1$.*

Remark. Sometimes there occur in a formula not the p functions (39), respect. (43) but only the first n of these functions, viz. the functions

$$G_{p,q}^{q,1}(z e^{(q-m-n-2\lambda+1)\pi i} \,||\, a_t) \qquad (t = 1, \ldots, n),$$

respect.

$$G_{p,p}^{p,1}(z e^{(p-m-n-2\lambda+1)\pi i} \,||\, a_t) \qquad (t = 1, \ldots, n).$$

In order that these functions are linearly independent, it is not necessary that the condition (38) is satisfied. We may replace (38) by the less stringent condition (20).

If we have to do with less than $q - p$ functions $G_{p,q}^{q,0}$, namely with the ν ($\nu < q - p$) functions

$$G_{p,q}^{q,0}(z e^{(q-m-n-2\psi)\pi i}) \qquad (\psi = \omega, \omega + 1, \ldots, \omega + \nu - 1),$$

then it is evident that the condition (37) needs not to be satisfied for $\psi = \omega, \omega + 1, \ldots, \omega + q - p - 1$, but only for $\psi = \omega, \omega + 1, \ldots, \omega + \nu - 1$.

§ 5. Contents of the paper.

The function $G_{p,q}^{m,n}(z)$ can by means of (7) be linearly expressed in terms of the functions of the fundamental system of (34) valid near $z = 0$. But now that we have constructed a system of fundamental solutions of (34) valid in the neighbourhood of $z = \infty$, it must also be possible to express the function $G_{p,q}^{m,n}(z)$ as a linear combination of these last fundamental solutions. That is to say: There must exist relations of the form

$$G_{p,q}^{m,n}(z) = \sum_{t=1}^{p} C_t G_{p,q}^{q,1}(z e^{(q-m-n-2\lambda+1)\pi i} \,||\, a_t) + \sum_{h=0}^{q-p-1} D_h G_{p,q}^{q,0}(z e^{(q-m-n-2h-2\omega)\pi i}), \quad (44)$$

where

$$G_{p,q}^{q,1}(z e^{(q-m-n-2\lambda+1)\pi i} \,||\, a_t) \text{ and } G_{p,q}^{q,0}(z e^{(q-m-n-2h-2\omega)\pi i})$$

are the in § 4 defined fundamental solutions in the neighbourhood of $z = \infty$ and where the coefficients C_t and D_h do not depend on z.

On account of the generality of the function $G_{p,q}^{m,n}(z)$ and the great number of important functions which are particular cases of it [25], it will

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[25] See the lists in [18]—[25]; these lists are not complete.

be interesting to investigate this subject more fully. In the present paper I will determine the coefficients C_t and D_h for all values of m, n, p and q, for which the function $G_{p,q}^{m,n}(z)$ has been defined and for all values of $|z|$ and arg z [26]). It appears necessary to distinguish several cases. In most cases the coefficients C_t and D_h are much more difficult to determine and much more complicated than the coefficients in formula (7).

§§ 6, 7, 8 and 13 supply some definitions and a great number of lemmas. In §§ 9, 10, 11, 12 and 14 I prove four expansion formulae (theorems 1, 2, 3 and 5) which render it possible to write the function $G_{p,q}^{m,n}$ in a special way as a linear combination of functions $G_{p,q}^{k,l}$ and $G_{p,q}^{k,l-1}$.

These expansion formulae are the most powerful instruments of the present paper. Their most important particular case appears when we put $l = 1$ and $k = q$. The in this manner specialized expansion formulae are written out in § 16. In §§ 17 and 19 I will show that the desired relations of the type (44) are particular cases of the specialized expansion formulae of § 16. § 17 is devoted to the case with $q > p$; the case with $q = p$ will be examined in § 19.

In § 2 I have mentioned BARNES' asymptotic expansion of the function $G_{p,q}^{m,n}(z)$ $(q > p)$. His results bear upon the two following cases only:

A. $1 \leqq n \leqq p < q$, $1 \leqq m \leqq q$, $m+n > \frac{1}{2}p+\frac{1}{2}q$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$.

B. $n=0$, $m=q$, $|\arg z| < (q-p+\varepsilon)\pi$.

But now that it is possible to express the function $G_{p,q}^{m,n}(z)$ for all values of m, n, p, q and arg z in terms of functions $G_{p,q}^{q,1}(\zeta \,|\, a_t)$ and $G_{p,q}^{q,0}(\zeta)$ of which the asymptotic expansions for large values of $|\zeta|$ can be written down by means of BARNES' formulae (18) and (26), it is also possible to deduce asymptotic expansions for the function $G_{p,q}^{m,n}(z)$ $(q > p)$ for all cases which do not come under those of BARNES. Hence we are now able, for all values of m, n, p, q and arg z, to determine asymptotic expansions of the function $G_{p,q}^{m,n}(z)$ $(q > p)$ as $|z| \to \infty$. These expansions are given in § 18; I have stated my results in the form of six theorems.

The function $G_{p,p}^{m,n}(z)$ has been defined in § 1 for $|z| < 1$ and any value of arg z. In § 3 (theorem E) I have given a formula for the analytic continuation of this function outside the circle $|z| = 1$ in the case

I. $m + n \geqq p + 1$, $|\arg z| < (m + n - p)\pi$.

Hence there still fail formulae for the analytic continuation of $G_{p,p}^{m,n}(z)$ in the two following cases:

II. $m + n \leqq p$, all values of arg z.

III. $m + n \geqq p + 1$, $|\arg z| > (m + n - p)\pi$.

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[26]) If $q = p$, I suppose $|z| > 1$; the values of z on the cross-cut are excluded.

As I have already stated in § 3, I will make in these cases a cross-cut along the real axis from $(-1)^{m+n-p}$ to ∞ . $(-1)^{m+n-p}$. Now it is possible to express the function $G_{p,p}^{m,n}(z)$ for all values of m, n, p and arg z (except values of z on the cross-cut) in terms of the fundamental solutions near $z = \infty$ of the differential equation satisfied by $G_{p,p}^{m,n}(z)$. Since the continuations for $|z| > 1$ of these fundamental solutions can be immediately written down by means of theorem F, it is also possible to determine in all cases (except when z lies on the cross-cut) the analytic continuation of $G_{p,p}^{m,n}(z)$ for $|z| > 1$. The formula in question is given in § 19.

A simple and important special case of the function $G_{p,q+1}^{m,n}(z)$ is the generalized hypergeometric function ${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_p; z)$. The behaviour of this function for large values of $|z|$ was a subject of investigation of several writers. Most of these investigations are fairly difficult. In § 20 I will shew that the asymptotic expansions of the function ${}_pF_q(z)$ can with slight labour be deduced from the analogous expansions of $G_{p,q+1}^{1,p}(z)$.

Now one would be apt to think, on account of (7), that it must on the contrary also be possible to deduce the asymptotic expansions of $G_{p,q}^{m,n}(z)$ from the known expansions of the function ${}_pF_{q-1}(z)$. In many cases however we then find for $G_{p,q}^{m,n}(z)$ an expansion wherein the coefficients all vanish. Now it is clear that a such expansion is worthless, so that in these cases it is not possible to deduce a suitable asymptotic expansion for $G_{p,q}^{m,n}(z)$ from the corresponding expansion of ${}_pF_{q-1}(z)$. For instance the asymptotic expansions of $G_{p,q}^{m,n}(z)$ which are exponentially small cannot be deduced from the asymptotic expansions of ${}_pF_{q-1}(z)$.

In § 15 I will examine some relations of the form

$$M(z) = \sum_{h=1}^{k} r_h M(z e^{(\gamma - 2h)\pi i}).$$

Finally, in § 21 I will determine for all values of arg z the asymptotic expansion of WHITTAKER's function $W_{k,m}(z)$ as $|z| \to \infty$.

§ 6. Definitions.

Definition 4. *If the numbers a_s are defined for $s = 0, \pm 1, \pm 2, \ldots$ and h is a positive integer or zero, the sums $\sum_{s=0}^{h} a_s$ and $\sum_{s=0}^{-1} a_s$ have the usual meaning*

$$\sum_{s=0}^{h} a_s = a_0 + a_1 + \ldots + a_h \text{ and } \sum_{s=0}^{-1} a_s = 0.$$

If k is an integer ≥ 2, the sum $\sum_{s=0}^{-k} a_s$ is defined as follows:

$$\sum_{s=0}^{-k} a_s = -a_{-1} - a_{-2} - \ldots - a_{-k+1}.$$

Definition 5. *Suppose that m, n and q are integers with*

$$0 \leqq m \leqq q \text{ and } n \geqq 0.$$

Then the coefficients $A^{m,n}_{q}$ *and* $\bar{A}^{m,n}_{q}$ *are defined as follows:*

$$A^{m,n}_{q} = (-2\pi i)^{m+n-q}\, e^{(a_1+\ldots+a_n-b_{m+1}-\ldots-b_q)\pi i}, \quad . \quad . \quad . \quad (45)$$

$$\bar{A}^{m,n}_{q} = (2\pi i)^{m+n-q}\, e^{-(a_1+\ldots+a_n-b_{m+1}-\ldots-b_q)\pi i}. \quad . \quad . \quad . \quad (46)$$

Definition 6. *Suppose that m, n and p are integers with*

$$0 \leqq n \leqq p \text{ and } m \geqq 0.$$

Then the coefficients $B^{m,n}_{p}$ *and* $\bar{B}^{m,n}_{p}$ *are defined as follows:*

$$B^{m,n}_{p} = (-2\pi i)^{m+n-p}\, e^{(b_1+\ldots+b_m-a_{n+1}-\ldots-a_p)\pi i} \quad . \quad . \quad . \quad (47)$$

$$\bar{B}^{m,n}_{p} = (2\pi i)^{m+n-p}\, e^{-(b_1+\ldots+b_m-a_{n+1}-\ldots-a_p)\pi i}. \quad . \quad . \quad . \quad (48)$$

Definition 7. *Suppose that m, n and q are integers with*

$$0 \leqq m \leqq q \text{ and } n \geqq 0;$$

suppose further that t is an arbitrary integer.

Then the coefficients $\Omega^{m,n}_{q}(t)$ *and* $\bar{\Omega}^{m,n}_{q}(t)$ *are defined as follows:* $\Omega^{m,n}_{q}(t)$, *respect.* $\bar{\Omega}^{m,n}_{q}(t)$ *is the coefficient of* x^t *in the expansion of*

$$\frac{\prod_{j=m+1}^{q}(1-xe^{2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{2\pi i a_j})}, \text{ respect. } \frac{\prod_{j=m+1}^{q}(1-xe^{-2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{-2\pi i a_j})}$$

in ascending powers of x.

Definition 8. *Suppose that m, n and p are integers with*

$$0 \leqq n \leqq p \text{ and } m \geqq 0;$$

suppose further that λ *is an arbitrary integer.*

Then the coefficients $\Gamma^{m,n}_{p}(\lambda)$ *and* $\bar{\Gamma}^{m,n}_{p}(\lambda)$ *are defined as follows:* $\Gamma^{m,n}_{p}(\lambda)$, *respect.* $\bar{\Gamma}^{m,n}_{p}(\lambda)$ *is the coefficient of* x^λ *in the expansion of*

$$\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}, \text{ respect. } \frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi i b_j})}$$

in ascending powers of x.

Definition 9. *Suppose that h, m, n, p and q are integers with*

$$0 \leqq n \leqq p,\ 0 \leqq m \leqq q \text{ and } h \geqq 1;$$

suppose further that λ *is an arbitrary integer.*

Then the coefficients $\Phi^{m,n}_{p,q}(h;\lambda)$ *and* $\bar{\Phi}^{m,n}_{p,q}(h;\lambda)$ *are defined as follows:*

$\Phi_{p,q}^{m,n}(h;\ \lambda)$, *respect.* $\overline{\Phi}_{p,q}^{m,n}(h;\ \lambda)$ *is the coefficient of* $x^{h+\lambda-1}$ *in the expansion of*

$$\left\{\sum_{t=0}^{h-1} \Omega_{q}^{0,p}(t)\, x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}\right\},$$

respect.

$$\left\{\sum_{t=0}^{h-1} \overline{\Omega}_{q}^{0,p}(t)\, x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi i b_j})}\right\}$$

in ascending powers of x.

Definition 10. *Suppose that* h, m, n, p *and* q *are integers with*

$$0 \leqq n \leqq p,\ 0 \leqq m \leqq q \ \text{ and } \ h \geqq 1;$$

suppose further that λ *is an arbitrary integer.*

Then the coefficients $\Psi_{p,q}^{m,n}(h;\lambda)$ *and* $\overline{\Psi}_{p,q}^{m,n}(h;\lambda)$ *are defined as follows:* $\Psi_{p,q}^{m,n}(h;\lambda)$, *respect.* $\overline{\Psi}_{p,q}^{m,n}(h;\lambda)$ *is the coefficient of* x^{λ} *in the expansion of*

$$\left\{\sum_{t=1}^{h} \overline{\Omega}_{q}^{0,p}(h-t)\, x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}\right\},$$

respect.

$$\left\{\sum_{t=1}^{h} \Omega_{q}^{0,p}(h-t)\, x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{-2\pi i b_j})}\right\}$$

in ascending powers of x.

Definition 11. *Suppose that* l, m, n *and* p *are integers with*

$$1 \leqq l \leqq p,\ 0 \leqq n \leqq p \ \text{ and } \ m \geqq 0;$$

suppose further that r *is an arbitrary integer.*

Then the coefficients $\Theta_{p}^{m,n}(l;r)$ *and* $\overline{\Theta}_{p}^{m,n}(l;r)$ *are defined as follows:* $\Theta_{p}^{m,n}(l;r)$, *respect.* $\overline{\Theta}_{p}^{m,n}(l;r)$ *is the coefficient of* x^r *in the expansion of*

$$\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{(1-xe^{2\pi i a_l})\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}, \ \textit{respect.} \ \frac{\prod_{j=n+1}^{p}(1-xe^{-2\pi i a_j})}{(1-xe^{-2\pi i a_l})\prod_{j=1}^{m}(1-xe^{-2\pi i b_j})}$$

in ascending powers of x.

Remark. From the above definitions it follows

$$\Omega^{m,n}_{q}(t)=\overline{\Omega}^{m,n}_{q}(t)=0 \text{ for } t=-1,-2,-3,\ldots, \quad (49)$$

$$\Omega^{m,n}_{q}(0)=\overline{\Omega}^{m,n}_{q}(0)=1, \quad (50)$$

$$\Gamma^{m,n}_{p}(\lambda)=\overline{\Gamma}^{m,n}_{p}(\lambda)=0 \text{ for } \lambda=-1,-2,-3,\ldots, \quad (51)$$

$$\Gamma^{m,n}_{p}(0)=\overline{\Gamma}^{m,n}_{p}(0)=1,$$

$$\Phi^{m,n}_{p,q}(h;\lambda)=\overline{\Phi}^{m,n}_{p,q}(h;\lambda)=0 \text{ for } \lambda=-h,-h-1,-h-2,\ldots, \quad (52)$$

$$\Psi^{m,n}_{p,q}(h;\lambda)=\overline{\Psi}^{m,n}_{p,q}(h;\lambda)=0 \text{ for } \lambda=0,-1,-2,\ldots, \quad (53)$$

$$\Theta^{m,n}_{p}(l;r)=\overline{\Theta}^{m,n}_{p}(l;r)=0 \text{ for } r=-1,-2,-3,\ldots, \quad (54)$$

and

$$\Theta^{m,n}_{p}(l;0)=\overline{\Theta}^{m,n}_{p}(l;0)=1.$$

§ 7. Lemmas.

Lemma 1.

$$G^{m,n}_{p,q}(z)=\frac{1}{2\pi i}\{e^{\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{-\pi i})-e^{-\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{\pi i})\}, \quad (55)$$

$$G^{m,n}_{p,q}(z)=\frac{1}{2\pi i}\{e^{\pi i a_{n+1}}\,G^{m,n+1}_{p,q}(ze^{-\pi i})-e^{-\pi i a_{n+1}}\,G^{m,n+1}_{p,q}(ze^{\pi i})\}. \quad (56)$$

Proof. From (2) it follows

$$\frac{1}{2\pi i}\{e^{\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{-\pi i})-e^{-\pi i b_{m+1}}\,G^{m+1,n}_{p,q}(ze^{\pi i})\}$$

$$=\frac{1}{2\pi i}\int_{C}\frac{e^{\pi i(b_{m+1}-s)}-e^{-\pi i(b_{m+1}-s)}}{2\pi i}\,\frac{\prod_{j=1}^{m+1}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+2}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds$$

$$=\frac{1}{2\pi i}\int_{C}\frac{\sin(b_{m+1}-s)\pi}{\pi}\,\frac{\prod_{j=1}^{m+1}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+2}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds$$

$$=\frac{1}{2\pi i}\int_{C}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds=G^{m,n}_{p,q}(z),$$

so that formula (55) has been established.

The proof of (56) is similar to that of (55).

Lemma 2. *If k, l, p, q, s and t are integers with*

$$1 \leqq l \leqq s \leqq p \leqq q,\ 1 \leqq t \leqq s-l+1 \text{ and } 0 \leqq k \leqq q,$$

then

$$G_{p,q}^{k,l,s}(z \,\|\, a_t) = e^{-2\pi i a_t}\, G_{p,q}^{k,l,s}(z e^{2\pi i} \,\|\, a_t) + 2\pi i e^{-\pi i a_t}\, G_{p,q}^{k,l-1,s}(z e^{\pi i}) \quad (57)$$

and

$$G_{p,q}^{k,l,s}(z \,\|\, a_t) = e^{2\pi i a_t}\, G_{p,q}^{k,l,s}(z e^{-2\pi i} \,\|\, a_t) - 2\pi i e^{\pi i a_t}\, G_{p,q}^{k,l-1,s}(z e^{-\pi i}). \quad (58)$$

Proof. Since $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_1, \ldots, a_\lambda$ and also of $a_{\lambda+1}, \ldots, a_p$, we have by (56) and the definitions of $G_{p,q}^{k,l,s}$ and $G_{p,q}^{k,l-1,s}$ (see the end of § 1)

$$2\pi i\, G_{p,q}^{k,l-1,s}(z) = e^{\pi i a_t}\, G_{p,q}^{k,l,s}(z e^{-\pi i} \,\|\, a_t) - e^{-\pi i a_t}\, G_{p,q}^{k,l,s}(z e^{\pi i} \,\|\, a_t).$$

From this formula follow (57) and (58) without difficulty.

The most important lemma is

Lemma 3. *Suppose that m, n and q are integers with $0 \leqq m \leqq q$ and $n \geqq 0$; further that r is an arbitrary integer; finally that the numbers $a_1, \ldots, a_n$ satisfy the condition*

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \ (j = 1, \ldots, n;\ t = 1, \ldots, n;\ j \neq t). \quad (20)$$

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{t=1}^{n} e^{(m+n-q+2r)\pi i a_t}\, \triangle_q^{m,n}(t) \\ &= -\frac{1}{2\pi i}\left\{A_q^{m,n}\, \Omega_q^{m,n}(r) - \overline{A}_q^{m,n}\, \overline{\Omega}_q^{m,n}(q-m-n-r)\right\}. \end{aligned}\right\} \quad (59)$$

Proof. Let us suppose that the numbers $\alpha_1, \ldots, \alpha_n$ satisfy the condition

$$\alpha_j^2 \neq \alpha_t^2 \qquad (j = 1, \ldots, n;\ t = 1, \ldots, n;\ j \neq t).$$

By U_r we denote the coefficient of x^r in the expansion of

$$\frac{\prod_{j=1}^{s}(1-\beta_j^2 x)}{\prod_{j=1}^{n}(1-\alpha_j^2 x)} \qquad (s \geqq 0;\ n \geqq 0)$$

in ascending powers of x. By V_r we denote the coefficient of x^r in the expansion of

$$\frac{\prod_{j=1}^{s}\left(1-\frac{x}{\beta_j^2}\right)}{\prod_{j=1}^{n}\left(1-\frac{x}{\alpha_j^2}\right)}$$

in ascending powers of x.

Finally we write

$$W_t = \frac{\prod_{j=1}^{s}\left(\frac{\beta_j}{\alpha_t} - \frac{\alpha_t}{\beta_j}\right)}{\prod_{\substack{j=1 \\ j\neq t}}^{n}\left(\frac{\alpha_j}{\alpha_t} - \frac{\alpha_t}{\alpha_j}\right)}.$$

From the definition of the coefficient U_r it follows that for small values of $|x|$ the following expansion holds

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} = (-1)^{n-s}\,\frac{\prod_{j=1}^{n}\alpha_j}{\prod_{j=1}^{s}\beta_j}\sum_{r=0}^{\infty} U_r x^r; \quad . \quad . \quad . \quad (60)$$

for large values of $|x|$ however we have according to the definition of the coefficient V_r

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} = \frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=0}^{\infty} V_h x^{s-n-h}. \quad . \quad . \quad . \quad . \quad (61)$$

The coefficients U_r and V_r satisfy further the relations

$$U_r = V_r = 0 \text{ for } r = -1, -2, -3, \ldots.$$

We now consider the expansion of the left-hand side of (61) in partial fractions. This expansion runs as follows [27])

$$\frac{\prod_{j=1}^{s}\left(\beta_j x - \frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x - \frac{1}{\alpha_j}\right)} - \frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=0}^{s-n} V_h x^{s-n-h} = \sum_{t=1}^{n}\frac{\alpha_t^{n-s-1} W_t}{\alpha_t x - \frac{1}{\alpha_t}}. \quad . \quad (62)$$

Now the coefficient of x^r in the expansion of the right-hand side of (62) in ascending powers of x is $-\sum_{t=1}^{n}\alpha_t^{n-s+2r} W_t$.

If $r \geqq 0$, we find therefore on account of (62) and (60)

$$\sum_{t=1}^{n}\alpha_t^{n-s+2r} W_t = (-1)^{n-s+1}\,\frac{\prod_{j=1}^{n}\alpha_j}{\prod_{j=1}^{s}\beta_j} U_r + \frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j} V_{s-n-r}, \quad . \quad (63)$$

this relation still being true when $r > s - n$, since $V_h = 0$ for $h = -1, -2, -3, \ldots$.

We may yet show that (63) also holds when $r < 0$.

---

[27]) Since $V_h = 0$ for $h < 0$, formula (62) also holds when $n > s$ (comp. definition 4).

For we have by (61)

$$\frac{\prod_{j=1}^{s}\left(\beta_j x-\frac{1}{\beta_j}\right)}{\prod_{j=1}^{n}\left(\alpha_j x-\frac{1}{\alpha_j}\right)}-\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=0}^{s-n}V_h x^{s-n-h}=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=s-n+1}^{\infty}V_h x^{s-n-h}.$$

From this formula and (62) it follows

$$\sum_{t=1}^{n}\frac{\alpha_t^{n-s-1}W_t}{\alpha_t x-\frac{1}{\alpha_t}}=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}\sum_{h=s-n+1}^{\infty}V_h x^{s-n-h}.$$

Since the coefficient of $x^{-\mu}$ in the expansion of the left-hand side of this relation is equal to $\sum_{t=1}^{n}\alpha_t^{n-s-2\mu}W_t$, we get for $\mu>0$

$$\sum_{t=1}^{n}\alpha_t^{n-s-2\mu}W_t=\frac{\prod_{j=1}^{s}\beta_j}{\prod_{j=1}^{n}\alpha_j}V_{s-n+\mu}.$$

If herein μ is replaced by $-r$, we find (63) (since $U_r=0$ for $r=-1, -2, -3, \ldots$) so that (63) is true for any integer.

We now put $s=q-m$, $\alpha_1=e^{\pi i a_1}, \ldots, \alpha_n=e^{\pi i a_n}$, $\beta_1=e^{\pi i b_{m+1}}, \ldots, \beta_s=e^{\pi i b_q}$. Formula (63) reduces then to (59), which proves the lemma.

Lemma 4. *Suppose that m, n, q and λ are integers with*

$$0\leqq m\leqq q,\quad n\geqq 1 \qquad (64)$$

and

$$0\leqq\lambda\leqq m+n-q-1; \qquad (65)$$

suppose further that the numbers $a_1, \ldots, a_{n+1}$ satisfy the condition

$$a_j-a_t\neq 0, \pm 1, \pm 2, \ldots \quad (j=1, \ldots, n+1;\ t=1, \ldots, n;\ j\neq t).$$

Then the following formula holds:

$$\left.\begin{aligned}&\sum_{t=1}^{n}e^{(m+n-q-2\lambda-1)\pi i a_t}\,\Delta_q^{m,n+1}(t)\\ &=-\pi^{m+n-q}\,e^{(m+n-q-2\lambda-1)\pi i a_{n+1}}\,\frac{\prod_{j=m+1}^{q}\sin(b_j-a_{n+1})\pi}{\prod_{j=1}^{n}\sin(a_j-a_{n+1})\pi}.\end{aligned}\right\} \qquad (66)$$

Proof. If λ satisfies (65), it follows from (59) with $r=-\lambda-1$ and $n+1$ instead of n, on account of (49)

$$\sum_{t=1}^{n+1}e^{(m+n-q-2\lambda-1)\pi i a_t}\,\Delta_q^{m,n+1}(t)=0.$$

Hence we have in view of (16)

$$\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \triangle_q^{m,n+1}(t) = -e^{(m+n-q-2\lambda-1)\pi i a_{n+1}} \triangle_q^{m,n+1}(n+1)$$

$$= -\pi^{m+n-q} e^{(m+n-q-2\lambda-1)\pi i a_{n+1}} \frac{\prod_{j=m+1}^{q} \sin(b_j - a_{n+1})\pi}{\prod_{j=1}^{n} \sin(a_j - a_{n+1})\pi},$$

so that the lemma has been proved.

Lemma 5. *Suppose that m, n, q and λ are integers which satisfy the conditions* (64) *and* (65); *suppose further that the numbers $a_1, \ldots, a_n$ fulfil the condition* (20) *and satisfy besides the inequality*

$$a_t - \beta \neq 0, \pm 1, \pm 2, \ldots \quad (t = 1, \ldots, n).$$

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \frac{\triangle_q^{m,n}(t)}{\sin(\beta - a_t)\pi} \\ &= -\pi^{m+n-q-1} e^{(m+n-q-2\lambda-1)\pi i \beta} \frac{\prod_{j=m+1}^{q} \sin(b_j - \beta)\pi}{\prod_{j=1}^{n} \sin(a_j - \beta)\pi} \end{aligned}\right\} \quad \ldots \quad (67)$$

Proof. From the definition of the coefficient $\triangle$ it follows that $\triangle_q^{m,n+1}(t)$ with $1 \leqq t \leqq n$ and $a_{n+1} = \beta$ is equal to

$$\frac{\pi \triangle_q^{m,n}(t)}{\sin(\beta - a_t)\pi}.$$

Formula (67) is therefore a particular case of (66).

Lemma 6. *Suppose that h, m, n, q and λ are integers with*

$$1 \leqq m+1 \leqq h \leqq q \text{ and } 0 \leqq \lambda \leqq m+n-q-1;$$

suppose further that the numbers $a_1, \ldots, a_n$ fulfil the condition (20) *and satisfy besides the inequality*

$$a_t - b_h \neq 0, \pm 1, \pm 2, \ldots \quad (t = 1, \ldots, n;\ m+1 \leqq h \leqq q).$$

Then the following formula holds:

$$\sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \frac{\triangle_q^{m,n}(t)}{\sin(b_h - a_t)\pi} = 0. \quad \ldots \quad (68)$$

Proof. Formula (68) is a particular case of (67), since the right-hand side of (67) vanishes for $\beta = b_h$ $(m+1 \leqq h \leqq q)$.

Lemma 7. *Suppose that m, n and p are integers with $0 \leqq n \leqq p$ and $m \geqq 0$; further that λ is an arbitrary integer; finally that the numbers $b_1, \ldots, b_m$ satisfy the condition*

$$b_j - b_s \neq 0, \pm 1, \pm 2, \ldots \quad (j = 1, \ldots, m;\ s = 1, \ldots, m;\ j \neq s).$$

Then the following formula holds:

$$\left.\begin{aligned} & B_p^{m,n}\,\Gamma_p^{m,n}(\lambda) - \overline{B}_p^{m,n}\,\overline{\Gamma}_p^{m,n}(p-m-n-\lambda) \\ & = -2i\pi^{m+n-p}\sum_{s=1}^{m} e^{(m+n-p+2\lambda)\pi i b_s}\,\frac{\prod_{j=n+1}^{p}\sin(a_j-b_s)\pi}{\prod_{\substack{j=1\\ j\neq s}}^{m}\sin(b_j-b_s)\pi} \end{aligned}\right\}. \quad (69)$$

Proof. If we replace m, n, q, r, a_j and b_j successively by n, m, p, λ, b_j and a_j, the coefficients $A^{m,n}_q$, $\overline{A}^{m,n}_q$, $\Omega^{m,n}_q(r)$ and $\overline{\Omega}^{m,n}_q(r)$ transform successively into $B_p^{m,n}$, $\overline{B}_p^{m,n}$, $\Gamma_p^{m,n}(\lambda)$ and $\overline{\Gamma}_p^{m,n}(\lambda)$; on account of (16) formula (59) reduces then to (69).

Mathematics. — *Primitief-symmetrische projectieve invarianten.* II. By P. G. MOLENAAR. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of February 23, 1946.)

§ 5. Beschouwt men a^{ikl}, b^{ikl}, c^{ikl} en d^{ikl} als aequivalente grootheden, en vervangt men dan in

$$R_E = R(abcd) = (ab)^2 (cd)^2 (ad)(bc) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

de b^{ikl}, c^{ikl} en d^{ikl} door a^{ikl}, dan verkrijgt men de *discriminant*

$$R = 2\{4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2) - (a_0 a_3 - a_1 a_2)^2\} =$$
$$= 12 a_0 a_1 a_2 a_3 - 8 a_0 a_2^3 - 2 a_0^2 a_3^2 - 8 a_1^3 a_3 + 6 a_1^2 a_2^2 \quad . \quad . \quad (1a)$$

van de enkele grondvorm

$$f = a_x^3 = a^{ikl} x_i x_k x_l = a_0 x_1^3 + 3 a_1 x_1^2 x_2 + 3 a_2 x_1 x_2^2 + a_3 x_2^3.$$

Volgens § 4(11) is $R(abcd) = R_E$ te splitsen in vier primitief-symmetrische invarianten. Vervangt men echter in $L_{(2)}$, $L_1^{(3)}$ en $L_2^{(3)}$ de b^{ikl}, c^{ikl} en d^{ikl} door a^{ikl}, dan wordt elk dezer vormen identiek nul, want volgens § 1(7) is

$$\zeta_{(1)}^1 \, \zeta_{(\alpha)}^i \, I = 0 \qquad (\alpha \neq 1)$$

dus

$$\zeta_{(1)}^1 \, I_{(\alpha)}^i = 0 \qquad (\alpha \neq 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Nu is

$$\zeta_{(1)}^1 = \tfrac{1}{6}(e_E + e_A + e_B + e_C + e_D + e_F)$$

zodat de E-component van (2) luidt

$$\tfrac{1}{6}\left(L_{i\,E}^{(\alpha)\,i} + L_{i\,A}^{(\alpha)\,i} + L_{i\,B}^{(\alpha)\,i} + L_{i\,C}^{(\alpha)\,i} + L_{i\,D}^{(\alpha)\,i} + L_{i\,F}^{(\alpha)\,i}\right) = 0 \qquad (\alpha \neq 1).$$

Vervangt men hierin echter b^{ikl}, c^{ikl} en d^{ikl} door a^{ikl}, dan worden deze zes isomeren van $L_{(\alpha)}^i$ aan elkaar gelijk, dus is

$$L_{(\alpha)}^i (a\,a\,a\,a) = 0 \qquad (\alpha \neq 1).$$

De *symmetrische* invariant

$$L_{(1)} = \tfrac{1}{6}(R_E + R_A + R_B + R_C + R_D + R_F) \quad . \quad . \quad . \quad . \quad (3)$$

levert dus bij gelijkstelling van de als aequivalent beschouwde coëfficiënten de discriminant van de enkele grondvorm.

Beschouwt men a_x^3, b_x^3, c_x^3 en d_x^3 als niet-aequivalente grondvormen, dan wordt de invariant $L_{(1)}$ in symbolische notatie gegeven door (3) en (1). We willen deze invariant in niet-symbolische vorm schrijven en vinden dan achtereenvolgens

$$\begin{aligned}R_E = {} & (-a_1b_2c_3 + a_1b_3c_2 + a_3b_1c_2 + 2a_2b_1c_3 - a_3b_0c_3 - 2a_2b_2c_2)\,d_0 + \\ & + (+4a_1b_2c_2 + 4a_2b_2c_1 - 5a_2b_1c_2 - 2a_1b_3c_1 - 2a_3b_1c_1 - 2a_1b_1c_3 + \\ & \qquad + 2a_3b_0c_2 + a_0b_2c_3 - a_0b_3c_2 + a_2b_0c_3)\,d_1 + \\ & + (+4a_2b_1c_1 + 4a_1b_1c_2 - 5a_1b_2c_1 - 2a_2b_0c_2 - 2a_0b_2c_2 - 2a_2b_2c_0 + \\ & \qquad + 2a_0b_3c_1 + a_3b_1c_0 - a_3b_0c_1 + a_1b_3c_0)\,d_2 + \\ & + (-a_2b_1c_0 + a_2b_0c_1 + a_0b_2c_1 + 2a_1b_2c_0 - a_0b_3c_0 - 2a_1b_1c_1)\,d_3\end{aligned}$$

en indien

$$\sum_{\binom{ikl}{123}},\ \sum_{\binom{ikl}{033}},\ \sum_{\binom{iklm}{0123}},\ \sum_{\binom{iklm}{0222}},\ \sum_{\binom{iklm}{0033}}$$

resp. beduiden sommaties over de 6 variaties van 123, de 3 variaties van 033, de 24 variaties van 0123, de 4 variaties van 0222 en de 6 variaties van 0033,

$$\begin{aligned}L_{(1)} = {} & \tfrac{1}{6}\Big[\Big\{3\sum_{\binom{ikl}{123}} a_i b_k c_l - 2\sum_{\binom{ikl}{033}} a_i b_k c_l - 12\, a_2 b_2 c_2\Big\}\,d_0 + \\ & + \Big\{3\sum_{\binom{ikl}{023}} a_i b_k c_l + 6\sum_{\binom{ikl}{122}} a_i b_k c_l - 12\sum_{\binom{ikl}{113}} a_i b_k c_l\Big\}\,d_1 + \\ & + \Big\{3\sum_{\binom{ikl}{013}} a_i b_k c_l + 6\sum_{\binom{ikl}{112}} a_i b_k c_l - 12\sum_{\binom{ikl}{022}} a_i b_k c_l\Big\}\,d_2 + \\ & + \Big\{3\sum_{\binom{ikl}{012}} a_i b_k c_l - 2\sum_{\binom{ikl}{003}} a_i b_k c_l - 12\, a_1 b_1 c_1\Big\}\,d_3\Big] = \\ = {} & \tfrac{1}{2}\sum_{\binom{iklm}{0123}} a_i b_k c_l d_m - \tfrac{1}{3}\sum_{\binom{iklm}{0033}} a_i b_k c_l d_m + \sum_{\binom{iklm}{1122}} a_i b_k c_l d_m + \\ & - 2\sum_{\binom{iklm}{0222}} a_i b_k c_l d_m - 2\sum_{\binom{iklm}{1113}} a_i b_k c_l d_m.\end{aligned}$$

Deze invariant bestaat uit 44 verschillende termen. Vervangt men b_i, c_i, d_i door a_i, dan vindt men weer

$$12a_0\,a_1\,a_2\,a_3 - 2a_0^2\,a_3^2 + 6a_1^2\,a_2^2 - 8a_0\,a_2^3 - 8a_1^3\,a_3 = R.$$

Omgekeerd kan men $L_{(1)}$ uit R afleiden, want

$$R = \sum_{i,k,l,m} \frac{\partial^4 R_E}{\partial a_i\,\partial b_k\,\partial c_l\,\partial d_m}\, a_i\,a_k\,a_l\,a_m$$

dus

$$\frac{\partial^4 R}{\partial a_i\,\partial a_k\,\partial a_l\,\partial a_m} = \frac{\partial^4 R_E}{\partial a_i\,\partial b_k\,\partial c_l\,\partial d_m} + \frac{\partial^4 R_E}{\partial a_k\,\partial b_i\,\partial c_l\,\partial d_m} + \ldots.$$

Nu is

$$L_{(1)} = \tfrac{1}{6}(R_E + R_A + \ldots + R_F) = \tfrac{1}{24} \times \text{de som der 24 isomeren van } R_E =$$

$$= \tfrac{1}{24} \sum_{i,k,l,m} \frac{\partial^4 R_E}{\partial a_i\, \partial b_k\, \partial c_l\, \partial d_m} (a_i\, b_k\, c_l\, d_m + b_i\, a_k\, c_l\, d_m + \ldots.) =$$

$$= \tfrac{1}{24} \sum_{i,k,l,m} \left(\frac{\partial^4 R_E}{\partial a_i\, \partial b_k\, \partial c_l\, \partial d_m} + \frac{\partial^4 R_E}{\partial a_k\, \partial b_i\, \partial c_l\, \partial d_m} + \ldots. \right) a_i\, b_k\, c_l\, d_m.$$

dus

$$L_{(1)} = \tfrac{1}{24} \sum_{i,k,l,m} \frac{\partial^4 R}{\partial a_i\, \partial a_k\, \partial a_l\, \partial a_m}\, a_i\, b_k\, c_l\, d_m. \quad . \quad . \quad . \quad . \quad (3a)$$

De *anti-symmetrische* invariant

$$L_{(2)} = \tfrac{1}{6}(R_E + R_A + R_B - R_C - R_D - R_F)$$

heeft een eenvoudige betekenis. Volgens § 4(4) is

$$\left.\begin{aligned} R_E - R_D &= (ab)^2\,(cd)^2\,(ad)\,(bc) - (ba)^2\,(cd)^2\,(bd)\,(ac) = \\ &= (ab)^2\,(cd)^2\,\{(ad)\,(bc) - (bd)\,(ac)\} = -(ab)^3\,(cd)^3 \end{aligned}\right\}$$

en evenzo

$$\left.\begin{aligned} R_A - R_C &= -(bc)^3\,(ad)^3 \\ R_B - R_F &= -(ca)^3\,(bd)^3 \end{aligned}\right\} \quad . \quad (4)$$

dus

$$L_{(2)} = -\tfrac{1}{6}\{(ab)^3\,(cd)^3 + (bc)^3\,(ad)^3 + (ca)^3\,(bd)^3\}. \quad . \quad . \quad . \quad (5)$$

Men herkent onmiddellijk het anti-symmetrisch karakter. Bovendien blijkt, dat deze primitief-symmetrische invariant uiteenvalt in invarianten van lagere graad. Deze zijn reeds primitief-symmetrisch. $L_{(2)}$ behoeft dus niet in de moduulbasis te worden opgenomen.

Verder is (5) te herleiden tot

$$L_{(2)} = \tfrac{1}{2} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix} = \tfrac{1}{2}(ab)\,(ac)\,(ad)\,(bc)\,(bd)\,(cd) = \tfrac{1}{2} P_E. \quad . \quad (6)$$

$L_{(2)} = 0$ betekent dus, dat de vier grondvormen *lineair afhankelijk* zijn. $L_{(2)}$ levert, in tegenstelling tot $L_{(1)}$, uitsluitend een invariant van de vier grondvormen.

De invarianten $L_{1}^{(3)}$ en $L_{2}^{(3)}$ bezitten *cyclische symmetrie*.
Evenals onder (2) bewijst men

$$\zeta_{1}^{1}{}_{(1)}\, I_{i}^{i}{}_{(3)} = 0 \quad \text{en} \quad \zeta_{1}^{1}{}_{(2)}\, I_{i}^{i}{}_{(3)} = 0$$

waaruit volgt

$$\left(\zeta_{1}^{1}{}_{(1)} + \zeta_{1}^{1}{}_{(2)}\right) I_{i}^{i}{}_{(3)} = 0 \quad \text{of} \quad \tfrac{1}{3}(e_E + e_A + e_B)\, I_{i}^{i}{}_{(3)} = 0.$$

De E-component hiervan geeft

$$L^{l}_{(3)\,l\,E} + L^{l}_{(3)\,l\,A} + L^{i}_{(3)\,i\,B} = 0.$$

Deze cyclische symmetrie $abc \to bca \to cab$ heeft betrekking op de drie coëfficiëntenrijen a^{ikl}, b^{ikl}, c^{ikl}. Uit de toelaatbare permutaties der viergroep van KLEIN volgt echter, dat deze cyclische symmetrie aanwezig is voor elk willekeurig drietal der vier coëfficiëntenrijen.

Evenals $L_{(2)}$ zijn ook deze cyclisch-symmetrische invarianten $L_{1}^{(3)}$ en $L_{2}^{(3)}$ uit te drukken in invarianten van lagere graad. Men heeft volgens (4)

$$R_E - R_D = -(ab)^3 (cd)^3 \quad \text{en} \quad R_F - R_B = (ac)^3 (db)^3$$

dus volgens § 4 (10a)

$$L_{1}^{(3)} = \tfrac{1}{8}\{(ac)^3 (db)^3 - (ab)^3 (cd)^3\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Evenzo vindt men

$$L_{2}^{(3)} = \tfrac{1}{8}\{ad)^3 \{(bc)^3 - (ac)^3 (db)^3\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Men controleert gemakkelijk het cyclisch-symmetrisch karakter van deze invarianten.

Verder is

$$L_{(3)} = L_{1}^{(3)} + L_{2}^{(3)} = \tfrac{1}{8}\{(ad)^3 (bc)^3 - (ab)^3 (cd)^3\}. \quad . \quad . \quad . \quad . \quad (9)$$

Ook aan deze invarianten kan men een betekenis geven.

De dubbelverhouding van vier gerangschikte punten P, Q, R, S, bepaald door $p_x = 0$, $q_x = 0$, $r_x = 0$, $s_x = 0$ is

$$\mu = \frac{(pr)\,(qs)}{(qr)\,(ps)}.$$

Naar analogie hiermede kan men onder de „*cubische dubbelverhouding*" van vier gerangschikte puntentripels $A_1 A_2 A_3$, $B_1 B_2 B_3$, $C_1 C_2 C_3$, $D_1 D_2 D_3$, bepaald door

$$a_x^3 = 0 \quad b_x^3 = 0 \quad c_x^3 = 0 \quad d_x^3 = 0$$

verstaan

$$\mu = \frac{(ac)^3\,(bd)^3}{(bc)^3\,(ad)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Is $L_{2}^{(3)} = 0$, dan is $(ac)^3\,(db)^3 = (ad)^3\,(bc)^3$ dus $\mu = -1$. De puntentripels $C_1 C_2 C_3$, $D_1 D_2 D_3$ liggen dan *cubisch harmonisch* ten opzichte van de tripels $A_1 A_2 A_3$, $B_1 B_2 B_3$.

Uit $L_{1}^{(3)} = 0$ en $L_{(3)} = 0$ volgen analoge liggingen voor andere paren puntentripels.

De dubbelverhouding (10) neemt bijzondere waarden aan, indien apolaire ligging van twee puntentripels optreedt. Ligt b.v. $A_1 A_2 A_3$ apolair ten opzichte van $C_1 C_2 C_3$, dan is $(ac)^3 = 0$, dus $\mu = 0$.

§ 6. Onder een *gemengde vorm* verstaat men een vorm

$$F = A^{i_1 \dots i_p, k_1 \dots k_\pi} x_{i_1} \dots x_{i_p} \xi_{k_1} \dots \xi_{k_\pi}$$

wier veranderlijken $x_1 \dots x_n$ en $\xi_1 \dots \xi_n$ tot twee verschillende gebieden behoren.

Deze twee rijen veranderlijken kunnen onderworpen worden aan twee van elkaar onafhankelijke lineaire homogene transformaties:

$$\left.\begin{aligned} x_l &= d_l^l \bar{x}_l \quad (i = 1 \dots n) \\ \xi_k &= \delta_k^h \bar{\xi}_h \quad (k = 1 \dots n) \end{aligned}\right\} \quad \dots\dots\dots \quad (1)$$

De coëfficiënten d_i^l zijn dus onafhankelijk van de coëfficiënten δ_k^h. Door deze transformatie (1) wordt in de coëfficiënten van F de transformatie:

$$\bar{A}^{l_1 \dots l_p, m_1 \dots m_\pi} = d_{i_1}^{l_1} \dots d_{i_p}^{l_p} \delta_{k_1}^{m_1} \dots \delta_{k_\pi}^{m_\pi} A^{i_1 \dots i_p, k_1 \dots k_\pi} \quad \dots \quad (2)$$

geïnduceerd.

$I(A)$ heet een projectieve invariant van F indien

$$I(\bar{A}) = d^g \delta^\gamma I(A) \quad \dots\dots\dots \quad (3)$$

waarin

$$d = |d_i^l| \quad \text{en} \quad \delta = |\delta_k^m|$$

de transformatiedeterminanten zijn.

F is symbolisch te schrijven als het product van de n-aire vormen

$$f = a_x^p \text{ en } \varphi = \alpha_\xi^n.$$

Het symbolische product

$$a^{i_1} a^{i_2} \dots a^{i_p} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_\pi}$$

geeft dan de coëfficiënt $A^{i_1, i_2, \dots i_p, k_1 k_2 \dots k_\pi}$.

Men heeft dan de fundamentele stelling: elke projectieve invariant van F is symbolisch voor te stellen door een som van producten van haakfactoren $(abc\dots)$ en $(\alpha\beta\gamma\dots)$.

Haakfactoren $(a\alpha\dots)$ behoeven dus niet voor te komen.

Omgekeerd stelt elk product van haakfactoren

$$(abc\dots) \dots\dots (\alpha\beta\gamma\dots) \dots\dots$$

mits het het juiste aantal symboolrijen bevat, een invariant van F voor. Hierin zijn $a\alpha$, $b\beta$, ... aequivalente symboolgroepen voor de coëfficiënten van F.

Bewijs: Is $I(A)$ in A van de graad t, dan voere men twee rijen van t gelijksoortige grondvormen in:

$$\left.\begin{aligned} f_1 &= a^{i_1 \dots i_p} x_{i_1} \dots x_{i_p}, \; f_2 = b^{i_1 \dots i_p} x_{i_1} \dots x_{i_p}, \dots, f_t = l^{i_1 \dots i_p} x_{i_1} \dots x_{i_p} \\ \varphi_1 &= \alpha^{k_1 \dots k_\pi} \xi_{k_1} \dots \xi_{k_\pi}, \; \varphi_2 = \beta^{k_1 \dots k_\pi} \xi_{k_1} \dots \xi_{k_\pi}, \dots, \varphi_t = \lambda^{k_1 \dots k_\pi} \xi_{k_1} \dots \xi_{k_\pi} \end{aligned}\right\} \quad (4)$$

Onderwerpt men dan $I(A)$ aan de *splitsing* S_1:

$$S_1(I) = \frac{1}{t} \frac{\partial I}{\partial A^{i_1 \ldots i_p, k_1 \ldots k_\pi}} a^{i_1 \ldots i_p} \alpha^{k_1 \ldots k_\pi} = K_1(A, a, \alpha) \quad . \quad . \quad (5)$$

dan is K_1 een invariant F, f_1 en φ_1, die lineair is in de coëfficiënten van f_1 en van φ_1 en die van de graad $t-1$ is in de coëfficiënten van F. Immers

$$K_1(A, a, \alpha) = \frac{1}{t} \sum \frac{\partial \frac{I(\bar{A})}{d^g \delta^\gamma}}{\partial A} a \alpha = \frac{1}{t d^g \delta^\gamma} \sum \frac{\partial I(\bar{A})}{\partial A} a \alpha =$$

$$= \frac{1}{t d^g \delta^\gamma} \sum \frac{\partial I(\bar{A})}{\partial \bar{A}} \sum \frac{\partial \bar{A}}{\partial A} a \alpha = \frac{1}{t d^g \delta^\gamma} \sum \frac{\partial I(\bar{A})}{\partial \bar{A}} \bar{a} \bar{\alpha} =$$

$$= \frac{1}{d^g \delta^\gamma} K_1(\bar{A}, \bar{a}, \bar{\alpha}).$$

Vervangt men in K_1 elk product $a^{i_1 \ldots i_p} \alpha^{k_1 \ldots k_\pi}$ door $A^{i_1 \ldots i_p, k_1 \ldots k_\pi}$, hetgeen bereikt wordt door de *contractie* $C_{a \alpha A}$

$$C_{a \alpha A}(K_1) = \frac{\partial^2 K_1}{\partial a^{i_1 \ldots i_p} \partial \alpha^{k_1 \ldots k_\pi}} A^{i_1 \ldots i_p, k_1 \ldots k_\pi} \quad . \quad . \quad . \quad . \quad (6)$$

dan verkrijgt men $I(A)$ weer.

Onderwerpt men K_1 vervolgens aan de splitsingen

$$S_2 = \frac{1}{t-1} \sum \frac{\partial}{\partial A} b \beta, \ldots, S_t = \sum \frac{\partial}{\partial A} l \lambda$$

dan verkrijgt men een vorm

$$K_t = K_t(a^{i_1 \ldots i_p}, \alpha^{k_1 \ldots k_\pi}, \ldots, l^{i_1 \ldots i_p}, \lambda^{k_1 \ldots k_\pi}) \quad . \quad . \quad . \quad . \quad (7)$$

die lineair en homogeen is in elk der rijen

$$a^{i_1 \ldots i_p}, \alpha^{k_1 \ldots k_\pi}, \ldots, l^{i_1 \ldots i_p}, \lambda^{k_1 \ldots k_\pi}$$

en die een invariant is van de vormen (4).

Door op K_t de contracties

$$C_{a \alpha A}, \; C_{b \beta A}, \ldots, C_{l \lambda A}$$

toe te passen, verkrijgt men $I(A)$ weer.

Gaat men deze vormen (4) nu symbolisch voorstellen

$$\left.\begin{aligned} f_1 &= a_x^p, \; f_2 = b_x^p, \ldots, f_t = l_x^p \\ \varphi_1 &= \alpha_\xi^\pi, \; \varphi_2 = \beta_\xi^\pi, \ldots, \varphi_t = \lambda_\xi^\pi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

waardoor $a^{i_1 \ldots i_p}$ vervangen wordt door een product van symbolen $a^{i_1} a^{i_2} \ldots a^{i_p}$, dan gaat (7) over in de vorm

$$K_t^* = K_t^*(a^i, \alpha^k \ldots\ldots l^i, \lambda^k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

die homogeen is in de symbolen a tot de macht p, in α tot de macht π, enz., en die een invariant is van de lineaire vormen

$$\left.\begin{matrix} a_x, b_x, \ldots, l_x \\ \alpha_\xi, \beta_\xi, \ldots, \lambda_\xi \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots \quad (10)$$

Onderwerpt men nu x en ξ aan de transformatie (1), dan worden de symbolen a en α, b en β, enz. getransformeerd in

$$\left.\begin{matrix} \bar{a}^l = d_i^l\, a^i \\ \bar{\alpha}^m = \delta_k^m\, \alpha^k \quad \text{enz.} \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots \quad (11)$$

Nu is

$$K_t^* (\bar{a}^l, \bar{\alpha}^m, \ldots) = d^g\, \delta^\gamma\, K_t^* (a^l, \alpha^m, \ldots)$$

dus

$$K_t^* (d_i^l\, a^i, \delta_k^m\, \alpha^k, \ldots.) = d^g\, \delta^\gamma\, K_t^* (a^l, \alpha^m, \ldots.) \ldots\ldots \quad (12)$$

Elke term van het linkerlid van (12) bestaat uit $tp = gn$ factoren, die lineair en homogeen zijn in d_i^l, en uit $t\pi = \gamma n$ factoren, die lineair en homogeen zijn in δ_k^m

Daar d_i^l en δ_k^m onafhankelijk van elkaar zijn, zal een g-voudige toepassing van het *proces van* CAYLEY:

$$\Omega_d = \Sigma \pm \frac{\partial^n}{\partial d_1^{i_1} \ldots \partial d_n^{i_n}}$$

gevolgd door een γ-voudige toepassing van het proces

$$\Omega_\delta = \Sigma \pm \frac{\partial^n}{\partial \delta_1^{i_1} \ldots \partial \delta_n^{i_n}}$$

uit het linkerlid van (12) een veelterm doen ontstaan, waarvan elke term bestaat uit g haakfactoren van de soort $(abc\ldots)$ en uit γ haakfactoren van de soort $(\alpha\beta\gamma\ldots)$, terwijl het rechterlid overgaat in $K^*(a^l, \alpha^m, \ldots)$, vermenigvuldigd met twee van nul verschillende constanten. Hiermede is het eerste deel der stelling bewezen. Het tweede deel volgt onmiddellijk uit het invariante karakter der haakfactoren.

Laat nu $L^* = (ab\ldots)\,(\ldots\ldots)\,\ldots$ de symbolische voorstelling zijn van een invariant

$$L = L (a^{i_1 \ldots i_p},\ b^{i_1 \ldots i_p}, \ldots)$$

die lineair en homogeen is in de coëfficiënten van de t gelijksoortige grondvormen

$$f_1, f_2, \ldots\ldots, f_t$$

en laat $\Lambda^* = (\alpha\beta\ldots)\,(\ldots\ldots)\,\ldots$ de symbolische voorstelling zijn van een

invariant

$$A = A\,(\alpha^{k_1 \ldots k_\pi},\ \beta^{k_1 \ldots k_\pi}, \ldots)$$

die lineair en homogeen is in de coëfficiënten van de t gelijksoortige grondvormen

$$\varphi_1, \varphi_2, \ldots\ldots, \varphi t$$

Past men dan op het product $L\,A$ de contracties

$$C_{a\alpha A},\ C_{b\beta A}, \ldots, C_{l\lambda A}$$

toe, dan verkrijgt men een t-de graads invariant $I(A)$ van de gemengde vorm $F = f\,.\,\varphi = A^{i_1 \ldots i_p, k_1 \ldots k_\pi}\, x_{l_1} \ldots x_{l_p}\, \xi_{k_1} \ldots \xi_{k_\pi}$.

Onderwerpt men $L\,A$ echter aan de contracties

$$C_{a\alpha A},\ C_{b\beta B}, \ldots C_{l\lambda L}$$

dan verkrijgt men een simultaaninvariant

$$[L\,A] = I(A, B, \ldots., L)$$

van t gelijksoortige grondvormen

$$F_1 = A^{i_1 \ldots i_p, k_1 \ldots k_\pi}\, x_{l_1} \ldots x_{l_p}\, \xi_{k_1} \ldots \xi_{k_\pi},$$

$$F_2 = B^{i_1 \ldots i_p, k_1 \ldots k_\pi}\, x_{l_1} \ldots x_{l_p}\, \xi_{k_1} \ldots \xi_{k_\pi}, \text{ enz.}$$

die lineair is in elk der coëfficiëntenrijen.

§ 7. Bij de contractie spelen de symmetrie-eigenschappen van L en A een belangrijke rol.

Dit moge blijken uit de twee volgende voorbeelden.

1. De binaire gemengde vorm

$$F = A^{ik}\, x_i\, \xi_k = A^{11}\, x_1\, \xi_1 + A^{12}\, x_1\, \xi_2 + A^{21}\, x_2\, \xi_1 + A^{22}\, x_2\, \xi_2$$

wordt symbolisch voorgesteld door $F = a_x\, \alpha_\xi$.

$f = a_x$ en $\varphi = \alpha_\xi$ hebben geen invarianten. Wel hebben $f_1 = a_x$ en $f_2 = b_x$ de anti-symmetrische invariant

$$L = (ab)$$

terwijl $\varphi_1 = \alpha_\xi$ en $\varphi_2 = \beta_\xi$ de anti-symmetrische invariant

$$A = (\alpha\beta)$$

bezitten. Dan is

$$I = [L\,A] = [(a\,b)\,(\alpha\,\beta)] = A^{11}\,B^{22} - A^{12}\,B^{21} - A^{21}\,B^{12} + A^{22}\,B^{11}$$

een simultaaninvariant van

$$F_1 = A^{ik}\, x_i\, \xi_k \quad \text{en} \quad F_2 = B^{ik}\, x_i\, \xi_k$$

die symmetrisch is in de coëfficiëntenrijen A^{ik} en B^{ik}.

Zijn F_1 en F_2 aequivalente grondvormen, dan volgt hieruit een quadratische invariant

$$I = 2(A^{11}\,A^{22} - A^{12}\,A^{21})$$

van de enkele grondvorm $F = A^{ik}\, x_i\, \xi_k$.

2. Is de binaire vorm

$$F = a_x^2\, a_\xi = A^{ik,l}\, x_i\, x_k\, \xi_l \qquad (i, k, l = 1, 2)$$

gegeven, dan is

$$I = (ab)^2\, (\alpha\beta)$$

de symbolische voorstelling van een simultaaninvariant van de gelijksoortige grondvormen

$$F_1 = a_x^2\, a_\xi \quad \text{en} \quad F_2 = b_x^2\, \beta_\xi$$

$L = (ab)^2$ is symmetrisch in de rijen a^{ik} en b^{ik}.
$\Lambda = (\alpha\beta)$ is daarentegen anti-symmetrisch in de rijen α^i en β^i.

$$I = A^{11,1}\, B^{22,2} - A^{11,2}\, B^{22,1} - 2\, A^{12,1}\, B^{12,2} + \\ + 2\, A^{12,2}\, B^{12,1} + A^{22,1}\, B^{11,2} - A^{22,2}\, B^{11,1}$$

is dan een anti-symmetrische invariant van F_1 en F_2.

Zijn F_1 en F_2 aequivalent, dan wordt I nul.

§ 8. Uit de n isomeren van $L(a, b, \ldots, l)$ en de n isomeren van $\Lambda(\alpha, \beta, \ldots, \lambda)$ kan men door contractie n^2 invarianten van de gemengde vorm afleiden. Deze willen we uitdrukken in primitief-symmetrische invarianten.

De invariant $L(a, b, \ldots, l)$ genereert in $\mathfrak{N}$ de vector

$$L = L_E\, e_E + L_A\, e_A + \ldots + L_G\, e_G.$$

Onderwerpt men de coëfficiëntenrijen $a, b, \ldots, l$ in elk der isomeren L_R aan de permutatie S^{-1}, dan verkrijgt men de componenten van de vector

$$L e_S = \overline{L}$$

want volgens § 1 (2) is

$$\overline{L}_R = L_{RS^{-1}} \quad \text{of} \quad L_R = \overline{L}_{RS}.$$

De vector L wordt dus volgens § 1 (3) aan de afbeelding S der reguliere voorstelling onderworpen.

Binnen elk der ruimten $\mathfrak{R}^{(\sigma)}_i$ wordt hierdoor echter een irreducibele voorstelling Γ_σ geïnduceerd.

Zijn nu

$$L^{(\sigma)\,i}_{k} = \frac{n_\sigma}{n} \sum_S s^{(\sigma)\,i}_{k}\, L_{S^{-1}} \quad \ldots\ldots\ldots \quad (1)$$

de primitief-symmetrische invarianten van $L(a, b, \ldots, l)$, dan heeft L langs $\zeta^{(\sigma)\,k}_{l}$ volgens § 3 (7) de component

$$\frac{n}{n_\sigma}\, L^{(\sigma)\,i}_{k}$$

en dan geeft de irreducibele voorstelling binnen $\mathfrak{R}^{(\sigma)}_i$ volgens § 1 (8) de betrekking

$$L^{i}_{k}{}^{(\sigma)} = \sum_{l=1}^{n_\sigma} s^{l}_{k}{}^{(\sigma)}\, \bar{L}^{i}_{l}{}^{(\sigma)} \qquad \begin{pmatrix} i \text{ vast} \\ k = 1, \ldots, n_\sigma \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hierin is $\frac{n}{n_\sigma} \bar{L}^{i}_{l}{}^{(\sigma)}$ de component van $L\, e_S$ langs $\zeta^{l}_{i}{}^{(\sigma)}$, dus

$$\bar{L}^{i}_{l}{}^{(\sigma)}_{R} = L^{i}_{l}{}^{(\sigma)}_{RS^{-1}}.$$

Derhalve volgt uit (2)

$$L^{i}_{k}{}^{(\sigma)} = \sum_{l=1}^{n_\sigma} s^{l}_{k}{}^{(\sigma)}\, L^{i}_{l}{}^{(\sigma)}_{S^{-1}} \quad \text{(niet sommeren over } S\text{)}. \quad . \quad . \quad . \quad (3)$$

Met behulp van dezelfde irreducibele voorstellingen kan men de primitief-symmetrische invarianten

$$A^{p}_{q}{}^{(\varrho)} = \frac{n_\varrho}{n} \sum_S s^{p}_{q}{}^{(\varrho)}\, A_{S^{-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

van $A(\alpha, \beta, \ldots, \lambda)$ definiëren.

Ook voor deze gelden dan betrekkingen analoog (3)

$$A^{p}_{q}{}^{(\varrho)} = \sum_{r=1}^{n_\varrho} s^{r}_{q}{}^{(\varrho)}\, A^{p}_{r}{}^{(\varrho)}_{S^{-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

De invarianten $\left[L^{i}_{k}{}^{(\sigma)}\, A^{p}_{q}{}^{(\varrho)}\right]$ zijn in het algemeen niet primitief-symmetrisch. Het is echter mogelijk, hiervoor een primitief-symmetrische basis te vinden.

Men kan de betrekkingen (3) (waarin $L^{i}_{1}{}^{(\sigma)}, \ldots, L^{i}_{n_\sigma}{}^{(\sigma)}$ volgens slot § 3 lineair onafhankelijk zijn) lezen als een voorstelling Γ_σ van $\mathfrak{S}_t$ in een n_σ-dimensionale ruimte $\mathfrak{R}^{(\sigma)}_i$ en de betrekkingen (5) als een voorstelling Γ_ϱ van $\mathfrak{S}_t$ in een n_ϱ-dimensionale ruimte $\mathfrak{R}^{(\varrho)}_p$.

Uit (3) en (5) volgt nu

$$L^{i}_{k}{}^{(\sigma)}\, A^{p}_{q}{}^{(\varrho)} = \sum_{l=1}^{n_\sigma} \sum_{r=1}^{n_\varrho} s^{l}_{k}{}^{(\sigma)}\, s^{r}_{q}{}^{(\varrho)}\, L^{i}_{l}{}^{(\sigma)}_{S^{-1}}\, A^{p}_{r}{}^{(\varrho)}_{S^{-1}} \qquad (S = E, A, \ldots, G).$$

Door contractie vindt men

$$\left[L^{i}_{k}{}^{(\sigma)}\, A^{p}_{d}{}^{(\varrho)}\right] = \sum_{l=1}^{n_\sigma} \sum_{r=1}^{n_\varrho} s^{l}_{k}{}^{(\sigma)}\, s^{r}_{q}{}^{(\varrho)} \left[L^{i}_{l}{}^{(\sigma)}\, A^{p}_{r}{}^{(\varrho)}\right]_{S^{-1}} \quad (S = E, A, \ldots, G). \quad . \quad (6)$$

Dit is een *productvoorstelling van* KRONECKER in een $n_\sigma\, n_\varrho$-dimensionale ruimte [8]). Een dergelijke voorstelling $\Gamma_\sigma \times \Gamma_\varrho$ is echter door transformatie met een matrix

$$M = \| m^{k,q}_{i} \| \qquad \begin{pmatrix} k = 1, \ldots, n_\sigma;\ q = 1, \ldots, n_\varrho \\ i = 1, \ldots, n_\sigma\, n_\varrho \end{pmatrix} \quad . \quad . \quad . \quad (7)$$

[8]) Zie B. L. V. D. WAERDEN, Moderne Algebra II, pag. 196.

van de graad en de rang $n_\sigma n_\varrho$ te reduceren tot

$$M(\Gamma_\sigma \times \Gamma_\varrho) M^{-1} = c_1 \Gamma_1 + c_2 \Gamma_2 + \ldots c_r \Gamma_r \ldots \quad (8)$$

De coëfficiënten c_i worden uitsluitend bepaald door de karakterrelaties der irreducibele voorstellingen.

De betrekkingen (6) gaan dan over in $c_1 + c_2 + \ldots + c_r$ groepen van betrekkingen

$$ {}_t P^{ip}_{(\tau)\,j} = \sum_{h=1}^{n_\tau} s^{h}_{(\tau)\,j}\, {}_t P^{ip}_{(\tau)\,h}\, S^{-1} \qquad \begin{pmatrix} S = E, A, \ldots, G \\ j = 1, \ldots, n_\tau;\ t = 1, \ldots, c_\tau \end{pmatrix} \quad (9)$$

waarbij

$$ {}_t P^{ip}_{(\tau)\,j} = \sum_{k=1}^{n_\sigma} \sum_{q=1}^{n_\varrho} m^{k,q}_{a_\tau + j} \left[L^{i}_{(\sigma)\,k}\, A^{p}_{(\varrho)\,q} \right] \begin{pmatrix} j = 1, \ldots, n_\tau \\ a_\tau = c_1 n_1 + \ldots + c_{\tau-1} n_{\tau-1} + (t-1) n_\tau \end{pmatrix} \quad (10)$$

Indien echter voor n_τ functies $P_{(\tau)\,j}$ $(j = 1, \ldots, n_\tau)$ van $A, B, \ldots, L$ de betrekkingen

$$P_{(\tau)\,j} = \sum_{h=1}^{n_\tau} s^{h}_{(\tau)\,j}\, P_{(\tau)\,h}\, S^{-1} \qquad (S = E, A, \ldots, G) \quad \ldots \quad (11)$$

gelden, dan zijn deze functies primitief-symmetrisch.

Want stelt men

$$\frac{1}{n_\tau} \sum_{h=1}^{n_\tau} \zeta^{h}_{(\tau)\,j}\, \underline{P}_{(\tau)\,h} = K_{(\tau)\,j}$$

(waarin $\underline{P}_{(\tau)\,h}$ de door $P_{(\tau)\,h}$ gegenereerde vector is), dan is

$$X = \zeta^{j}_{(\tau)\,j}\, K_{(\tau)\,j} = \frac{1}{n_\tau} \sum_{h=1}^{n_\tau} \zeta^{j}_{(\tau)\,j}\, \zeta^{h}_{(\tau)\,j}\, \underline{P}_{(\tau)\,h} = \frac{1}{n_\tau} \sum_{h=1}^{n_\tau} \zeta^{h}_{(\tau)\,j}\, P_{(\tau)\,h} \qquad \text{(niet sommeren over } j\text{)}$$

dus

$$X_E = \frac{1}{n_\tau} \sum_{h=1}^{n_\tau} \frac{n_\tau}{n} \sum_{S} s^{h}_{(\tau)\,j}\, P_{(\tau)\,h}\, S^{-1} = \frac{1}{n} \sum_{S} \sum_{h=1}^{n_\tau} s^{h}_{(\tau)\,j}\, P_{(\tau)\,h}\, S^{-1} = \text{volgens (11)} = P_{(\tau)\,j}\,.$$

$P_{(\tau)\,j}$ wordt dus voorgesteld door $\zeta^{j}_{(\tau)\,j}\, K_{(\tau)\,j}$, en is derhalve een primitief-symmetrische functie van $A, B, \ldots, L$.

Door de betrekkingen (10) worden dus primitief-symmetrische invarianten verkregen, omdat deze voldoen aan (9).

Tenslotte kan men de invarianten $\left[L^{i}_{(\sigma)\,k}\, A^{p}_{(\varrho)\,q} \right]$ uitdrukken in de primitief-symmetrische invarianten ${}_t P^{ip}_{(\tau)\,j}$.

Is $\| M \|$ de inverse matrix van $\| m \|$, dan volgt uit (10)

$$\left.\begin{aligned} \left[L^{i}_{(\sigma)\,k}\, A^{p}_{(\varrho)\,q} \right] = \sum_{\tau=1}^{r} \sum_{t=1}^{c_\tau} \sum_{j=1}^{n_\tau} M^{a_\tau + j}_{k,q}\, {}_t P^{ip}_{(\tau)\,j} \\ \begin{pmatrix} k = 1, \ldots, n_\sigma \\ q = 1, \ldots, n_\varrho \\ a_\tau = c_1 n_1 + \ldots + c_{\tau-1} n_{\tau-1} + (t-1) n_\tau \end{pmatrix} \end{aligned}\right\} \quad \ldots \quad (12)$$

Men kan de primitief-symmetrische invarianten P ook vinden, door elk der invarianten $\left[L^{l}_{k(\sigma)}\, \Lambda^{p}_{q(\varrho)}\right]$, voor zover niet reeds primitief-symmetrisch, met behulp van § 3 (9) te splitsen in primitief-symmetrische delen. Hiervoor moet men kunnen beschikken over de volledig gereduceerde reguliere voorstelling. Het opstellen van de matrices $\| m^{k,q}_{l} \|$ is dan overbodig.

De isomeren

$$\left[L^{l}_{k(\sigma)}\, \Lambda^{p}_{q(\varrho)}\right]_S = \left[L^{l}_{k\,S(\sigma)}\, \Lambda^{p}_{q\,S(\varrho)}\right]$$

kan men vinden uit (3). Hieruit toch volgt

$$L^{l}_{k\,S(\sigma)} = \sum_{l=1}^{n_\sigma} s^{l}_{k(\sigma)}\, L^{l}_{l(\sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Deze methode heeft echter het bezwaar, dat men meerdere malen dezelfde primitief-symmetrische invariant vindt.

Cristallography. — *Afleiding van het symbool van een kristalvlak, wanneer de symbolen van vier andere vlakken, benevens de hoeken van het beschouwde vlak met twee dier vlakken, bekend zijn.* By W. F. DE JONG. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of February 23, 1946.)

1. *Inleiding.* Wanneer van vier tautozonale vlakken de onderlinge hoeken bekend zijn en van drie dezer vlakken zijn de symbolen gegeven, dan kan het symbool van het vierde vlak met behulp van de dubbelverhoudingen van GAUSZ–MILLER [1]) afgeleid worden zonder dat de kristalelementen of andere constanten vooraf bepaald worden. Het meer algemeene probleem, waarbij de onderlinge hoeken van vijf willekeurige en de symbolen van vier dezer vlakken gegeven zijn en het symbool van het vijfde vlak wordt gevraagd, kan op overeenkomstige wijze worden opgelost met behulp van dubbelverhoudingen, welke wij in deze korte verhandeling zullen afleiden (zie nr. 3). In nr. 4 wordt een toepassing van het vraagstuk gegeven.

2. *Vier tautozonale vlakken.* Een dubbelverhouding van de sinussen der hoeken tusschen vier tautozonale vlakken $h_1\ (h_1\,k_1\,l_1)$, $h_2\ (h_2\,k_2\,l_2)$, $h_3\ (h_3\,k_3\,l_3)$ en $h_4\ (h_4\,k_4\,l_4)$ kan geschreven worden [2]):

$$\frac{\sin(1\,2)}{\sin(1\,4)} : \frac{\sin(3\,2)}{\sin(3\,4)} = \frac{\begin{vmatrix} h_1 k_1 l_1 \\ h_2 k_2 l_2 \end{vmatrix}}{\begin{vmatrix} h_1 k_1 l_1 \\ h_4 k_4 l_4 \end{vmatrix}} : \frac{\begin{vmatrix} h_3 k_3 l_3 \\ h_2 k_2 l_2 \end{vmatrix}}{\begin{vmatrix} h_3 k_3 l_3 \\ h_4 k_4 l_4 \end{vmatrix}}$$

of ook

$$\frac{\sin(1\,2)\sin(3\,4)}{\sin(2\,3)\sin(1\,4)} = \frac{\begin{vmatrix} h_1 k_1 l_1 \\ h_2 k_2 l_2 \end{vmatrix} \begin{vmatrix} h_3 k_3 l_3 \\ h_4 k_4 l_4 \end{vmatrix}}{\begin{vmatrix} h_2 k_2 l_2 \\ h_3 k_3 l_3 \end{vmatrix} \begin{vmatrix} h_1 k_1 l_1 \\ h_4 k_4 l_4 \end{vmatrix}} \quad \ldots\ldots \quad (1)$$

waarbij van elk der matrices dezelfde kolom geschrapt wordt, zoodat de dubbelverhouding der sinussen op drie wijzen in determinanten met indices van de vlakken wordt uitgedrukt; sin (12) geeft aan den sinus van den hoek tusschen de vlakken h_1 en h_2; enz.

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1) C. F. GAUSZ, Werke II, 308 (1831).
W. H. MILLER, A. Treatise on Crystallography, 12 (1839).

2) P. TERPSTRA, Leerboek der geometrische kristallografie, 111 e.v. (1927).

Voor het bewijs kunnen wij van de reciproke tralie gebruik maken [3]); in deze tralie, waarvan de assen A^*, B^* en C^* en de aslengten a^*, b^* en c^* mogen zijn, worden de vier tautozonale vlakken voorgesteld door vier punten van één tralievlak, dat ook den oorsprong O^* bevat (fig. 1); de voerstralen van O^* naar de punten zijn de loodlijnen op de overeenkomstige kristalvlakken. Wordt, door vermenigvuldiging van de indices van elk vlak met eenzelfde getal, de eerste index in de vier symbolen gelijk gemaakt, n.l. aan $h_1\ h_2\ h_3\ h_4$, dan liggen de traliepunten, welke de vier vlakken voorstellen, op een rechte tralielijn, evenwijdig aan het coördinatenvlak $B^*\,O^*\,C^*$. Projecteert men nu die punten, evenwijdig aan de C^*-as, op het vlak $A^*\,O^*\,B^*$, dan zijn de symbolen van de projecties $(h_1\ h_2\ h_3\ h_4 \quad k_1\ h_2\ h_3\ h_4 \quad 0)$, enz.

Elke dubbelverhouding der sinussen van de hoeken tusschen de voerstralen naar deze projecties is gelijk aan die der sinussen van de hoeken tusschen de voerstralen naar de oorspronkelijke punten $(h_1\ k_1\ l_1)$, enz., en gelijk aan die van de overeenkomstige afstanden tusschen de projecties. Wij krijgen dus:

$$\left.\begin{aligned}\frac{\sin(1\,2)\sin(3\,4)}{\sin(2\,3)\sin(1\,4)} &= \frac{\text{afst}(1\,2)\,.\,\text{afst}(3\,4)}{\text{afst}(2\,3)\,.\,\text{afst}(1\,4)} = \\ &= \frac{\{(k_1\ h_2\ h_3\ h_4 - k_2\ h_1\ h_3\ h_4)\,b^*\}\,\{(k_3\ h_1\ h_2\ h_4 - k_4\ h_1\ h_2\ h_3)\,b^*\}}{\{(k_2\ h_1\ h_3\ h_4 - k_3\ h_1\ h_2\ h_4)\,b^*\}\,\{(k_1\ h_2\ h_3\ h_4 - k_4\ h_1\ h_2\ h_3)\,b^*\}} = \\ &= \frac{\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix}\begin{vmatrix} h_3 & k_3 \\ h_4 & k_4 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 \\ h_3 & k_3 \end{vmatrix}\begin{vmatrix} h_1 & k_1 \\ h_4 & k_4 \end{vmatrix}}\end{aligned}\right\}. \qquad (2)$$

Op overeenkomstige wijze worden de twee andere uitdrukkingen van de beschouwde dubbelverhouding bepaald.

3. *Vijf willekeurige vlakken.* Vermenigvuldigt men de indices van vijf willekeurige vlakken $h_0\ (h_0\ k_0\ l_0)$, $h_1\ (h_1\ k_1\ l_1)$, $h_2\ (h_2\ k_2\ l_2)$, $h_3\ (h_3\ k_3\ l_3)$ en $h_4\ (h_4\ k_4\ l_4)$, met zoodanige getallen, dat in alle symbolen de eerste indices gelijk worden, n.l. aan $h_0\ h_1\ h_2\ h_3\ h_4$, dan liggen de vijf punten H_0, H_1, H_2, H_3 en H_4, welke in de reciproke tralie de vlakken voorstellen, in één tralievlak, evenwijdig aan het coördinatenvlak $B^*\,O^*\,C^*$ (fig. 2). Er worden dan door die punten en den oorsprong vier viervlakken bepaald, t.w. $O^*\,H_0\,H_1\,H_2$, $O^*\,H_0\,H_2\,H_3$, $O^*\,H_0\,H_3\,H_4$ en $O^*\,H_0\,H_1\,H_4$.

Nu is de inhoud van een viervlak (fig. 3):

$$I = {}^1/_6\ r_1\ r_2\ r_3\ \text{Sin}\,(1\,2\,3). \qquad (3)$$

waarin Sin (1 2 3) het argument of de ruimtesinus van den drievlakshoek

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[3]) Zt. f. Krist. (A) **101**, 317 (1939).

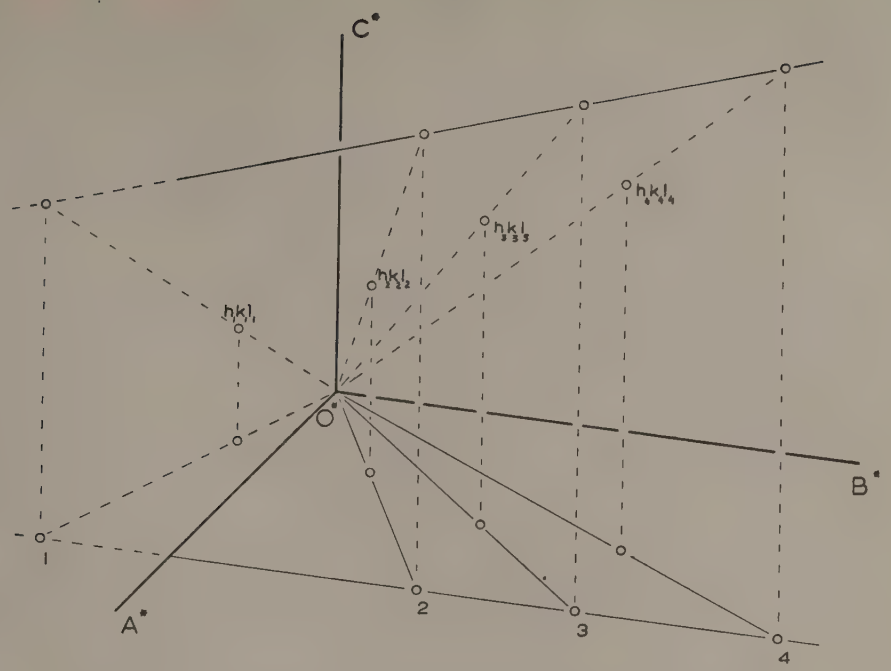

Fig. 1. Het met punt 1 overeenkomende kristalvlak heeft tot symbool $(h_1\ h_2\ h_3\ h_4\quad k_1\ h_2\ h_3\ h_4\quad 0)$.

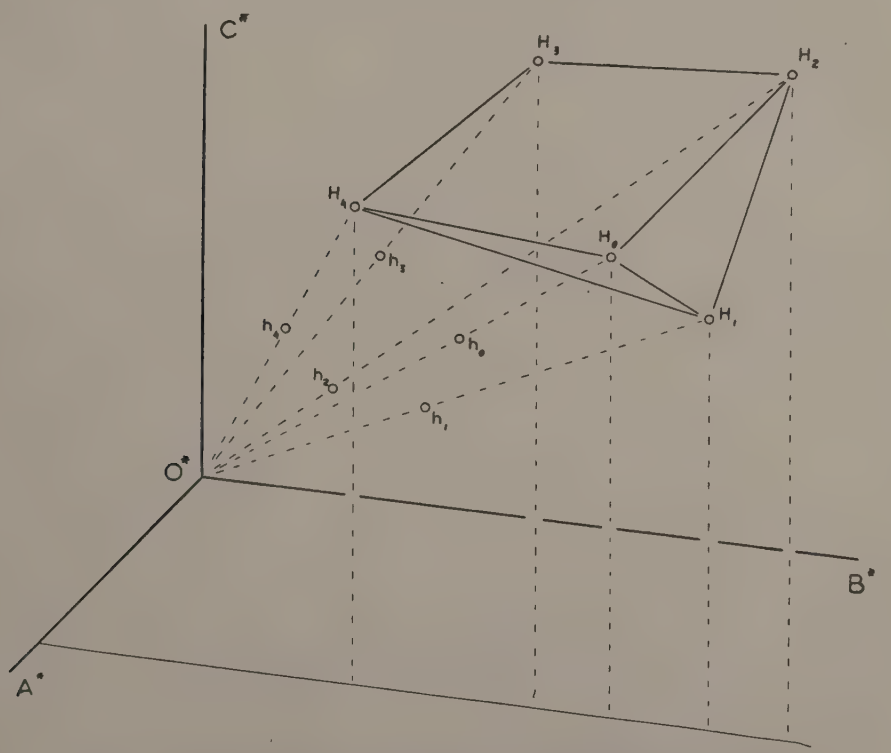

Fig. 2. Het met punt H_0 overeenkomende kristalvlak heeft tot symbool $(h_0\ h_1\ h_2\ h_3\ h_4\quad k_0\ h_1\ h_2\ h_3\ h_4\quad l_0\ h_1\ h_2\ h_3\ h_4)$.

$r_1\ r_2\ r_3$ is:

$$\left.\begin{aligned}&\text{Sin}\,(1\,2\,3) = \\ &= \sqrt{\{1 - \cos^2(1\,2) - \cos^2(2\,3) - \cos^2(3\,1) + 2\cos(1\,2)\cos(2\,3)\cos(3\,1)\}} = \\ &= \sin(1\,2)\sin(2\,3)\sin(1\,2\,3)\end{aligned}\right\} \quad (4)$$

sin (1 2 3) is de sinus van den standhoek op de ribbe r_2.

Voor onze vier viervlakken geldt derhalve:

$$\left.\begin{aligned}&\frac{{}^1/_6\, r_1\, r_0\, r_2\, \text{Sin}\,(1\,0\,2) \times {}^1/_6\, r_3\, r_0\, r_4\, \text{Sin}\,(3\,0\,4)}{{}^1/_6\, r_2\, r_0\, r_3\, \text{Sin}\,(2\,0\,3) \times {}^1/_6\, r_1\, r_0\, r_4\, \text{Sin}\,(1\,0\,4)} = \\ &= \frac{\text{Sin}\,(1\,0\,2)\,\text{Sin}\,(3\,0\,4)}{\text{Sin}\,(2\,0\,3)\,\text{Sin}\,(1\,0\,4)} = \frac{\text{opp}\,\triangle\, H_1\, H_0\, H_2 \times \text{opp}\,\triangle\, H_3\, H_0\, H_4}{\text{opp}\,\triangle\, H_2\, H_0\, H_3 \times \text{opp}\,\triangle\, H_1\, H_0\, H_4}\end{aligned}\right\}. \quad (5)$$

Het oppervlak van een driehoek, waarvan de hoekpunten gevormd worden door drie punten $m_1\, n_1$; $m_2\, n_2$ en $m_3\, n_3$ van een tralievlak met netparameters a en b en netmaasoppervlak O_p, wordt voorgesteld door

$$\text{opp}\,(1\,2\,3) = {}^1/_2 \begin{vmatrix} m_1 & n_1 & 1 \\ m_2 & n_2 & 1 \\ m_3 & n_3 & 1 \end{vmatrix} O_p$$

De bovenstaande verhouding wordt dus:

$$\frac{\text{Sin}\,(1\,0\,2)\,\text{Sin}\,(3\,0\,4)}{\text{Sin}\,(2\,0\,3)\,\text{Sin}\,(1\,0\,4)} =$$

$$= \frac{\begin{vmatrix} k_1\, h_0\, h_2\, h_3\, h_4 & l_1\, h_0\, h_2\, h_3\, h_4 & 1 \\ k_0\, h_1\, h_2\, h_3\, h_4 & l_0\, h_1\, h_2\, h_3\, h_4 & 1 \\ k_2\, h_0\, h_1\, h_3\, h_4 & l_2\, h_0\, h_1\, h_3\, h_4 & 1 \end{vmatrix} \begin{vmatrix} k_3\, h_0\, h_1\, h_2\, h_4 & l_3\, h_0\, h_1\, h_2\, h_4 & 1 \\ k_0\, h_1\, h_2\, h_3\, h_4 & l_0\, h_1\, h_2\, h_3\, h_4 & 1 \\ k_4\, h_0\, h_1\, h_2\, h_3 & l_4\, h_0\, h_1\, h_2\, h_3 & 1 \end{vmatrix}}{\begin{vmatrix} k_2\, h_0\, h_1\, h_3\, h_4 & l_2\, h_0\, h_1\, h_3\, h_4 & 1 \\ k_0\, h_1\, h_2\, h_3\, h_4 & l_0\, h_1\, h_2\, h_3\, h_4 & 1 \\ k_3\, h_0\, h_1\, h_2\, h_4 & l_3\, h_0\, h_1\, h_2\, h_4 & 1 \end{vmatrix} \begin{vmatrix} k_1\, h_0\, h_2\, h_3\, h_4 & l_1\, h_0\, h_2\, h_3\, h_4 & 1 \\ k_0\, h_1\, h_2\, h_3\, h_4 & l_0\, h_1\, h_2\, h_3\, h_4 & 1 \\ k_4\, h_0\, h_1\, h_2\, h_3 & l_4\, h_0\, h_1\, h_2\, h_3 & 1 \end{vmatrix}} =$$

$$= \frac{\begin{vmatrix} h_1\, k_1\, l_1 \\ h_0\, k_0\, l_0 \\ h_2\, k_2\, l_2 \end{vmatrix} \begin{vmatrix} h_3\, k_3\, l_3 \\ h_0\, k_0\, l_0 \\ h_4\, k_4\, l_4 \end{vmatrix}}{\begin{vmatrix} h_2\, k_2\, l_2 \\ h_0\, k_0\, l_0 \\ h_3\, k_3\, l_3 \end{vmatrix} \begin{vmatrix} h_1\, k_1\, l_1 \\ h_0\, k_0\, l_0 \\ h_4\, k_4\, l_4 \end{vmatrix}}$$

De rationaliteit van deze dubbelverhouding is reeds door GAUSZ aangegeven [4]), zonder dat echter de waarde werd bepaald.

Vervangt men nog de ruimtesinussen door hun waarden (4), dan vindt

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[4]) Vgl. TH. LIEBISCH, Geometrische Krystallographie, 69 (1881).

men tenslotte:

$$\frac{\sin(1\,0\,2)\sin(3\,0\,4)}{\sin(2\,0\,3)\sin(1\,0\,4)} = \frac{\begin{vmatrix} h_1 & k_1 & l_1 \\ h_0 & k_0 & l_0 \\ h_2 & k_2 & l_2 \end{vmatrix} \cdot \begin{vmatrix} h_3 & k_3 & l_3 \\ h_0 & k_0 & l_0 \\ h_4 & k_4 & l_4 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 & l_2 \\ h_0 & k_0 & l_0 \\ h_3 & k_3 & l_3 \end{vmatrix} \cdot \begin{vmatrix} h_1 & k_1 & l_1 \\ h_0 & k_0 & l_0 \\ h_4 & k_4 & l_4 \end{vmatrix}} \quad \ldots \ldots \quad (6)$$

Hierin is sin (1 0 2) de sinus van den standhoek op de snijlijn van de vlakken door de loodlijnen op h_1 en h_0 eenerzijds en h_0 en h_2 anderzijds,

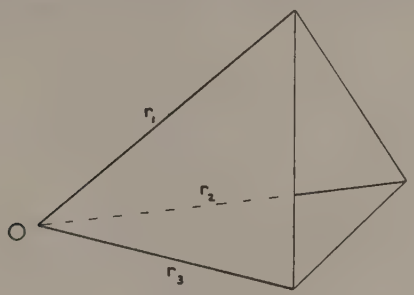

Fig. 3.

m.a.w. de sinus van den hoek der bundelvlakken $[h_1\ h_0]$ en $[h_0\ h_2]$, in de stereografische projectie die van den hoek der bundelcirkels $[h_1\ h_0]$ en $[h_0\ h_2]$.

4. *Toepassing.* Wij keeren nu terug tot het in den aanhef genoemde vraagstuk. Zijn in stereografische projectie gegeven de vier vlakken h_1 $(h_1\ k_1\ l_1)$, h_2 $(h_2\ k_2\ l_2)$, h_3 $(h_3\ k_3\ l_3)$ en h_4 $(h_4\ k_4\ l_4)$, waarvan niet drie

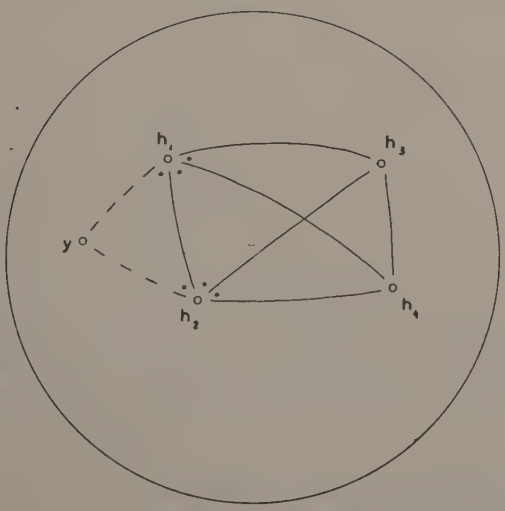

Fig. 4.

in één bundel zijn gelegen (fig. 4), en wordt gevraagd het symbool van het vlak y $(h_y\, k_y\, l_y)$, waarvan de hoeken met h_1 en h_2 zijn gemeten, dan berekent men trigonometrisch de met een punt voorziene hoeken tusschen de bundelcirkels en past in de punten h_1 en h_2 de formule der dubbelverhouding (6) toe; voor h_1 krijgt men dan:

$$\frac{\sin(y\,1\,2)\sin(3\,1\,4)}{\sin(2\,1\,3)\sin(y\,1\,4)} = \frac{\begin{vmatrix} h_y & k_y & l_y \\ h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \end{vmatrix} \cdot \begin{vmatrix} h_3 & k_3 & l_3 \\ h_1 & k_1 & l_1 \\ h_4 & k_4 & l_4 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 & l_2 \\ h_1 & k_1 & l_1 \\ h_3 & k_3 & l_3 \end{vmatrix} \cdot \begin{vmatrix} h_y & k_y & l_y \\ h_1 & k_1 & l_1 \\ h_4 & k_4 & l_4 \end{vmatrix}} \quad \ldots \quad (7)$$

en voor h_2 een overeenkomstige uitdrukking.

Uit de twee vergelijkingen kunnen de verhoudingen $h_y : k_y : l_y$ worden berekend.

Voorbeeld. Van axiniet zijn gegeven [5]) de hoeken tusschen de vlakken a (1 0 0); x (1 1 1); r (1 $\bar{1}$ 1) en s (2 0 1) en verder $a \wedge y = 28°\ 55'$ en $r \wedge y = 45°\ 15'$.

In a en r worden de hoeken tusschen de bundelcirkels met behulp van boldriehoeksmeting berekend, men vindt dan de waarden als in fig. 5

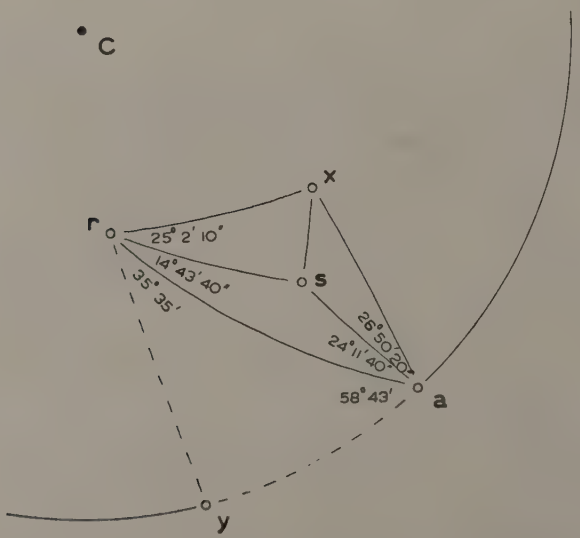

Fig. 5. Het vlak y wordt door DANA [5]) M genoemd.

aangegeven.

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[5]) J. D. DANA, A System of Mineralogy, 527 (1914).

Er geldt dan in *a*:

$$\frac{\sin 58^\circ 43' \sin 26^\circ 50' 20''}{\sin 24^\circ 11' 40'' \sin 109^\circ 45'} = \frac{\begin{vmatrix} h_y & k_y & l_y \\ 1 & 0 & 0 \\ 1 & \bar{1} & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \bar{1} & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} h_y & k_y & l_y \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}}$$

Hieruit volgt

$$\frac{k_y + l_y}{k_y - l_y} = 1{,}0003$$

In *r* wordt overeenkomstig gevonden:

$$\frac{k_y + l_y}{l_y - h_y} = 1{,}0012$$

Na geringe afronding komt er

$$h_y : k_y : l_y = 1 : \bar{1} : 0$$

Grafisch kan de waarde der dubbelverhouding sneller worden gevonden, in *a* leest men de waarde 1 af (fig. 6).

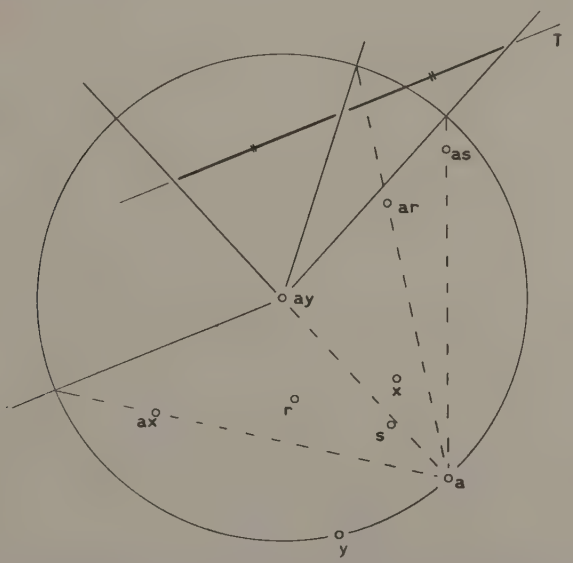

Fig. 6. De punten *ax*, enz. zijn de polen van de bundels [*ax*], enz. De transversaal *T* is evenwijdig aan één der vier stralen uit het middelpunt getrokken; de drie andere stralen bepalen stukken die zich verhouden als 1.

Prof. Ir. J. A. GRUTTERINK dank ik hartelijk voor zijn belangstelling en ondersteuning.

Delft. *Laboratorium voor Delfstofkunde der Technische Hoogeschool.*

Physiology. — *The haemoglobins of the equids and of some other mammals.* By C. ROMIJN. (From the Laboratory for Veterinary Physiology, Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of February 23, 1946.)

In two previous communications (1942, 1943) the author described a photo-electric method, by which the rate of alcalidenaturation of oxyhaemoglobin could be estimated and the specificity of the pigments from foetal and adult blood in the cow had been studied.

In the present article a spectrophotometric method will be described by which more accurate data could be obtained especially at low haemoglobin concentrations, when only 10 % or less undenaturated oxyhaemoglobin is still present.

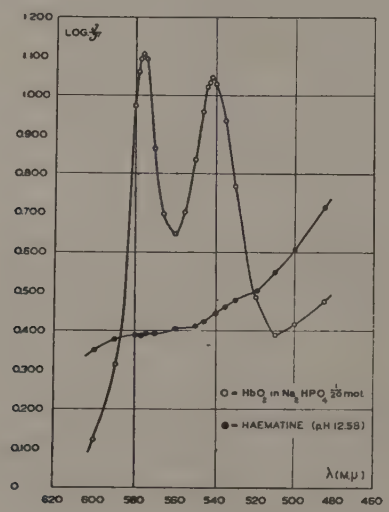

Fig. 1.

Absorption spectra of oxyhaemoglobin of the elephant and its denaturation product. (Spectrophotometer after KÖNIG–MARTENS.)

It is a wellknown fact that the two absorption bands characteristic in the spectrum of oxyhaemoglobin, disappear when the pigment denaturates at an alcaline reaction and the spectrum of alcaline haematine appears (fig. 1).

At a wave-length of about 540 m μ and of about 580 m μ the difference in extinction of the two pigments of the same concentration is a maximal one and since VIERORDT has shown that the extinction of a mixture of pigments is the algebraic sum of the extinctions of the pigments alone, we can calculate the relative concentrations of oxyhaemoglobin and of alcaline

haematine when the extinction of the mixture has been estimated and the absorption coefficients of the two pigments are known.

$$E = e_1 + e_2$$

$$E = \frac{c_1}{A_1} + \frac{c_2}{A_2}$$

A_1 and A_2 are the absorption coefficients of oxyhaemoglobin and of alcaline haematine.

When $c_1 = x\,\%$ then $c_2 = (100 - x)\,\%$

$$E = \frac{x}{A_1} + \frac{100 - x}{A_2}$$

$$x = \frac{A_1 A_2 E - 100 A_1}{A_2 - A_1}$$

For the estimation of the denaturation-rate of oxyhaemoglobin I used a spectrophotometer of KÖNIG, MARTENS and GRÜNBAUM (1903) at a wavelength of 540 m μ and cuvettes of 10 cm length. These cuvettes had a water-jacket of a constant temperature of 22° C which remained constant within 0.1° C.

Procedure:

The blood, withdrawn from the jugular vein was shaken with air and after oxygenation it was haemolysed by addition of saponine and then diluted with Na_2HPO_4 $^1/_{20}$ Mol. till the haemoglobin concentration was 0.22 mg in 1 cc of solution. After centrifugation during half an hour with 5000 rotations pro minute 20 cc of the clear oxyhaemoglobin solution were pipetted into the 10 cm photometercuvettes and the temperature adjusted on 22.0° C. The extinction of the solution was estimated then at 540 m μ and A_1 calculated ($c_1 = 100\,\%$ HbO_2). 2 cc of a 1 N. NaOH solution were added and thoroughly mixed with the oxyhaemoglobin. From this moment each 5 minutes the extinction was estimated. After addition of the soda lime the concentration of the pigment is 0.2 mg in each cubic centimeter. When the measured extinction remains constant, denaturation in the cuvette is complete and the pigment is 100 % alcaline haematine. The pH of the solution had been measured with the hydrogen electrode and was in all cases pH 12.58.

From the measured extinctions the percentages of undenaturated oxyhaemoglobin were calculated and the logarithms of these data plotted against the time (see previous articles).

Blood of several equids was used e.g. of one pure-blood Arabian horse, of two French half-blooded horses and of a cold-blooded horse in comparison with that of six horses from Norwegian origin, the so-called

fjord-horses [1]). Further, blood was collected from one common donkey, from two Nubian wild donkeys, from a Chapman zebra and from one mule [2]).

The results, obtained with blood of the equids are collected in Fig. 2.

Fig. 2 shows that all the investigated equids have only one haemoglobin in their blood, then the logarithmic alcalidenaturationcurve is a straight line. This is not in accordance with the results of HAUROWITZ (1938) who studied the sulfurcontent of horse-haemoglobin and who suggested that more than one haemoglobin should be present in horse-blood.

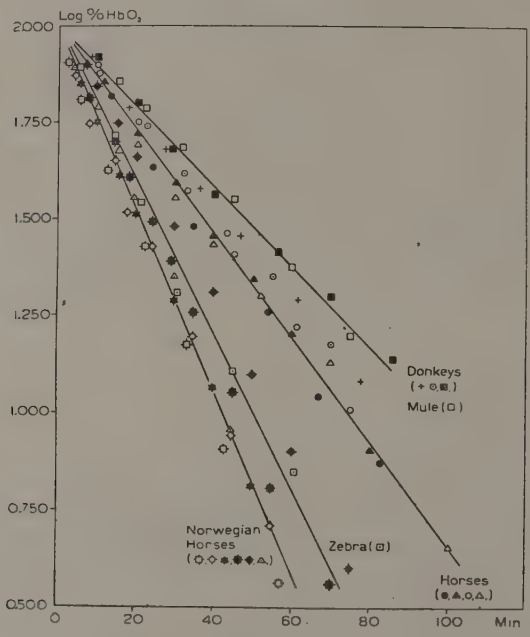

Fig. 2.

Alcalidenaturation-rate of the blood-haemoglobins of several equids.
$t = 22°$ C. pH = 12.58.

Furthermore the haemoglobins of the true horses are quite identical, from the warm-blooded animals as well as from the cold-blooded ones.

There is a striking difference between the bloodpigments of the Norwegian horses and the zebra on one side and the haemoglobins of the donkeys and the mule on the other. First animals have haemoglobins which are more sensitive to alcali and last animals have bloodpigments which are

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1) I wish to thank the managing board of the circus "Strassburger" who placed these animals at my disposal.

2) I am indepted to the direction of the zoological gardens "Blijdorp" in Rotterdam and "Artis" in Amsterdam for their worthful assistance.

less sensitive to alcali than horse-haemoglobin. The haemoglobin of the mule is identical with donkey haemoglobin.

Since the Norwegian fjord-horses have still many characteristic resemblances with the Asiatic wild horses, the s.c. Przewalsky horses, it would be very interesting to examine in this respect the blood haemoglobins of these rare animals.

The haemoglobins of several other mammals have been examined in relation to their alcali-sensibility and the results are collected in fig. 3.

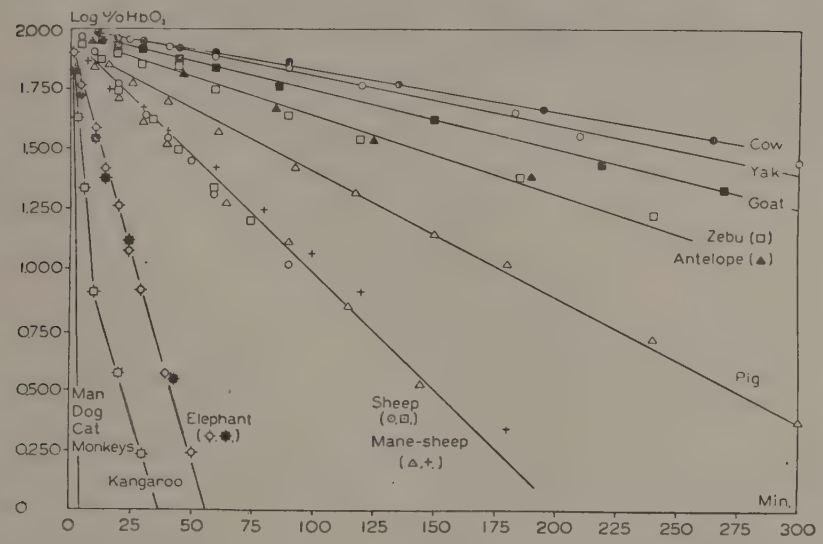

Fig. 3.

Alcalidenaturation-rate of the haemoglobins of some mammals.
$t = 22°$ C. pH = 12.58.

Fig. 3 shows that the haemoglobins of different mammals differ in respect to their alcali-sensibility. While in the cow and in other ruminants there is a very resistent form of haemoglobin, there are animals such as the elephant, the kangaroo, the carnivores and the apes, with a very sensitive bloodpigment. Especially the haemoglobins of the carnivores, the apes and of man denaturate immediately after addition of the soda lime and should be examined therefore at a much lower pH. In the blood of the kangaroo are two haemoglobins with different sensibility to alcali, a situation which the author described in a previous communication (1943) for the blood of some new-born calves.

An interesting result is the fact that haemoglobin of the African mane-sheep is quite identical with the bloodpigment of the common sheep, whereas in zoölogical respect this curious animal takes an intermediate position between the goat and the sheep. It is evident therefore that the affinity of some mammals to each other can be established by means of the examination of the bloodpigments.

Summary.

The alcali-denaturation of the haemoglobins of several mammals has been examined by means of a spectrophotometric method. There are great differences in sensibility of the bloodpigments of different animals and the possibility of establishing their affinity to each other is discussed.

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ROMIJN, C. Versl. Ned. Akad. v. Wetensch., Amsterdam, **52**, 689 (1943).

Mathematics. — *Inequalities concerning polynomials and trigonometric polynomials.* By J. G. VAN DER CORPUT and C. VISSER.

(Communicated at the meeting of March 30, 1946.)

Introduction.

1. This note is a continuation of that on the same subject by VISSER [1]). Its purpose is to show that the method applied by VISSER is of a somewhat larger scope than might be concluded from the simple applications to which he restricted himself, and that, in a slightly generalized form, it leads to a number of inequalities of which those considered by him are special instances.

2. Let

$$F(t) = \sum a_k e^{ikt}$$

be a finite trigonometric sum. Let s and m be two integers of which the first is positive. Then

$$\sum_{p=0}^{s-1} e^{-\frac{2\pi i p m}{s}} F\left(t + \frac{2\pi p}{s}\right) = s \sum_{k \equiv m \pmod s} a_k e^{ikt}, \quad . \quad . \quad . \quad (1)$$

and

$$\sum_{p=0}^{s-1} (-1)^p \int_{\tau + \frac{2\pi p}{s}}^{\tau + \frac{2\pi(p+1)}{s}} e^{-i\left(\frac{s}{2}+m\right)t} F(t)\, dt = 2is \sum_{k \equiv m \pmod s} \frac{a_k}{-\frac{s}{2} - m + k} e^{i\left(-\frac{s}{2}-m+k\right)\tau}. \quad . \quad . \quad (2)$$

We obtain (2) when we multiply both sides of (1) by the factor $e^{-i\left(\frac{s}{2}+m\right)t}$ and integrate over the interval $\tau \leqq t \leqq \tau + \frac{2\pi}{s}$.

It follows from (1) and (2) that

$$\text{Max} |F(t)| \geqq \text{Max} \left| \sum_{k \equiv m \pmod s} a_k e^{ikt} \right|, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\int_0^{2\pi} |F(t)\ dt \geqq 2s\, \text{Max} \left| \sum_{k \equiv m \pmod s} \frac{a_k}{-\frac{s}{2} - m + k} e^{ikt} \right| \quad . \quad . \quad (4)$$

In (4) there is equality if and only if for a value $t = \tau$ for which the

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[1]) C. VISSER, A simple proof of certain inequalities concerning polynomials, Kon. Ned. Akad. v. Wetensch., Amsterdam, **47**, 276—281 (1945).

right-hand side of (2) has a maximum absolute value, the function

$$\varphi(t) = \frac{e^{-imt} F(t)}{e^{ist} - e^{is\tau}}$$

is a trigonometric polynomial with a constant argument (we say that a function has a constant argument if all its values which differ from zero have the same argument; a function which vanishes identically has, accordingly, a constant argument). To see this, consider the function $\psi(t)$ defined by

$$\psi(t) = (-1)^p e^{\frac{1}{2}is\tau}\left(e^{\frac{1}{2}is(t-\tau)} - e^{-\frac{1}{2}is(t-\tau)}\right) = (-1)^p e^{-\frac{1}{2}ist}\left(e^{ist} - e^{is\tau}\right)$$

for

$$\tau + \frac{2\pi p}{s} \leqq t < \tau + \frac{2\pi(p+1)}{s} \quad (p = 0, 1, \ldots, s-1).$$

The function $\psi(t)$ has a constant argument, and

$$\varphi(t)\,\psi(t) = (-1)^p e^{-i\left(\frac{s}{2}+m\right)t} F(t).$$

If $\varphi(t)$ is a trigonometric polynomial with a constant argument, then $\varphi(t)\,\psi(t)$ has a constant argument, and the left-hand side of (2) is equal to

$$\int_0^{2\pi} |F(t)|\,dt,$$

so that there is equality in (4). Conversely, if there is equality in (4), then $\varphi(t)\,\psi(t)$ has a constant argument, and $F(t)$ vanishes in the points $t = \tau + \frac{2\pi p}{s}$ $(p = 0, 1, \ldots, s-1)$. Then $F(t)$ is divisible by

$$\prod_{p=0}^{s-1}\left(e^{it} - e^{i\left(\tau + \frac{2\pi p}{s}\right)}\right) = e^{ist} - e^{is\tau},$$

so that $\varphi(t)$ is a trigonometric polynomial, and since $\varphi(t)\,\psi(t)$ and $\psi(t)$ both have a constant argument, the same is true for $\varphi(t)$.

In what follows we shall consider some special cases of (3) and (4). We shall further apply our results to polynomials in a real variable, and obtain some inequalities related to those of W. MARKOF [2]).

Inequalities with two coefficients.

3. Suppose that there are two coefficients a_u, a_v $(u < v)$ such that for no other coefficient $a_k \neq 0$ we have $k \equiv u \pmod{v-u}$. Put $s = v-u$ and $m = u$. Then the right-hand side of (3) becomes

$$\text{Max}\,|a_u e^{iut} + a_v e^{ivt}| = |a_u| + |a_v|.$$

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[2]) W. MARKOF, Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen, Mathematische Ann. **77**, 213—258 (1915).

Hence

$$\text{Max } |F(t)| \geqq |a_u| + |a_v|.$$

The right-hand side of (4) becomes

$$4 \text{ Max } |a_u e^{iut} + a_v e^{ivt}| = 4|a_u| + 4|a_v|.$$

Hence

$$\int_0^{2\pi} |F(t)|\, dt \geqq 4|a_u| + 4|a_v|.$$

Denote the arguments of a_u and a_v by α and β respectively. Let τ be such that $u\tau + \alpha = v\tau + \beta$. Then

$$a_u e^{iut} + a_v e^{ivt}$$

attains its maximum absolute value for $t = \tau$. There is equality in the second inequality if and only if

$$\varphi(t) = \frac{e^{-iut} F(t)}{e^{ist} - e^{is\tau}} = \frac{e^{i\beta} F(t)}{e^{i(vt+\beta)} - e^{i(ut+\alpha)}}$$

is a trigonometric polynomial with a constant argument.

For a trigonometric polynomial

$$F(t) = \sum_{-n}^{n} a_k e^{ikt}$$

or order n, we have in particular

$$\text{Max } |F(t)| \geqq |a_{-k}| + |a_k|,$$

and

$$\int_0^{2\pi} |F(t)|\, dt \geqq 4|a_{-k}| + 4|a_k|$$

for all $k > \frac{n}{3}$. In the last inequality the sign of equality holds if and only if

$$\frac{F(t)}{e^{i(kt+\beta)} - e^{-i(kt-\alpha)}},$$

where $\alpha = \arg a_{-k}$ and $\beta = \arg a_k$, is a trigonometric polynomial with a constant argument. For a real-valued $F(t)$ this means that

$$\frac{F(t)}{\sin(kt + \beta)}$$

is a trigonometric polynomial with a constant sign.

Inequalities with three coefficients.

4. Suppose that for three coefficients a_u, a_v, a_w we have $v - u = w - v > 0$,

while there is no other coefficient $a_k \neq 0$ for which $k \equiv u \pmod{v-u}$. Put $s = v-u$ and $m = v$. The right-hand side of (3) becomes

$$\text{Max} \left| a_u e^{-ist} + a_v + a_w e^{ist} \right|$$

Suppose that a_v is real and that a_u and a_w are conjugate complex numbers. Then this maximum is $|a_u| + |a_v| + |a_w|$. Hence

$$\text{Max} \, |F(t)| \geqq |a_u| + |a_v| + |a_w|.$$

For a real-valued trigonometric polynomial

$$F(t) = \sum_{-n}^{n} a_k e^{ikt}$$

of order n, we have therefore

$$\text{Max} \, |F(t)| \geqq |a_0| + 2|a_k|$$

for all $k > \frac{n}{2}$.

This can also be stated as follows. For any trigonometric polynomial

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

with real coefficients

$$\text{Max} \, |F(t)| \geqq \left| \frac{a_0}{2} \right| + \sqrt{a_k^2 + b_k^2}$$

for all $k > \frac{n}{2}$.

5. Under the same assumptions (4) becomes

$$\int_0^{2\pi} |F(t)| \, dt \geqq 4 \, \text{Max} \left| -\frac{a_u}{3} e^{-ist} + a_v + a_w e^{ist} \right|.$$

Assuming also again that a_v is real and that a_u and a_w are conjugate complex numbers, we conclude that

$$\int_0^{2\pi} |F(t)| \, dt \geqq 4 |a_v| + \tfrac{8}{3} |a_w|.$$

Hence for a real-valued trigonometric polynomial

$$F(t) = \sum_{-n}^{n} a_k e^{ikt}$$

we have

$$\int_0^{2\pi} |F(t)| \, dt \geqq 4 |a_0| + \tfrac{8}{3} |a_k|$$

for all $k > \frac{n}{2}$. It is easily seen that there is equality only if $F(t)$ vanishes identically.

Polynomials in a real variable.

6. Let

$$P(u) = \sum_{k=0}^{n} A_k u^k$$

be a polynomial in the real variable u. We denote by

$$T_0(u),\ T_1(u),\ \ldots$$

the polynomials of TCHEBYCHEF. $T_k(u)$ is defined by

$$T_k(u) = \cos kt, \text{ where } u = \cos t.$$

$P(u)$ has a unique expansion

$$P(u) = \sum_{k=0}^{n} c_k T_k(u).$$

We put

$$F(t) = P(\cos t) = \sum_{k=0}^{n} c_k \cos kt = \sum_{k=0}^{n} c_k \frac{e^{ikt} + e^{-ikt}}{2}.$$

Now by sec. 3 we have

$$\operatorname{Max} |F(t)| \geqq |c_k|$$

for all $k > \frac{n}{3}$. Hence also

$$\operatorname*{Max}_{-1 \leqq u \leqq 1} |P(u)| \geqq |c_k|$$

for all $k > \frac{n}{3}$. In particular

$$\operatorname*{Max}_{-1 \leqq u \leqq 1} |P(u)| \geqq |c_n| = \frac{|A_n|}{2^{n-1}} \text{ (TCHEBYCHEF);}$$

$$\operatorname*{Max}_{-1 \leqq u \leqq 1} |P(u)| \geqq |c_{n-1}| = \frac{|A_{n-1}|}{2^{n-2}} \text{ for } n \geqq 2 \text{ (W. MARKOF);}$$

$$\operatorname*{Max}_{-1 \leqq u \leqq 1} |P(u)| \geqq |c_{n-2}| = \frac{|nA_n + 4A_{n-2}|}{2^{n-1}} \text{ for } n \geqq 4.$$

7. Let us further denote by

$$U_0(u),\ U_1(u),\ \ldots$$

the polynomials defined by

$$U_k(u) = \frac{\sin(k+1)t}{\sin t}, \text{ where } u = \cos t.$$

$P(u)$ has a unique expansion

$$P(u) = \sum_{k=0}^{n} d_k U_k(u).$$

We put

$$G(t) = P(\cos t) \sin t = \sum_{k=0}^{n} d_k \sin(k+1)t = \sum_{k=0}^{n} d_k \frac{e^{i(k+1)t} - e^{-i(k+1)t}}{2i}.$$

By sec. 3 again we have

$$\int_0^{2\pi} |G(t)|\, dt \geqq 4 |d_k|$$

for all k for which $k+1 > \frac{n+1}{3}$.

Hence for these k

$$\int_{-1}^{1} |P(u)|\, du = \frac{1}{2} \int_0^{2\pi} |G(t)|\, dt \geqq 2 |d_k|.$$

In particular

$$\int_{-1}^{1} |P(u)|\, du \geqq 2 |d_n| = \frac{|A_n|}{2^{n-1}};$$

$$\int_{-1}^{1} |P(u)|\, du \geqq 2 |d_{n-1}| = \frac{|A_{n-1}|}{2^{n-2}} \text{ for } n \geqq 1;$$

$$\int_{-1}^{1} |P(u)|\, du \geqq 2 |d_{n-2}| = \frac{|(n-1)A_n + 4A_{n-2}|}{2^{n-1}} \text{ for } n \geqq 3.$$

Another inequality.

8. If

$$F(t) = \Sigma a_k e^{ikt},$$

then

$$|F(t)|^2 = \Sigma a_k \bar{a}_l e^{i(k-l)t},$$

where $\bar{a}_l$ denotes the conjugate of a_l. Hence

$$\sum_{p=0}^{s-1} \left| F\left(t + \frac{2\pi p}{s}\right) \right|^2 = s \sum_{k-l \equiv 0 \,(\mathrm{mod}.\, s)} a_k \bar{a}_l e^{i(k-l)t}.$$

Suppose that there are two coefficients a_u, a_v $(u < v)$ such that for no other coefficient $a_k \neq 0$ we have $k \equiv u \pmod{v-u}$. Then, if $s = v - u$,

$$\frac{1}{s} \sum_{p=0}^{s-1} \left| F\left(t + \frac{2\pi p}{s}\right) \right|^2 = \Sigma |a_k|^2 + (a_u \bar{a}_v e^{-ist} + \bar{a}_u a_v e^{ist}).$$

It follows that

$$\begin{aligned}\operatorname{Max} |F(t)|^2 &\geqq \Sigma |a_k|^2 + \operatorname{Max} |a_u \bar{a}_v e^{-ist} + \bar{a}_u a_v e^{ist}| \\ &= \Sigma |a_k|^2 + 2 |a_u a_v|,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Min} |F(t)|^2 &\leqq \Sigma |a_k|^2 - \operatorname{Max} |a_u \bar{a}_v e^{-ist} + \bar{a}_u a_v e^{ist}| \\ &= \Sigma |a_k|^2 - 2 |a_u a_v|.\end{aligned}$$

This we apply, with $u = -n$ and $v = n$, to a trigonometric polynomial

$$F(t) = \sum_{-n}^{n} a_k e^{ikt}$$

of order n. We find

$$\operatorname{Max} |F(t)|^2 \geqq \sum_{-n}^{n} |a_k|^2 + 2 |a_n \bar{a}_{-n}|;$$

$$\operatorname{Min} |F(t)|^2 \leqq \sum_{-n}^{n} |a_k|^2 - 2 |a_n a_{-n}|$$

Since

$$\Sigma |a_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |F(t)|^2 dt = \mathfrak{M}\{|F(t)|^2\},$$

we can also write

$$\operatorname{Max} |F(t)|^2 - \mathfrak{M}\{|F(t)|^2\} \geqq 2 |a_n a_{-n}|;$$

$$\operatorname{Min} |F(t)|^2 - \mathfrak{M}\{|F(t)|^2\} \leqq -2 |a_n a_{-n}|.$$

9. Now consider a polynomial

$$P(u) = A_0 + A_1 u + \ldots + A_n u^n$$

in a real variable u. Put

$$\begin{aligned}F(t) &= A_0 + A_1 \left(\frac{e^{it} + e^{-it}}{2}\right) + \ldots + A_n \left(\frac{e^{it} + e^{-it}}{2}\right)^n \\ &= \frac{A_n}{2^n} e^{-int} + \ldots + \frac{A_n}{2^n} e^{int}.\end{aligned}$$

Then

$$\operatorname*{Max}_{-1 \leqq u \leqq 1} |P(u)|^2 = \operatorname{Max} |F(t)|^2;$$

$$\operatorname*{Min}_{-1 \leqq u \leqq 1} |P(u)|^2 = \operatorname{Min} |F(t)|^2;$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{|P(u)|^2}{\sqrt{1-u^2}} du = \frac{1}{2\pi} \int_0^{2\pi} |F(t)|^2 dt,$$

and so

$$\operatorname*{Max}_{-1\leqq u\leqq 1} |P(u)|^2 \geqq \frac{1}{\pi}\int_{-1}^{1} \frac{|P(u)|^2}{\sqrt{1-u^2}}\,du + \frac{|A_n|^2}{2^{2n-1}};$$

$$\operatorname*{Min}_{-1\leqq u\leqq 1} |P(u)|^2 \leqq \frac{1}{\pi}\int_{-1}^{1} \frac{|P(u)|^2}{\sqrt{1-u^2}}\,du - \frac{|A_n|^2}{2^{2n-1}}.$$

It follows that

$$\operatorname*{Max}_{-1\leqq u\leqq 1} |P(u)|^2 - \operatorname*{Min}_{-1\leqq u\leqq 1} |P(u)|^2 \geqq \frac{|A_n|^2}{2^{2n-2}},$$

which includes, of course, TCHEBYCHEF's inequality.

Appendix.

We shall derive from (1) for a real-valued trigonometric polynomial a more precise result than the inequality (3).

First we prove:

Lemma. Let σ be a positive integer. Then

$$\operatorname*{Max}_{-1\leqq b_q\leqq 1} \left|\sum_{q=0}^{\sigma-1} b_q e^{\frac{2\pi i q}{\sigma}}\right| = \begin{cases} \dfrac{2}{\sin\dfrac{\pi}{\sigma}} & \text{for } \sigma \text{ even;} \\[2ex] \dfrac{1}{\sin\dfrac{\pi}{2\sigma}} & \text{for } \sigma \text{ odd.} \end{cases}$$

Suppose that the maximum of the absolute value of

$$S(b) = \sum_{q=0}^{\sigma-1} b_q e^{\frac{2\pi i q}{\sigma}}$$

in the closed domain $-1 \leqq b_q \leqq 1$ is attained in $(b_0^*, \ldots, b_{\sigma-1}^*)$. We assert that each b_q^* such that $\left|\dfrac{2\pi q}{\sigma} - \arg S(b^*)\right| \leqq \dfrac{\pi}{2}$ is equal to 1. Suppose on the contrary that for some q

$$b_q^* < 1; \qquad \left|\frac{2\pi q}{\sigma} - \arg S(b^*)\right| \leqq \frac{\pi}{2}.$$

If for this q we replace b_q^* in $S(b^*)$ by 1, then $S(b^*)$ changes into

$$T = S(b^*) + (1 - b_q^*)e^{\frac{2\pi i q}{\sigma}}$$

and $|T| > |S(b^*)|$, which is a contradiction. In the same manner we find that each b_q^* such that $\left|\dfrac{2\pi q}{\sigma} - \arg S(b^*)\right| \geqq \dfrac{\pi}{2}$ is equal to -1.

It follows that for σ even the absolute value of $S(b^*)$ is equal to

$$\left| \sum_{q=0}^{\frac{1}{2}\sigma-1} e^{\frac{2\pi i q}{\sigma}} - \sum_{q=\frac{1}{2}\sigma}^{\sigma-1} e^{\frac{2\pi i q}{\sigma}} \right| = \frac{2}{\sin \frac{\pi}{\sigma}},$$

and that for σ odd it is equal to

$$\left| \sum_{q=0}^{\frac{1}{2}(\sigma-1)} e^{\frac{2\pi i q}{\sigma}} - \sum_{q=\frac{1}{2}(\sigma+1)}^{\sigma-1} e^{\frac{2\pi i q}{\sigma}} \right| = \frac{1}{\sin \frac{\pi}{2\sigma}}.$$

Next we prove:

Lemma. If the integer μ is prime to the positive integer σ, and if $-M \leqq c_p \leqq M$, then

$$\left| \sum_{p=0}^{\sigma-1} c_p e^{\frac{2\pi i p\mu}{\sigma}} \right| \leqq \begin{cases} \dfrac{2M}{\sin \frac{\pi}{\sigma}} & \text{for } \sigma \text{ even;} \\ \dfrac{M}{\sin \frac{\pi}{2\sigma}} & \text{for } \sigma \text{ odd.} \end{cases}$$

For if

$$q \equiv p\mu \pmod{\sigma},\ 0 \leqq q < \sigma;\ c_p = Mb_q,$$

then

$$\sum_{p=0}^{\sigma-1} c_p e^{\frac{2\pi i p\mu}{\sigma}} = M \sum_{q=0}^{\sigma-1} b_q e^{\frac{2\pi i q}{\sigma}},$$

and the result follows by the first lemma.

Now consider the formula (1). Let d be the greatest common divisor of s and m. Put $s = d\sigma$. The left-hand side of (1) can be divided into d sums of the form

$$\Sigma e^{-\frac{2\pi i p m}{s}} F\left(t + \frac{2\pi p}{s}\right) = \Sigma e^{-\frac{2\pi i p\mu}{\sigma}} F\left(t + \frac{2\pi p}{s}\right),$$

in each of which p runs through σ consecutive integers, and where $m = d\mu$. The absolute value of each of these d sums is, by the preceding lemma, at most

$$\begin{cases} \dfrac{2}{\sin \frac{\pi}{\sigma}} \operatorname{Max} |F(t)| & \text{for } \sigma \text{ even;} \\ \dfrac{1}{\sin \frac{\pi}{2\sigma}} \operatorname{Max} |F(t)| & \text{for } \sigma \text{ odd.} \end{cases}$$

We have thus proved:

If d is the greatest common divisor of s and m, and if $s = d\sigma$, then any real-valued trigonometric polynomial

$$F(t) = \Sigma a_k e^{ikt}$$

satisfies the inequality

$$\operatorname{Max} |F(t)| \geqq \lambda \operatorname{Max} \Big| \sum_{k \equiv m \,(\mathrm{mod.}\, s)} a_k e^{ikt} \Big|,$$

where $\lambda = \frac{1}{2}\sigma \sin \frac{\pi}{\sigma}$ for σ even, and $\lambda = \sigma \sin \frac{\pi}{2\sigma}$ for σ odd.

Mathematics. — *On Unities and Dimensions.* III. By H. B. DORGELO and J. A. SCHOUTEN.

(Communicated at the meeting of March 30, 1946.)

This table leads to the following relative dimensions for the four systems mentioned above (rationalized as well as not rationalized) [1]:

	Electromagn.; Giorgi		*Electrostatic*	*Gauss*	
	c. g. s.	*m. kg. s.*	*c. g. s.*	*c. g. s.*	
	$[h'] = [p] = [l^{-2}\, t^2]$		$[p'] = [l^{-2}\, t^2]$	$[k] = [k'] = [l\, t^{-1}]$	
Q	$[m^{1/2}\, l^{1/2}]$	$[q]$	$[m^{1/2}\, l^{1/2}\, t^{-1}]$	$[m^{1/2}\, l^{3/2}\, t^{-1}]$;	$[q]$
F	$[m^{1/2}\, l^{1/2}\, t^{-2}]$	$[m\, l\, t^{-2}\, q^{-1}]$	$[m^{1/2} l^{-1/2}\, t^{-1}]$	$[m^{1/2} l^{-1/2} t^{-1}]$;	$[m\, l\, t^{-2}\, q^{-1}]$
D	$[m^{1/2}\, l^{-3/2}]$	$[l^{-2}\, q]$	$[m^{1/2}\, l^{-1/2}\, t^{-1}]$	$[m^{1/2} l^{-1/2} t^{-1}]$;	$[l^{-2}\, q]$
H	$[m^{1/2}\, l^{-1/2}\, t^{-1}]$	$[l^{-1}\, t^{-1}\, q]$	$[m^{1/2}\, l^{1/2}\, t^{-2}]$	$[m^{1/2} l^{-1/2} t^{-1}]$;	$[l^{-2}\, q]$
B	$[m^{1/2}\, l^{-1/2}\, t^{-1}]$	$[m\, t^{-1}\, q^{-1}]$	$[m^{1/2}\, l^{-3/2}]$	$[m^{1/2} l^{-1/2} t^{-1}]$;	$[m\, l\, t^{-2}\, q^{-1}]$
ε	$[l^{-2}\, t^2]$	$[m^{-1}\, l^{-3}\, t^2\, q^2]$	$[1]$	$[1]$	$[m^{-1} l^{-3} t^2 q^3]$
μ	$[1]$	$[m\, l\, q^{-2}]$	$[l^{-2}\, t^2]$	$[1]$	$[m\, l^3\, t^{-2}\, q^{-2}]$
E	$[m^{1/2}\, l^{1/2}\, t^{-2}]$	$[m l^2\, t^{-2}\, q^{-1}]$	$[m^{1/2}\, l^{1/2}\, t^{-1}]$	$[m^{1/2}\, l^{1/2}\, t^{-1}]$;	$[m\, l^3\, t^{-2}\, q^{-1}]$
I	$[m^{1/2}\, l^{1/2}\, t^{-1}]$	$[t^{-1}\, q]$	$[m^{1/2}\, l^{3/2}\, t^{-2}]$	$[m^{1/2}\, l^{3/2}\, t^{-2}]$;	$[t^{-1}\, q]$
R	$[l\, t^{-1}]$	$[m\, l^2\, t^{-1}\, q^{-2}]$	$[l^{-1}\, t]$	$[l^{-1}\, t]$	$[m\, l^2\, t^{-1}\, q^{-2}]$

(5.2)

From this table we see that the relative dimensions, expressed in terms of m, l, t and q are the same for the three systems to the left. This is due to the special values of the seven constants in these systems and leads us to the question if it is possible, by introducing more fundamental units, to get dimension formulas valid for all systems. There being six constants left after the introduction of q it would be necessary to introduce six more fundamental units. If we make the condition that the constants φ and ψ in (4. 1 g, h) have the dimension $[1]$, a reasonable condition, satisfied by all usual systems, we get the relations

$$[a\, h\, h'\, p^{-1}] = [1]; \qquad [k\, k'\, a^{-1}\, h'^{-1}\, p'^{-1}] = [l^2\, t^{-2}] \quad . \quad . \quad (5.3)$$

and there are only four additional fundamental units needed. Another not unreasonable condition, satisfied in all usable systems, is that a be dimensionless

$$[a] = [1] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

[1] It is useful to remark that the relative dimensions in m, l, t, q indicate how the orthogonal components change, if not only the fundamental units of mass, length and time are changed but also the fundamental unit of charge, i.e. if h' is changed but a, k, k' and h keep their values.

Then only three additional fundamental units are needed. It is plausible to take for the first two the units of ε and μ. For the third unit can be taken for instance the unit of the magnetic flux Φ with the dimension $[ml^2t^{-1}q^{-1}]\,[k'a^{-1}]$ (KALANTAROFF) or the unit of magnetical pole-strength P (surface integral of **H**) with the dimension $[lt^{-1}q]\,[k^{-1}]$ (SOMMERFELD). From

$$[\varepsilon] = [p]; \qquad [\mu] = [p'] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

$$[k'^{-1}\,a] = [m\,l^2\,t^{-1}\,q^{-1}\,\Phi^{-1}] \text{ or } [k] = [l\,t^{-1}\,q\,P^{-1}] \quad . \quad . \quad (5.6)$$

five of the constants a, k, k', h, p and p' can be eliminated (h' was already eliminated with the aid of q). Using (5.5) for the elimination of p and p', (5.3) gets the form

$$[a\,h] = [m\,l^3\,t^{-2}\,q^{-2}\,\varepsilon]; \qquad [k\,k'\,a^{-1}] = [m^{-1}\,l^{-1}\,q^2\,\mu] \quad . \quad . \quad (5.7)$$

The constant a could be eliminated using (5.4). From (5.6) either h and k or h and k' (but not k and k') can be solved:

$$[h] = [ml^3\,t^{-2}\,q^{-2}\,\varepsilon]\,[a^{-1}] \left\{ \begin{array}{c} [k] = [m^{-1}\,l^{-1}\,q^2\,\mu]\,[k'^{-1}\,a] \\ \text{or} \\ [k'] = [m^{-1}\,l^{-1}\,q^2\,\mu]\,[k^{-1}\,a] \end{array} \right\} \quad . \quad (5.8)$$

and this leads to the dimension formulas

$$\left.\begin{array}{ll} \mathbf{F}: & [m\,l\,t^{-2}\,q^{-1}]\,[a^{-1}] \\ \mathbf{D}: & [m\,l\,t^{-2}\,q^{-1}\,\varepsilon]\,[a^{-1}] \\ \mathbf{H}: & [m\,t^{-1}\,q^{-1}\,\mu^{-1}]\,[k'\,a^{-1}] \quad \text{or} \quad [l^{-1}\,t^{-1}\,q]\,[k^{-1}] \\ \mathbf{B}: & [m\,t^{-1}\,q^{-1}]\,[k'\,a^{-1}] \quad \text{or} \quad [l^{-1}\,t^{-1}\,q\,\mu]\,[k^{-1}] \\ \varepsilon: & [m\,l^2\,t^{-2}\,q^{-1}]\,[a^{-1}] \\ I: & [t^{-1}\,q] \\ R: & [m\,l^2\,t^{-1}\,q^{-2}]\,[a^{-1}] \end{array}\right\} \quad . \quad . \quad (5.9)$$

only containing the constants a, k or a, k'. Using Φ or P to eliminate k' or k respectively we get

$$\left.\begin{array}{ll} \mathbf{H}: & [l^{-2}\,\Phi\,\mu^{-1}] \text{ or } [l^{-2}\,P] \\ \mathbf{B}: & [l^{-2}\,\Phi] \quad \text{or } [l^{-2}\,P\,\mu] \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.10)$$

It has to be remarked that it is not possible to use one of the quantities **F**, **D**, E, I or R instead of Φ or P because in their dimensions neither k nor k' occur. a can be eliminated without using (5.4) by introducing a last fundamental unit, for instance the unit of R and then we get dimension formulas containing m, l, t, q, ε, μ, Φ or P and R, *valid for all possible systems satisfying the conditions that φ and ψ be dimensionless.*

Now the introduction of a fourth fundamental unit, for instance Q, is certainly very useful because by this introduction we can get rid of fractional exponents. But it is highly questionable if the introduction of

more than four fundamental units yields any practical profit. Till yet no practical profit seems to have appeared. Instead of q another unit can be chosen. In order to facilitate comparation we give here a table of the relative dimensions for different choices of the four fundamental units. This table is valid for the electromagnetic system, the electrostatic system and the system of GIORGI.

	$mltq$	$mltR$ (*Giorgi*)	$ltFH$ (*Maxwell*)	$ltEI$ (*Oberdorfer*)	$ltq\Phi$ (*Kalantoroff*)
Q:	q	$m^{1/2} l t^{-1/2} R^{-1/2}$	ltH	tI	q
F:	$mlt^{-2} q^{-1}$	$m^{1/2} t^{-3/2} R^{1/2}$	F	$l^{-1} E$	$l^{-1} t^{-1} \Phi$
D:	$l^{-2} q$	$m^{1/2} l^{-1} t^{-1/2} R^{-1/2}$	$l^{-1} tH$	$l^{-2} tI$	$l^{-2} q$
H:	$l^{-1} t^{-1} q$	$m^{1/2} t^{-3/2} R^{-1/2}$	H	$l^{-1} I$	$l^{-1} t^{-1} q$
B:	$mt^{-1} q^{-1}$	$m^{1/2} l^{-1} t^{-1/2} R^{1/2}$	$l^{-1} tF$	$l^{-2} tE$	$l^{-2} \Phi$
ε:	$m^{-1} l^{-3} t^2 q^2$	$l^{-1} tR^{-1}$	$l^{-1} tF^{-1} H$	$l^{-1} tE^{-1} I$	$l^{-1} tq\Phi^{-1}$
μ:	$ml q^{-2}$	$l^{-1} tR$	$l^{-1} tFH^{-1}$	$l^{-1} tEI^{-1}$	$l^{-1} tq^{-1} \Phi$
E:	$ml^2 t^{-2} q^{-1}$	$m^{1/2} l t^{-3/2} R^{-1/2}$	lF	E	$t^{-1} \Phi$
I:	$t^{-1} q$	$m^{1/2} l t^{-3/2} R^{-1/2}$	lH	I	$t^{-1} q$
R:	$ml^2 t^{-1} q^{-2}$	R	FH^{-1}	EI^{-1}	$q^{-1} \Phi$
Φ:	$ml^2 t^{-1} q^{-1}$	$m^{1/2} l t^{-1/2} R^{1/2}$	ltF	tE	Φ
C:	$m^{-1} l^{-2} t^2 q^2$	tR^{-1}	$tF^{-1} H$	$tE^{-1} I$	$tq\Phi^{-1}$
L:	$ml^2 q^{-2}$	tR	tFH^{-1}	tEI^{-1}	$tq^{-1} \Phi$
M:	m	m	$t^3 FH$	$l^{-2} t^3 EI$	$l^{-2} tq\Phi$

. (5. 11)

The system with m, l, t, R is not recommendable because of the fractional exponents. In the other systems without m the dimensions of the electromagnetic quantities have a remarkably simple form and there is a certain charm in the duality between **F** and **H** or E and I or q and Φ. But the dimension of m in these systems appears highly artifical and not fit for practical use. Therefore, in our opinion, the system with m, l, t and q is certainly the best one. There is another fact that tells strongly in favour of taking the unit of electric charge as the fourth fundamental unit. The electric charge represents a property of material phenomena as essential as mass, length and time and independent of these three.

NOW SOMMERFELD [2]), reasoning in a similar way, proposes to introduce a fifth fundamental unit, basing his opinion on the fact that a neutron has a magnetic moment but no electric charge. He takes five fundamental units m, l, t, q and P (pole strength). Then k can be eliminated [3]):

$$[k] = [l t^{-1} q P^{-1}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

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[2]) Ann. der Physik V **36**, 335—339 (1939).

[3]) SOMMERFELD writes V instead of k and F. HUND, Eine Übersicht über die elektromagnetischen Einheiten, Phys. Zeitschrift **39** (1938) p. 376, writes γ instead of k.

After that, chosing

$$[a] = [1]; \quad [h] = [1]; \quad [k] = [k']; \quad [\varphi] = [\psi] = [1] \quad . \quad (5.13)$$

as in the four (not rationalized) usual systems, the dimension formulas take the form

$$\left.\begin{aligned} \mathbf{F} &: [m\, l\, t^{-2} q^{-1}] = [\mathrm{dyn}]\,[q^{-1}] \\ \mathbf{D} &: [l^{-2} q] \\ \mathbf{H} &: [l^{-2} P] \\ \mathbf{B} &: [m\, l\, t^{-2} P^{-1}] = [\mathrm{dyn}]\,[P^{-1}] \\ \varepsilon &: [m^{-1} l^3 t^2 q^2] = [\tilde{h}^{-1} \tilde{c}^{-1} q^2] \\ \frac{1}{\mu} &: [m^{-1} l^{-3} t^2 P^2] = [\tilde{h}^{-1} \tilde{c}^{-1} P^2] \end{aligned}\right\} \quad . \quad . \quad (5.14)$$

expressing a rather fine duality between $\mathbf{F}$, $\mathbf{D}$, Q, ε and $\mathbf{B}$, $\mathbf{H}$, P, $\frac{1}{\mu}$.

If we take instead of P the magnetic flux Φ, k' can be eliminated:

$$[k'] = [m^{-1} l^{-2} t\, q\, \Phi]\,[a] \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.15)$$

and with the same assumption (5. 13) we get

$$\left.\begin{aligned} \mathbf{F} &: [m\, l\, t^{-2} q^{-1}] = [\mathrm{dyn}]\,[q^{-1}] \\ \mathbf{D} &: [l^{-2} q] \\ \mathbf{H} &: [m\, l\, t^{-2} \Phi^{-1}]\,[\mathrm{dyn}]\,[\Phi^{-1}] \\ \mathbf{B} &: [l^{-2} \Phi] \\ \varepsilon &: [m^{-1} l^3 t^2 q^2] = [\tilde{h}^{-1} \tilde{c}^{-1} q^2] \\ \mu &: [m^{-1} l^3 t^2 \Phi^2] = [\tilde{h}^{-1} \tilde{c}^{-1} \Phi^2] \end{aligned}\right\} \quad . \quad . \quad (5.16)$$

expressing a duality between $\mathbf{F}$, $\mathbf{D}$, Q, ε and $\mathbf{H}$, $\mathbf{B}$, Φ, μ.

The system of five fundamental of SOMMERFELD as well as the other system satisfy really SOMMERFELD's condition that the dimension formulas be "möglichst einfach und sinnvoll". But, $k = k'$ being no longer dimensionless the equations (4. 13a, c) take the form (for $h = 1$)

$$\left.\begin{aligned} k \nabla \times \mathbf{F} + \dot{\mathbf{B}} &= 0 \\ \dot{\mathbf{D}} - k \nabla \times \mathbf{H} &= -4\pi \varrho\, \bar{u} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.17)$$

with a dimensionated constant. This appears rather artificial. If these systems of five fundamental units do not yield any more practical profit we prefer the system with four fundamental units in m, l, t and q.

§ 6. *The electromagnetic equations independent of the choice of the coordinatesystem.*

We start from the equations (4. 13, *a*, *b*, *c*, *d*), written in full

$$\left.\begin{array}{ll} a) & k'\,\partial_2 F^3 - k'\,\partial_3 F^2 + c\,\partial_4 B^1 = 0; \qquad \partial_4 \overset{\text{def}}{=} \frac{1}{c}\frac{\partial}{\partial t} \\ b) & \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0 \\ c) & \frac{c}{4\pi h}\partial_4 D^1 - \frac{k}{4\pi}\partial_2 H^3 + \frac{k}{4\pi}\partial_3 H^2 = -\varrho\, u^1 \\ d) & \frac{1}{4\pi h}\partial_1 D^1 + \frac{1}{4\pi h}\partial_2 D^2 + \frac{1}{4\pi h}\partial_3 D^3 = \varrho \\ & \qquad\qquad\qquad\qquad \text{cycl. } 1,\ 2,\ 3. \end{array}\right\} \quad . \quad . \quad (6.1)$$

and try to transform them in such a way that they contain only invariant differential operations. (From now on we take the fundamental tensor negative definite in view of the relativistic formulas in § 7.) In (6. 1 *a*, *b*) this is only possible if **F** is looked upon as a covariant vector and **B** as a covariant bivector, both ordinary and not pseudo because the electric potential has no screw sense. ϱ in (6. 1 *c*, *d*) can have no screw sense and can only be a (WEYLian) density of weight + 1. That implies that D is a contravariant (WEYLian) vectordensity of weight + 1 and **H** a contravariant (WEYLian) bivectordensity of weight + 1. Hence (6. 1) can be written in the form:

$$\left.\begin{array}{ll} a) & 2\frac{k'}{c}\,\partial_{[c} F_{b]} + \partial_4 B_{cb} = 0 \\ b) & \partial_{[d} B_{cb]} = 0 \\ c) & \frac{1}{4\pi h}\partial_4 \mathfrak{D}^a + \frac{k}{4\pi c}\partial_c \mathfrak{H}^{ca} = -\frac{\varrho}{c}\,u^a \qquad ;\ a, b, c, d = 1, 2, 3 \\ d) & \frac{1}{4\pi h}\partial_c \mathfrak{D}^c = \varrho \end{array}\right\} \quad . \quad (6.2)$$

containing only invariant differential operations and this is the only way to do it. (6. 2) having the invariant form it can be written in general coordinates:

$$\left.\begin{array}{l} 2\frac{k'}{c}\,\partial_{[\gamma} F_{\beta]} + \partial_4 B_{\gamma\beta} = 0 \\ \partial_{[\delta} B_{\gamma\beta]} = 0 \\ \frac{1}{4\pi h}\partial_4 \mathfrak{D}^\alpha + \frac{k}{4\pi c}\partial_\gamma \mathfrak{H}_{\gamma\alpha} = -\frac{\varrho}{c}\,u^\alpha \qquad ;\ \alpha, \beta, \gamma, \delta = 1, 2, 3. \\ \frac{1}{4\pi h}\partial_\gamma \mathfrak{D}^\gamma = \varrho \end{array}\right\} \quad . \quad (6.3)$$

The figures of the four quantities F_β, $\mathfrak{D}^\alpha$, $B_{\gamma\delta}$, $\mathfrak{H}^{\alpha\beta}$ are (see Fig. 4):

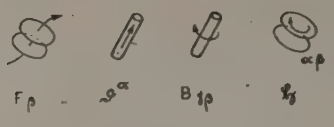

Fig. 4.

From these figures we see that the relation between **F** and **D** and between **H** and **B** in an isotropic medium can only be established by means of the fundamental tensor because it requires the notion "perpendicular".

With respect to general coordinates the equations (4. 13 g, h) take the form

$$\left.\begin{aligned} \mathfrak{D}^\alpha &= -\frac{a\,h\,h'}{p}\,\varepsilon\,\mathfrak{g}^{1/2}\,g^{\alpha\beta}\,F_\beta \\ B_{\gamma\beta} &= \frac{k\,k'}{a\,h'\,p'\,c^2}\,\mu\,\mathfrak{g}^{-1/2}\,g_{\gamma\alpha}\,g_{\beta\delta}\,\mathfrak{H}^{\alpha\delta} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.4)$$

in an isotropic medium. In a not isotropic medium we get

$$\left.\begin{aligned} \mathfrak{D}^\alpha &= -\frac{a\,h\,h'}{p}\,\varepsilon^{\alpha\beta}\,F_\beta \\ B_{\gamma\beta} &= \frac{k\,k'}{a\,h'\,p'\,c^2}\,\mu_{\gamma\beta\alpha\delta}\,\mathfrak{H}^{\alpha\delta} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

where $\varepsilon^{\alpha\beta}$ and $\mu_{\gamma\beta\alpha\delta}$ are affinordensities of weight + 1 and — 1 respectively.

In order to derive the absolute dimension, i.e. the dimension of the general components, from the relative dimension, we remark that the orthogonal components of the radiusvector get a factor l if the unit of length is transformed but that the general components remain unchanged. That implies that the absolute dimension can be derived from the relative dimension by adding for each contra- or covariant index a factor l^{-1} or l respectively and an additional factor $l^{3\mathfrak{f}}$ for a density of weight $\mathfrak{f}$. Hence $g^{\lambda\varkappa}$ has the absolute dimension l^{-2} and $g_{\lambda\varkappa}$ the absolute dimension l^2. From (5. 1) we deduce the following table for the absolute dimensions (the absolute dimension of a scalar is equal to the relative dimension):

$$\left.\begin{aligned} F_\beta &: [m^{1/2}l^{1/2}t^{-1}]\,[h'^{-1/2}a^{-1}] = [m\,l^2\,t^{-1}\,q^{-1}]\,[a^{-1}] \\ \mathfrak{D}^\alpha &: [m^{1/2}l^{3/2}t^{-1}]\,[h\,h'^{1/2}] = [q]\,[h] \\ \mathfrak{H}^{\alpha\beta} &: [m^{1/2}l^{3/2}t^{-2}]\,[k^{-1}\,h'^{1/2}] = [t^{-1}\,q]\,[k^{-1}] \\ B_{\gamma\beta} &: [m^{1/2}l^{1/2}]\,[k'\,h'^{-1/2}a^{-1}] = [m\,l^2\,t^{-1}\,q^{-1}]\,[k'\,a^{-1}] \\ \varepsilon^{\alpha\beta} &: [l]\,[h\,h'\,a] = [m^{-1}\,l^{-2}t^2\,q^2]\,[h\,a] \\ \mu_{\delta\gamma\beta\alpha} &: [l^{-1}\,t^2]\,[k\,k'\,h'^{-1}\,a^{-1}] = [m\,l^2\,q^{-2}]\,[k\,k'\,a^{-1}] \end{aligned}\right\} \quad . \quad (6.6)$$

From this table we get the following absolute dimensions for the four systems mentioned above

	Electromagn.; Giorgi *c.g.s.*	*m.kg.s.*	*Electrostatic* *c.g.s.*	*Gauss* *c.g.s.*
F_β :	$[m^{1/2} l^{1/2} t^{-2}]$	$[ml^2t^{-2}t^{-2}q^{-1}]=[\tilde{h}t^{-1}q^{-1}]$	$[m^{1/2} l^{1/2} t^{-1}]$	$[m^{1/2} l^{1/2} t^{-1}]$; $[m l^2 t^{-2} q^{-1}]$
$\mathfrak{D}^\alpha$:	$[m^{1/2} l^{1/2}]$	$[q]$	$[m^{1/2} l^{3/2} t^{-1}]$	$[m^{1/2} l^{3/2} t^{-1}]$; $[q]$
$\mathfrak{H}^{\alpha\beta}$:	$[m^{1/2} l^{1/2} t^{-1}]$	$[t^{-1} q]$	$[m^{1/2} l^{3/2} t^{-2}]$	$[m^{1/2} l^{1/2} t^{-1}]$; $[l^{-1} q]$
$B_{\gamma\beta}$:	$[m^{1/2} l^{3/2} t^{-1}]$	$ml^2t^{-1}q^{-1}]=[\tilde{h}q^{-1}]$	$[m^{1/2} l^{1/2}]$	$[m^{1/2} l^{3/2} t^{-1}]$; $[m l^3 t^{-2} q^{-1}]$
$\varepsilon^{\alpha\beta}$:	$[l^{-1} t^2]$	$[m^{-1} l^{-2} t^2 q^2]=[\tilde{h}^{-1} tq^2]$	$[l]$	$[l]$; $[m^{-1} l^{-2} t^2 q^2]$
$\mu_{\delta\gamma\beta\alpha}$:	$[l]$	$[ml^2q^{-2}]=[\tilde{h}tq^{-2}]$	$[l^{-1} t^2]$	$[l]$; $[m l^4 t^{-2} q^{-2}]$

The absolute dimensions, expressed in m, l, t, q are the same for the three systems to the left and they differ from the absolute dimensions for the system of GAUSS only by factors lt^{-1}.

Here follow several examples of the relations between the absolute dimension and the possibilities of the construction of the figure of a quantity.

1. *Electric field strength* $F_\beta : [\tilde{h}\, t^{-1}\, q^{-1}]$.

If an electron is at rest in O in a field F_β in a gas with a radiation frequence ν, the radiation begins where the energy of the electrons is $h\nu$. If a magnetic field is added this does not change the energy of the electron because the force due to this field is perpendicular to the velocity. Hence, if the magnetic field is changed, the locus of the points where radiation sets in is just the second plane of the covariant vector $\frac{e}{h\nu} F_\beta$ if the first plane is taken through O. That means *that the radiation itself draws in space the covariant vector* $\frac{e}{h\nu} F_\beta$.

2. *Magnetic induction* $B_{\gamma\beta} : [\tilde{h}\, q^{-1}]$.

a. A moving electron in a magnetic field is subjected to the force

$$\mathbf{K} = e\,\mathbf{v} \times \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.7)$$

perpendicular to the velocity. According to the theorem of HUYGENS this force is $\frac{mv^2}{R}$, R being the radius of curvature of the path. That implies that the electron describes a helical line with an axis in the direction of **B**. If the cylinder of this helical line has a perpendicular section with an area O, the moment of momentum is $\frac{1}{\pi} e\,BO$. According to quantum theory this

moment of momentum has to be a multiple of $\frac{h}{2\pi}$. Hence, φ being the quantum number of the movement we have

$$2\,e\,B\,O = \varphi\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

From this it follows that the cylinder of the path represents just the covariant bivector $\frac{2e}{\pi\varphi h} B_{\gamma\beta}$. Hence *the moving electron itself draws in space the bivector $B_{\gamma\beta}$ provided with a factor that can be computed if the quantum number of the movement is known.*

b. From

$$[\tilde{h}\, q^{-1}] = [R]\,[q] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.9)$$

it follows that the bivector $B_{\gamma\beta}$ can be constructed if the units of resistance and electric charge are known. We can use a flat coil, connected with a ballistic galvanometer. The sense indicating the positive sense of the galvanometer makes from this coil a contravariant bivector $f^{\alpha\beta}$ (cf. fig. 1). The scalar

$$^1/_2\, f^{\alpha\beta} B_{\alpha\beta} = \frac{\text{area coil}}{\text{area section of } B_{\alpha\beta} \text{ with plane of coil}} \quad . \quad (6.10)$$

is the magnetic flux through the coil. This flux is + if the sense of $f^{\alpha\beta}$ is the same as the sense of $B_{\alpha\beta}$. If R is the resistance of the coil and N the number of windings we have

$$\frac{R}{N} \int I\,dt = {}^1/_2\, f^{\alpha\beta} B_{\alpha\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.11)$$

if the coil is withdrawn suddenly from the field. Accordingly

$$\text{area section of } B_{\alpha\beta} \text{ with plane of coil} = \frac{N}{R} \frac{\text{area coil}}{\int I\,dt} \quad . \quad (6.12)$$

Hence this method enables us to measure the section of the tube of $B_{\alpha\beta}$ with every plane. $B_{\alpha\beta}$ is fixed by measuring in three different planes.

3. *Magnetic field strength* $\mathfrak{H}^{\alpha\beta} : [t^{-1}\, q]$ *and electric displacement* $\mathfrak{D}^{\alpha} : [q]$.

These quantities are mere calculating quantities and cannot be measured *directly*. In fact a moving charge is only subjected to forces depending on **F** and **B**. Hence the measuring of **H** and **D** is always based on a measuring of directly measurable quantities as mass, length, time, charge, **F** and **B**, from which **H** and **D** can be derived by calculation. *Notwithstanding this the construction of the figures of* **H** *and* **D** *does not require more units than is indicated by the absolute dimension.* For instance, to construct $\mathfrak{D}^{\alpha}$ in a condensator it suffices to measure the charge of the condensator by means of a ballistic galvanometer, requiring only the unit of electric charge. If then the lines of force are drawn (this requires only the knowledge of the *form* of the condensator and not of its absolute size,

hence no unit of length) and if a definite tube has the charge $\triangle Q$, a little piece of this tube represents the vector density $\frac{Q}{\triangle Q}\mathfrak{D}^{\alpha}$ so much the better according as $\triangle Q$ is smaller. Of course, to determine the *orthogonal* components of **D** the unit of length is required. Similarly the construction of the figure of the magnetic field strength $\mathfrak{H}^{\alpha\beta}$ of a definitely given current requires only the unit of electric current. This is in accordance with the absolute dimension $[t^{-1}q]$ of $\mathfrak{H}^{\alpha\beta}$ and I.

§ 7. *The electromagnetic equations in fourdimensional invariant form.* If, starting from the equations (6. 2) in *orthogonal* components we write

$$\left.\begin{aligned} \frac{c}{k'}B_{4b} = -\frac{c}{k'}B_{b4} \overset{\text{def}}{=} F_b;\ 4\pi h\,\mathfrak{F}^{4a} = -4\pi h\,\mathfrak{F}^{a4} \overset{\text{def}}{=} \mathfrak{D}^a;\ \mathfrak{F}^{ab} \overset{\text{def}}{=} \frac{k}{4\pi c}\mathfrak{H}^{ab} \\ \mathfrak{s}^a \overset{\text{def}}{=} \frac{\varrho}{c}u^a;\quad \mathfrak{s}^4 \overset{\text{def}}{=} \varrho; \\ a, b, c = 1, 2, 3 \end{aligned}\right\} \quad (7.1)$$

the equations (6. 2) can be written in the form

$$\left.\begin{aligned} \partial_{[j}B_{ih]} &= 0 \\ \partial_j \mathfrak{F}^{jh} &= -\mathfrak{s}^h \end{aligned}\right\};\quad h, i, j = 1, 2, 3, 4 \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

having the fourdimensional invariant form if B_{ih} is a fourdimensional covariant bivector and $\mathfrak{F}^{jk}$ a fourdimensional contravariant bivector density of weight + 1. (7. 2) having the invariant form it can be written in general coordinates:

$$\left.\begin{aligned} \partial_{[\omega}B_{\mu\lambda]} &= 0 \\ \partial_\mu \mathfrak{F}^{\mu\varkappa} &= -\mathfrak{s}^{\varkappa} \end{aligned}\right\};\quad \varkappa, \mu, \lambda, \omega = 1, 2, 3, 4 \quad . \quad . \quad . \quad . \quad (7.3)$$

From (5. 1) we get for the relative dimensions of B_{ih}, $\mathfrak{F}^{hi}$ and $\mathfrak{s}^h$

$$\left.\begin{aligned} B_{ih} &: [m t^{-1} q^{-1}]\,[k' a^{-1}] \\ \mathfrak{F}^{hi} &: [l^{-2} q] \\ \mathfrak{s}^h &: [l^{-3} q] \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.4)$$

In order to derive the absolute dimensions from the relative dimensions we have in four dimensions to add for every covariant index a factor l, for every contravariant index a factor l^{-1} and for a density of weight + 1 a factor l^4. Hence the absolute dimensions are

$$\left.\begin{aligned} B_{\lambda\varkappa} &: [m l^2 t^{-1} q^{-1}]\,[k' a^{-1}] = [\tilde{h} q^{-1}]\,[k' a^{-1}] \\ \mathfrak{F}^{\varkappa\lambda} &: [q] \\ \mathfrak{s}^{\varkappa} &: [q] \end{aligned}\right\} \quad . \quad . \quad . \quad (7.5)$$

From this we see that $\mathfrak{F}^{\varkappa\lambda}$ and $\mathfrak{s}^{\varkappa}$ have the absolute dimension $[q]$ in all possible systems and that $B_{\lambda\varkappa}$ has the absolute dimension $[\tilde{h} q^{-1}]$ in the

electromagnetic and the electrostatic system and the system of GIORGI and the dimension $[\tilde{c}\ \tilde{h}\, q^{-1}]$ in the system of GAUSS.

§ 8. *Conclusions.*

I. As we have seen in § 6, **F** and **B** are ordinary quantities and **D** and **H** densities. **F** and **B** are measurable and **D** and **H** only calculating quantities. In § 7 we have seen that **F** and **B** are components of one fourdimensional bivector and **D** and **H** components of one fourdimensional bivectordensity. Hence there is an intimate relationship between **F** and **B** and also between **D** and **H**. This relationship is also to be seen in the formulas for the relative dimensions (cf. (5.1))

$$\left.\begin{aligned} \mathbf{F} &: [m l t^{-2} q^{-1}]\,[a^{-1}] \\ \mathbf{D} &: [l^{-2} q]\,[h] \\ \mathbf{H} &: [l^{-1} t^{-1} q]\,[k^{-1}] \\ \mathbf{B} &: [m t^{-1} q^{-1}]\,[k' a^{-1}] \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8.1)$$

Apart from the constants a, h, k, k' the dimensions of **F** and **B** differ only by a factor $[l t^{-1}]$ and so do the dimensions of **D** and **H**. If we choose $[k] = [l t^{-1}]$ and $[a] = [h] = [1]$ as in the system of GAUSS the dimensions of **F** and **B** and also of **D** and **H** are equal.

The relationship mentioned does not hinder that the figures of **F** and **B** and also of **D** and **H** are quite different (cf. fig. 4). But that is due to the fact that in both cases the two components arise in quite a different way. **F** and **D** arise by dropping the index 4 in $B_{\beta 4}$ and $\mathfrak{F}^{4\alpha}$ and **B** and **H** by simply equalizing orthogonal components.

Anyhow there is no reason to conclude that **F** and **D** or **H** and **B** are intimately related. This conclusion is not in any way affected by the fact that, using orthogonal coordinatesystems and $\varepsilon_0 = 1$; $\mu_0 = 1$, the components of the calculating quantities **D** and **H** in vacuum happen to be equal to the components of the measurable quantities **F** and **B** respectively. The only resemblance between **F** and **D** and between **H** and **B** is that in the derivation of **F** and **D** from the fourdimensional quantities the time-axis is used whilst **H** and **B** are derived without using the time-axis. (That makes clear why in the absolute dimensions of F_β and $\mathfrak{H}^{\alpha\beta}$ next to $\tilde{h}$ and q also t occurs and in the absolute dimensions of $\mathfrak{D}^\alpha$ and $B_{\gamma\beta}$ only $\tilde{h}$ and q (cf. (6.6)).

II. In connection with a publication of SOMMERFELD [4]), WESTPHAL [5]) has proposed to call **B** (and not **H**) the magnetic fieldstrength and to introduce **H** (called by him magnetic "Erregung" or magnetic displacement) as a surface density of a magnetic "charge" ($H = \frac{P}{O}$; $P =$ pole strength). Now the name magnetic fieldstrength is not so bad, in fact

[4]) Zeitschr. f. Techn. Physik **16**, 420 (1935).
[5]) Zeitschr. f. Physik **119**, 164 (1941).

the formula of the force an electron in a magnetic field is subjected to contain **B** and not **H**. However it has to be remarked that a change of names mostly leads to mistakes if it cannot be carried through thoroughly and generally. Better a bad name generally accepted and understood as a good name used only by a part of the authors. But there are serious objections against calling **H** a magnetic displacement. The figure of **H** (cf. fig. 4) is not the figure of a displacement (a tube) and magnetic "charges" do not exist. More important than the giving of names is that it becomes absolutely clear, that **F** and **B** are measurable quantities and **D** and **H** mere calculating quantities.

III. There is only one reasonable method of rationalisation (the other leading to units unfit for use), and, using the system of GIORGI this rationalization is certainly recommendable. It not only simplifies the formulas but the unit of **H**, the ampere winding pro meter, is more fit for practical use than the oerstedt. But it may be remarked that apart from ε and μ only **H** (a calculating quantity) gets another unit by the rationalization and that, therefore, the whole question of rationalization or not rationalization is not so very important. The amount of time spent in discussions on this subject seems not to be in any way justified by its real importance.

IV. The systems with three fundamental units are unpractical because fractional exponents are a nuisance in all calculations. Therefore we think it absolutely necessary to introduce a fourth fundamental unit. The unit of resistance cannot be chosen because then the fractional exponents remain. There is no such strong motive for the introduction of a fifth fundamental unit, and at present it is not to be seen if this introduction can yield any practical profit. There does not exist a system of four fundamental units complying with all wishes of investigators because these wishes are different according to the many different fields of investigation. In ordinary mechanics one would certainly prefer to write for the dimension of an energy $[m\,l^2t^{-2}]$, but an electrotechnical ingeneer will prefer sometimes $[t\,E\,I]$ and in investigations concerning atoms the form $[\tilde{h}\,t^{-1}]$ is convenient. Now here is a way out, we can adopt *one* system of fundamental units and use *next to* the ordinary dimension formulas belonging to this system, other formulas more convenient for special purposes. In fact, once a system of fundamental units being accepted all changes of dimension formulas are effected by factorization of a product in different ways.

Concerning the choice of the system to be accepted, personally we prefer the rationalized system of GIORGI with m, l, t and q. Till yet the question is not settled internationally. For the present the subdepartment of technical physics of the Technical University of Delft has decided to use the not rationalized system of GIORGI and in one of the last meetings of the Netherlands Physical Committee the Committee has also provisionally accepted this system.

Mathematics. — *Sur l'approximation de fonctions abstraites.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1946.)

1. Chaque fonction réelle bornée (c.à.d. une fonction bornée dont les valeurs sont des nombres réels), définie sur un ensemble arbitraire V, peut s'écrire comme limite d'une suite uniformement convergente de fonctions, dites simples, dont chacune ne prend sur V qu'un nombre fini de valeurs distinctes.

Dans ce qui suit nous étudions ce que devient de cette propriété si on remplace la fonction réelle par une transformation univoque arbitraire de V en un espace métrique arbitrairement donné E.

La distance de deux points a et b de E sera désignée par $d(a, b)$.

Définition. *Une transformation univoque de V en un ensemble fini de E s'appelle une fonction simple.*

Soit $f(x)\,(x \in V)$ une transformation univoque arbitraire de V en E.

Problème. *On demande des conditions nécessaires et suffisantes pour-que $f(x)$ soit approximable uniformement par des fonctions simples. Donc, sous quelles conditions existe, $\varepsilon > 0$ étant donné, une fonction simple $\varphi(x)$* telle que

$$d(f(x),\ \varphi(x)) < \varepsilon$$

pour tout $x \in V$.

Disons qu'une fonction qui possède cette propriété est de première classe. En désignant par A l'ensemble des valeurs de $f(x)$ pour $x \in V$ (donc $A \subset E$), on a:

Théorème 1. *Une condition nécessaire et suffisante pourque $f(x)$ soit de 1-re classe est que A est totalement borné.*

Démonstration. 1. A soit totalement borné. Cela signifie que pour tout $\varepsilon > 0$ donné A est la somme d'un nombre fini d'ensembles A_i disjoints avec diamètre $< \varepsilon$:

$$A = A_1 + \ldots + A_n,\ ,\ A_i A_j = 0\ (i \neq j).$$

Soit V_i le sous-ensemble de V qui par $f(x)$ est transformé en A_i; on a $V_1 + \ldots + V_n = V,\ V_i V_j = 0\ (i \neq j)$. Soit $a_i \in A_i$. Alors la fonction

$$\varphi(x) = a_i \quad \text{pour } x \in V_i \quad (i = 1, \ldots, n)$$

est l'approximation désirée.

2. Supposons que $f(x)$ est approximable uniformement au sens indiqué. Pour $\varepsilon > 0$ donné il existe une fonction simple $\varphi(x)$ telle que pour tout $x \in V$

$$d(f(x),\ \varphi(x)) < \tfrac{1}{2}\varepsilon.$$

Supposons que $\varphi(x)$ prend les valeurs distinctes $a_1, \ldots, a_n$ resp. sur $V_1, \ldots, V_n$; $V_1 + \ldots + V_n = V$. Par $f(x)$, $V_1, \ldots, V_n$ sont transformés en $A_1, \ldots, A_n$; $A = A_1 + \ldots + A_n$. En désignant par $\alpha_i, \beta_i, \ldots$ des points de A_i, on a donc

$$d(\alpha_i, a_i) < \tfrac{1}{2}\varepsilon \quad (i = 1, \ldots, n).$$

Il s'ensuit

$$d(\alpha_i, \beta_i) < d(\alpha_i, a_i) + d(a_i, \beta_i) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

donc la diamètre de A_i est $< \varepsilon$. La décomposition désirée est donc $A = A_1 + \ldots + A_n$ de sorte que A est totalement borné.

Conséquence. Si A est totalement borné, A est séparable; en vertu du théorème d'URYSOHN le théorème précédent peut donc s'exprimer comme il suit:

Pourque $f(x)$ soit de 1-re classe il faut et il suffit que A est homéomorphe à un sous-ensemble du cube fondamental $I^{\aleph_0}$ de HILBERT.

En effet, chaque pareil ensemble est totalement borné, puisque $I^{\aleph_0}$ lui-même est totalement borné. Inversement, chaque espace métrique séparable, à fortiori chaque espace métrique totalement borné, est topologiquement contenu dans $I^{\aleph_0}$. Si E est de dimension finie, on a même: A est homéomorphe à un ensemble borné d'un espace euclidien à n dimensions.

Quant aux fonctions qui ne sont pas de 1-re classe, on a le théorème suivant.

Théorème 2. *Une condition nécessaire et suffisante pourque $f(x)$ soit approximable uniformement par des fonctions, qui sont chacune limite d'ordre fini ou transfini — pourvu que dans ce dernier cas l'ordre de la limite est dénombrable — de fonctions de 1-re classe, est que A est séparable.*

Remarques. On peut considérer le système de fonctions

$$f^{(n)}(x) = \lim_{i_1} \lim_{i_2} \ldots \lim_{i_n} f_{i_1 \ldots i_n}(x) \qquad (n = 1, 2, \ldots)$$

oP $f_{i_1 \ldots i_n}(x)$ est une fonction de 1-re classe. Ensuite on construit les fonctions

$$f^{(\omega)}(x) = \lim_{n \to \infty} f^{(n)}(x),$$

$$f^{(\omega+1)}(x) = \lim_{i \to \infty} f_i^{(\omega)}(x).$$

On continuera ce procédé transfiniment sous la condition de se restreindre à des nombres ordinaux dénombrables. Alors le théorème exprime que la séparabilité de A est nécessaire et suffisante pourque $f(x)$ soit approximable uniformement par des fonctions appartenant au système ainsi construit. D'ailleurs, on verra immédiatement que chaque fonction qui est approximable d'une telle façon, l'est déjà par des fonctions qui ne prennent sur V qu'une infinité dénombrable de valeurs distinctes dans E, donc qui transforment V en un sous-ensemble dénombrable de E.

Démonstration. 1. A soit séparable. $\varepsilon > 0$ étant donné, construisons pour chaque point a de A le voisinage $d(a, a) < \varepsilon$ $(a \in A)$. En remarquant que, à cause de la séparabilité de A, chaque recouvrement de A contient un recouvrement dénombrable, on voit comme dans la démonstration du théorème 1 que $f(x)$ est approximable uniformement par des fonctions qui ne prennent qu'une infinité dénombrable de valeurs distinctes de E.

2. Inversement, soit d'abord

$$f(x) = \lim_{i \to \infty} f_i(x),$$

où $f_i(x)$ est de 1-re classe. A cause du théorème 1, A_i — l'ensemble des valeurs de f_i — est totalement borné, donc séparable. Pour chaque i soit $a_n^{(i)}$ $(n = 1, 2, \ldots)$ un ensemble dénombrable partout dense dans A_i. On voit alors que chaque point de A est point d'accumulation de l'ensemble dénombrable des points $a_n^{(i)}$ $(i, n = 1, 2, \ldots)$. Il s'ensuit que A est séparable. Puisque un ensemble dénombrable d'ensembles dénombrables est dénombrable, on voit que la séparabilité est conservée après chaque passage à la limite, pourvu que l'ordre de la limite soit dénombrable. A est donc séparable pour chaque limite dénombrable transfinie. Il suit alors de la partie 1 de la démonstration que $f(x)$ est déjà approximable uniformement par des fonctions qui ne prennent qu'une infinité dénombrable de valeurs distinctes de E.

Selon le théorème de métrisation de URYSOHN la condition du théorème 2 peut être remplacée aussi par la suivante: A est homéomorphe à un sous-ensemble du cube fondamental de HILBERT.

Nous dirons que les fonctions, dont il s'agit dans le théorème 2, sont de la deuxième classe.

Il y a une relation entre les deux classes.

Soit $f^{(2)}(x)$ une fonction de la 2-me classe; l'ensemble correspondant des valeurs $A^{(2)}$ est donc séparable. Chaque espace métrique séparable est homéomorphe à un espace métrique totalement borné; il existe donc une fonction biunivoque, bicontinue F qui transforme $A^{(2)}$ en un espace totalement borné $A^{(1)}$. Il s'ensuit que la fonction, définie sur V,

$$F(f^{(2)}(x)) = f^{(1)}(x)$$

est de la 1-re classe.

On peut continuer la classification aux fonctions dont l'ensemble des valeurs A est non-séparable. On voit qu'il faut classificer selon le plus petit nombre cardinal, possédant la propriété qu'il existe un ensemble dense dans A ayant ce nombre comme puissance, en ce sens que des fonctions qui ont ce même cardinal appartiennent à la même classe.

3. Supposons qu'il existe dans E un point O tel qu'on a pour chaque couple a et b de points de E

$$d(a, O) \leqq \max(d(b, O), d(a, b)). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

La métrique est donc en quelques égards non-archimédienne. Posons pour abrévier

$$d(a, O) = \| a \|,$$

et appelons $\| a \|$ la norme de a. Dans un pareil espace on a le théorème suivant.

Théorème 3. *La limite d'une suite convergente de fonctions, dont chacune a un ensemble des normes des valeurs de la fonction au plus dénombrable, a elle-même un ensemble de normes au plus dénombrable.*

Démonstration. Soit $f(x) = \lim_{n \to \infty} f_n(x)$ et soit $\varepsilon > 0$ donné. Considérons l'ensemble e_ε des points $x \in V$ où l'on a $\| f(x) \| \geqq \varepsilon$. En chaque point de e_ε on a pour $n \geqq N_x$

$$\| f_n(x) \| = \max (\| f(x) \|, d(f(x), f_n(x)) = \| f(x) \|,$$

puisque $\| f(x) \| \geqq \varepsilon$ et $d(f(x), f_n(x)) < \varepsilon$ pour $n \geqq N_x$. Soit $e_\varepsilon^{(k)}$ le sous-ensemble de e_ε où

$$\| f(x) \| = \| f_n(x) \|$$

pour $n \geqq k$, cependant non pour $n \geqq k - 1$. Alors

$$e_\varepsilon = e_\varepsilon^{(1)} + e_\varepsilon^{(2)} + \dots$$

Sur chaque $e_\varepsilon^{(k)}$ $f(x)$ a un ensemble de normes au plus dénombrable; donc $f(x)$ admet au plus une infinité dénombrable de normes $\geqq \varepsilon$. En faisant parcourir ε une suite $\varepsilon_i \to 0$, il suit que $f(x)$ admet au plus une infinité dénombrable de normes distinctes, puisque chaque $x \in V$, sauf les points où $\| f(x) \| = 0$, est contenu dans un e_{ε_i}.

Soit de nouveau E un espace métrique, dont la métrique ne satisfait pas nécessairement à la condition non-archimédienne (1).

Soit $\triangle(a, b)$ la borne inférieure des nombres ϱ, pour lesquels les points a et b de E puissent être liés par une ϱ- chaîne (ϱ-Kette). Si a, b, c sont trois points de E, on a

$$\triangle(a, b) \leqq \max (\triangle(a, c), \triangle(b, c)),$$

$$\triangle(a, b) = \triangle(b, a).$$

Cependant on peut avoir $\triangle(a, b) = 0$ sans que $a = b$. On a $\triangle(a, b) = 0$ pour tous les points a et b qui appartiennent à une même composante connexe de E. $\triangle(a, b)$ détermine une métrique dans l'espace des composantes connexes $Q(E)$ de E, c'est l'espace dont les points sont les composantes connexes de E. On obtient une transformation de E en $Q(E)$, en faisant correspondre à chaque point a de E le point A de $Q(E)$ qui représente la composante connexe de E où appartient le point a considéré. Cette transformation est continue. En effet, soit a_n une suite de points de E avec limite a, donc

$$d(a_n, a) \to 0.$$

Soit A l'image de a, A_n l'image de a_n (il est bien possible que $A_n = A$, en

effet si a_n et a appartiennent à une même composante connexe). Il suit de la définition de la distance $\triangle$ en $Q(E)$

$$\triangle(A_n, A) \leqq d(a_n, a) \to 0.$$

Comme il est connu, on en tire la continuité de la transformation.

L'application des théorèmes 2 et 3 à l'espace $Q(E)$ donne les propriétés suivantes.

S'il existe en $Q(E)$ un point O tel que l'ensemble des normes en $Q(E)$ — ce sont les $\triangle$-distances des point de $Q(E)$ à O — est non-dénombrable, alors $Q(E)$ est non-séparable par rapport à la métrique $\triangle$.

En effet, il existent alors des transformations en $Q(E)$, qu'on ne peut pas obtenir comme limite au sens du théorème 2; selon ce théorème $Q(E)$ contient alors un ensemble non-séparable et l'espace lui-même est alors non-séparable.

Revenons alors à l'espace E et remarquons que l'image continue d'un espace séparable est séparable; on obtient la propriété suivante:

Théorème 4. *Si E contient un point O tel que l'ensemble des nombres $\triangle(a, O)$ avec $a \in E$ est non-dénombrable, alors E est non-séparable.*

L'inversion de ce théorème n'est pas vraie: si l'espace E ne contient qu'une infinité dénombrable de valeurs $\triangle(a, O)$ quelque soit O dans E, l'espace E peut bien être non-séparable. Il suffit de considérer l'union d'une infinité dénombrable d'espaces connexes non-séparables.

On tire du théorème 4:

Si E est un espace métrique séparable, l'ensemble des nombres $\triangle(a, O)$ avec $a \in E$ est dénombrable, quelque soit $O \in E$.

Si la métrique donnée de E est non-archimédienne, c.à.d. si l'on a pour tout a, b, c dans E

$$d(a, b) \leqq \max\,(d(a, c), d(b, c)),$$

on voit que

$$d(a, b) = \triangle(a, b).$$

Cependant il n'est pas permis de conclure que dans un pareil espace séparable l'ensemble des distances mutuelles des points est dénombrable. On peut conclure cela si de plus l'espace est linéaire (par rapport à un certain corps des coefficients) et si de plus

$$d(a, b) = d(a - b, \theta) = \| a - b \|.$$

J'ai étudié de tels espaces dans un article précédent.

Mathematics. — *On the theory of linear integral equations.* IV. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the first two papers bearing the same title, to which we shall refer with I and II [1]). In this paper we shall consider linear integral equations with kernel $K(x, y) \in L_2^{(2m)}(\triangle)$, having the property that there exists a positive Hermitean kernel $H(x, y) \in L_2^{(2m)}(\triangle)$, such that

$$P(x, y) = \int_{\triangle} H(x, z)\, K(z, y)\, dz$$

(then also belonging to $L_2^{(2m)}(\triangle)$) is Hermitean. Defining the completely continuous, linear transformations K, H and $P = HK$ in the space $L_2^{(m)}(\triangle)$ by

$$Kf = \int_{\triangle} K(x, y)\, f(y)\, dy,$$

$$Hf = \int_{\triangle} H(x, y)\, f(y)\, dy,$$

$$Pf = \int_{\triangle} P(x, y)\, f(y)\, dy,$$

H is therefore positive self-adjoint and $P = HK$ is self-adjoint, so that K is symmetrisable relative to H. Making now the additional assumption that every $f(x) \in L_2$, satisfying $Hf = 0$, satisfies also $Kf = 0$, we shall call the kernel $K(x, y)$ a *Marty-kernel*. We observe that this last condition is certainly satisfied if $H(x, y)$ is *definite*, that is, if

$$(Hf, f) = \int_{\triangle \times \triangle} H(x, y)\, \overline{f(x)}\, f(y)\, dx\, dy = 0$$

implies $f(x) = 0$, since then $Hf = 0$ implies $f = 0$, so that also $Kf = 0$. It was this case that was considered for the first time by J. MARTY [2]) for bounded kernels.

To terminate this paragraph we mention that the condition of H-orthogonality for two functions $f(x)$ and $g(x)$, belonging to L_2, takes the form

$$(Hf, g) = \int H(x, y)\, \overline{g(x)}\, f(y)\, dx\, dy = 0,$$

[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49** (1946).

[2]) J. MARTY, Valeurs singulières d'une équation de FREDHOLM, C. R. Acad. sc. Paris **150**, 1499—1502 (1910).

and that $f(x)$ is H-normal if

$$(Hf, f) = \int_{\Delta\times\Delta} H(x, y)\, \overline{f(x)}\, f(y)\, dx\, dy = 1.$$

§ 2. *Integral equation with Marty-kernel.*

We consider now the linear integral equation $Kf - \lambda f = g$ or

$$\int_{\Delta} K(x, y)\, f(y)\, dy - \lambda f(x) = g(x), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f(x)$, $g(x)$ belong to L_2, and $K(x, y)$ is a Marty-kernel. Supposing that

$$\| H(x, y) \|_{2m}^2 = \int_{\Delta\times\Delta} | H(x, y) |^2\, dx\, dy \neq 0,$$

so that $H \neq O$, all theorems proved in I may therefore be applied to this equation.

Theorem 1. *The characteristic values of* (1) *are real and characteristic functions belonging to different characteristic values are H-orthogonal.*

Proof. Follows from I, Theorem 5.

Theorem 2. *If $\lambda \neq 0$ is a characteristic value of* (1), *this equation has, for a given function $g(x) \in L_2$, a solution $f(x) \in L_2$ for those and only those functions $g(x)$ that are H-orthogonal to all characteristic functions, belonging to the characteristic value λ. If $\lambda \neq 0$ is no characteristic value, so that it is a regular value, the equation has a uniquely determined solution for every $g(x) \in L_2$.*

Proof. Follows from I, Theorem 12 and Theorem 3.

Theorem 3. *If*

$$\| P(x, y) \|_{2m}^2 = \int_{\Delta\times\Delta} | P(x, y) |^2\, dx\, dy \neq 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$P(x, y) = \int_{\Delta} H(x, z)\, K(z, y)\, dz,$$

the equation (1) *has a characteristic value $\neq 0$.*

Proof. Since, by (2), $P = HK \neq O$, the result follows from I, Theorem 6.

Let now $\lambda_i (|\lambda_1| \geq |\lambda_2| \geq \ldots)$ be the sequence of all characteristic values $\neq 0$, each of them occurring as many times as denoted by its multiplicity, and $\psi_i(x)$ a corresponding H-orthonormal sequence of characteristic func-

tions. These functions satisfy therefore the relations

$$\int_{\Delta\times\Delta} H(x,y)\,\overline{\psi_i(x)}\,\psi_i(y)\,dx\,dy = 1,$$

$$\int_{\Delta\times\Delta} H(x,y)\,\overline{\psi_i(x)}\,\psi_j(y)\,dx\,dy = 0 \text{ for } i\neq j,$$

or, writing

$$\int_{\Delta} H(x,y)\,\psi_j(y)\,dy = \chi_j(x),$$

$$\int_{\Delta} \psi_i(x)\,\overline{\chi_j(x)}\,dx = \begin{cases} 1 \text{ for } i=j, \\ 0 \text{ for } i\neq j, \end{cases}$$

so that the sequences $\psi_i(x)$ and $\chi_i(x)$ are biorthogonal.

Theorem 4. (*Expansion Theorem*). *Writing*

$$a_i = (f,\chi_i) = \int_{\Delta} f(x)\,\overline{\chi_i(x)}\,dx,$$

and introducing the notation

$$N(f) = (Hf,f)^{1/2} = \left(\int_{\Delta\times\Delta} H(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy\right)^{1/2},$$

we have

$$\lim_{k\to\infty} N\left(Kf - \sum_{i=1}^{k} \lambda_i a_i \psi_i\right) = 0;$$

in other words

$$\lim_{k\to\infty} \int_{\Delta\times\Delta} H(x,y)\left[\overline{\int_{\Delta} K(x,z)\,f(z)\,dz - \sum_{i=1}^{k} \lambda_i a_i \psi_i(x)}\right] \left[\int_{\Delta} K(y,z)\,f(z)\,dz - \sum_{i=1}^{k} \lambda_i a_i \psi_i(y)\right] dx\,dy = 0$$

for any $f(x)\in L_2$. Furthermore

$$\int_{\Delta\times\Delta} P(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy = \Sigma \lambda_i |a_i|^2.$$

Proof. Follows from I, Theorem 9.

Theorem 5. *Let $\lambda_{n_i} (i = 1, 2, \ldots)$ be the subsequence of all positive characteristic values where $\lambda_{n_1} \geq \lambda_{n_2} \geq \ldots$, let the functions $p_1(x), \ldots, p_{i-1}(x) \in L_2$ be arbitrarily given, and let*

$$\mu_i = \text{upper bound} \int_{\Delta\times\Delta} P(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy \Big/ \int_{\Delta\times\Delta} H(x,y)\,\overline{f(x)}\,f(y)\,dx\,dy$$

for all $f(x) \in L_2$ *satisfying* $\int_{\Delta\times\Delta} H(x,y)\overline{f(x)}f(y)dx\,dy \neq 0$ *and*

$$\int_{\Delta\times\Delta} H(x,y)\,\overline{p_1(x)}\,f(y)\,dx\,dy = \ldots = \int_{\Delta\times\Delta} H(x,y)\,\overline{p_{i-1}(x)}\,f(y)\,dx\,dy = 0.$$

The number μ_i *depends on* $p_1(x), \ldots, p_{i-1}(x)$. *Letting now these functions run through the whole space* L_2, *we have* $\lambda_{n_i} = \min \mu_i$.

A similar statement holds for the negative characteristic values.

Proof. Follows from I, Theorem 11.

Theorem 6. *Let* $\lambda \neq 0$, *and let* $g(x) \in L_2$ *be H-orthogonal to all characteristic functions of* (1), *belonging to the characteristic value* λ (*If* λ *is no characteristic value,* $g(x)$ *is therefore arbitrary*). *Then every solution* $f(x)$ *of* (1) *satisfies a relation of the form*

$$\lim_{k\to\infty} N\left(f + \frac{g}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)}\,a_i\,\psi_i\right) = 0$$

or

$$\lim_{k\to\infty} \int_{\Delta\times\Delta} H(x,y)\left[\overline{f(x) + \frac{g(x)}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)}\,a_i\,\psi_i(x)}\right]$$

$$\left[f(y) + \frac{g(y)}{\lambda} + \sum_{i=1}^{k}{}' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)}\,a_i\,\psi_i(y)\right] dx\,dy = 0,$$

where $a_i = \int_{\Delta} g(x)\overline{\chi_i(x)}dx$ *for* $\lambda_i \neq \lambda$, *and where* Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* ψ_i *has the value* $-\int_{\Delta} f(x)\overline{\chi_i(x)}dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 13.

Theorem 7. (*Expansion Theorem*).

$$\lim_{k\to\infty} \int_{\Delta\times\Delta\times\Delta} H(x,y)\left[\overline{K(x,z) - \sum_{i=1}^{k} \lambda_i\,\psi_i(x)\,\overline{\chi_i(z)}}\right]$$

$$\left[K(y,z) - \sum_{i=1}^{k} \lambda_i\,\psi_i(y)\,\overline{\chi_i(z)}\right] dx\,dy\,dz = 0.$$

Proof. This theorem is the analogue of II, Theorem 8, (4), and the proof, though more complicated, is also analogous. We begin by introducing a HILBERT space Z, the elements $[f]$ of which are classes of elements of L_2. We define the element $[f]$ to contain f and all elements g for which $Hg = Hf$ (equivalent with $N(f-g) = 0$). Furthermore we define $[f] + [g] = [f+g]$, $a[f] = [af]$ for every complex a, and

$([f], [g]) = (Hf, g)$, so that $\|[f]\| = (Hf, f)^{1/2} = N(f)$. It is not difficult to prove that these definitions are without contradiction, and it is clear that, since $(Hf, g) = 0$ is equivalent with $([f], [g]) = 0$, two functions $f(x)$ and $g(x) \in L_2$ are H-orthogonal if and only if $[f]$ and $[g]$ are orthogonal in Z. In the same way, since $(Hf, f)^{1/2} = 1$ is equivalent with $\|[f]\| = 1$, the function $f(x)$ is H-normal if and only if the element $[f]$ is normal. The space Z is not necessarily complete, that is, $\lim \|[f]_m - [f]_n\| = 0$ as $m, n \to \infty$ will not always imply the existence of an element $[f]$ satisfying $\lim_{n\to\infty} \|[f] - [f]_n\| = 0$. By adjunction of ideal elements, however, we can obtain from Z the complete HILBERT space $\overline{Z}$, the closure of Z. Evidently every bounded, linear transformation in Z may be continued on $\overline{Z}$ as a linear transformation with the same bound. A certain class of linear transformations C in the space L_2 corresponds now with linear transformations in the space Z. We define $[C][f] = [Cf]$; this definition however is only then without contradiction if $[f] = [g]$ implies $[Cf] = [Cg]$, or, in other words, if $Hf = 0$ implies $HCf = 0$. We shall therefore consider only transformations C satisfying this condition. Amongst them is our symmetrisable transformation K, since $Hf = 0$ implies $Kf = 0$, hence certainly $HKf = 0$. Furthermore we observe that the linear transformation $[K]$ in the space Z is bounded and self-adjoint. The boundedness follows from I, Theorem 8, by which

$$|\lambda_1| = \max N(Kf)/N(f)$$

for all $f(x) \in L_2$ satisfying $N(f) \neq 0$, hence

$$|\lambda_1| = \max \|[Kf]\|/\|[f]\| = \max \|[K][f]\|/\|[f]\|$$

for all $[f] \in Z$ satisfying $[f] \neq [0]$ or

$$\|[K][f]\| \leq |\lambda_1| \cdot \|[f]\|;$$

the self-adjointness follows from

$$([K][f], [g]) = (HKf, g) = (f, HKg) = (Hf, Kg) = ([f], [K][g]).$$

The transformation $[K]$ has the characteristic values λ_i with the orthonormal characteristic elements $[\psi_i]$, since $K\psi_i = \lambda_i\psi_i$ implies $[K][\psi_i] = \lambda_i[\psi_i]$. Moreover, if the element $[g] \in Z$ is orthogonal to all $[\psi_i]$, we have $[K][g] = [0]$. To prove this statement, we observe that, if $([g], [\psi_i]) = (Hg, \psi_i) = (g, \chi_i) = 0$ $(i = 1, 2, \ldots)$, we find by Theorem 4 that $N(Kg) = 0$, hence also $HKg = 0$ or $[K][g] = [Kg] = [0]$. Since $[K]$ is bounded, it may be continued on the closure $\overline{Z}$, and it is not difficult to prove that for an element $[g] \in \overline{Z}$, orthogonal to all $[\psi_i]$, we have also $[K][g] = [0]$.

After these preliminaries we observe that the relation $K\psi_i = \lambda_i\psi_i$ implies, for every $f \in L_2$,

$$(K^* H\psi_i, f) = (H\psi_i, Kf) = (\psi_i, HKf) = (HK\psi_i, f) = (\lambda_i H\psi_i, f),$$

hence $K^*H\psi_i = \lambda_i H\psi_i$ or $K^*\chi_i = \lambda_i\chi_i$, or, since the adjoint transformation K^* is given by

$$K^* f = \int_\triangle \overline{K(y, x)}\, f(y)\, dy,$$

the relation

$$\int_\triangle \overline{K(x, y)}\, \chi_i(x)\, dx = \lambda_i\, \chi_i(y) \quad \ldots\ldots\ldots \quad (3)$$

for almost every $y \in \triangle$. Writing now $f(x) = K(x, z)$, the integral $\int_\triangle |f(x)|^2\, dx = \int_\triangle |K(x, z)|^2\, dx$ is finite for almost every $z \in \triangle$; hence $f(x) \in L_2$ for almost every $z \in \triangle$. With $f(x) \in L_2$ corresponds the element $[f]$ in the space Z, and we have, by (3),

$$([f], [\psi_i]) = (Hf, \psi_i) = (f, H\psi_i) = (f, \chi_i) = \int_\triangle K(x, z)\, \overline{\chi_i(x)}\, dx = \lambda_i\, \overline{\chi_i(z)}$$

for almost every $z \in \triangle$.

Considering now an element $[g] \in \overline{Z}$, orthogonal to all $[\psi_i]$, there exists a sequence of elements $[g_n] \in Z$ such that $\lim [g_n] = [g]$. Since $[K][g] = [0]$, we have

$$\lim [K]\, [g_n] = \lim [Kg_n] = [0]$$

or

$$\lim \| [Kg_n] \| = \lim N(Kg_n) = \lim (HKg_n, Kg_n)^{1/2} = 0,$$

so that certainly $\lim \| HKg_n \| = 0$. Writing

$$HKg_n = Pg_n = \int_\triangle P(z, x)\, g_n(x)\, dx = p_n(z),$$

the relation $\lim \|HKg_n\| = 0$ is equivalent with $\lim \int_\triangle |p_n(z)|^2 dz = 0$; the sequence of functions $p_n(z)$ converges therefore in mean to 0. Then, as well-known, there exists a subsequence $n_k (k = 1, 2, \ldots)$ of indices such that $p_{n_k}(z)$ converges point-wise to 0 as $k \to \infty$ for almost every $z \in \triangle$. Consequently, since

$$([f], [g]) = \lim ([f], [g_n]) = \lim (Hf, g_n) =$$

$$= \lim \int_{\triangle \times \triangle} H(x, y)\, K(y, z)\, \overline{g_n(x)}\, dx\, dy = \lim \int_\triangle P(x, z)\, g_n(x)\, dx =$$

$$= \lim \int_\triangle \overline{P(z, x)\, \overline{g_n(x)}\, dx} = \lim \overline{p_n(z)}$$

for almost every $z \in \Delta$, and therefore certainly

$$([f], [g]) = \lim_{k \to \infty} \overline{p_{n_k}(z)}$$

for these values of z, we have $([f], [g]) = 0$ for almost every $z \in \Delta$.

From the relations $([f], [\psi_i]) = \lambda_i \overline{\chi_i(z)}$ and $([f], [g]) = 0$ for any $[g] \in \overline{Z}$ orthogonal to all $[\psi_i]$, both holding for almost every $z \in \Delta$, we deduce now that $[f] = \Sigma \lambda_i \overline{\chi_i(z)} [\psi_i]$ for almost every $z \in \Delta$, hence

$$\| [f] - \sum_{i=1}^{k} \lambda_i \overline{\chi_i(z)} [\psi_i] \|^2 = \sum_{i=k+1} \lambda_i^2 |\chi_i(z)|^2$$

or

$$N^2 \left(K(x, z) - \sum_{i=1}^{k} \lambda_i \overline{\chi_i(z)} \psi_i(x) \right) = \sum_{i=k+1} \lambda_i^2 |\chi_i(z)|^2$$

or

$$\int_{\Delta \times \Delta} H(x, y) \overline{\left[K(x, z) - \sum_{i=1}^{k} \lambda_i \psi_i(x) \overline{\chi_i(z)} \right]} \left[K(y, z) - \sum_{i=1}^{k} \lambda_i \psi_i(y) \overline{\chi_i(z)} \right] dx\, dy = \sum_{i=k+1} \lambda_i^2 |\chi_i(z)|^2$$

for almost every $z \in \Delta$. Taking $k = 0$ in this relation, we find

$$\left.\begin{aligned} \sum_{i=1} \lambda_i^2 |\chi_i(z)|^2 &= \int_{\Delta \times \Delta} H(x, y) \overline{K(x, z)} K(y, z)\, dx\, dy = \\ &= \int_{\Delta \times \Delta} \overline{H(y, x) K(x, z)} K(y, z)\, dx\, dy = \int_{\Delta} \overline{P(y, z)} K(y, z)\, dy = \\ &= \int_{\Delta} P(z, y) K(y, z)\, dy \leqslant \left(\int_{\Delta} |P(z, y)|^2 dy \right)^{1/2} \cdot \left(\int_{\Delta} |K(y, z)|^2 dy \right)^{1/2} \end{aligned}\right\} \quad (4)$$

for almost every $z \in \Delta$, so that

$$\int_{\Delta} \sum_{i=1} \lambda_i^2 |\chi_i(z)|^2 dz = \sum_{i=1} \lambda_i^2 \int_{\Delta} |\chi_i(z)|^2 dz$$

is finite. This shows that

$$\int_{\Delta} \sum_{i=k+1} \lambda_i^2 |\chi_i(z)|^2 dz = \sum_{i=k+1} \lambda_i^2 \int_{\Delta} |\chi_i(z)|^2 dz$$

tends to 0 as $k \to \infty$, hence also

$$\lim_{k \to \infty} \int_{\Delta \times \Delta \times \Delta} H(x, y) \overline{\left[K(x, z) - \sum_{i=1}^{k} \lambda_i \psi_i(x) \overline{\chi_i(z)} \right]} \left[K(y, z) - \sum_{i=1}^{k} \lambda_i \psi_i(y) \overline{\chi_i(z)} \right] dx\, dy\, dz = 0.$$

The completely continuous transformations $K^n (n = 2, 3, \ldots)$, defined by

$$K^n f = \int_{\Delta} K_n(x, y) f(y)\, dy,$$

where $K_n(x, y) \in L_2^{(2m)}$ is the n-th iterated kernel of $K(x, y)$, are also symmetrisable relative to H. Indeed, if it has already been proved that HK^{n-1} is self-adjoint, we have

$$(HK^n f, g) = (HK^{n-1} Kf, g) = (Kf, HK^{n-1} g) =$$
$$= (HKf, K^{n-1} g) = (f, HKK^{n-1} g) = (f, HK^n g)$$

for every pair of functions $f(x), g(x) \in L_2$; the transformation HK^n is therefore self-adjoint, in other words, K^n is symmetrisable.

Theorem 8. *The sequence $\lambda_i^n (i = 1, 2, \ldots)$ is the sequence of all characteristic values $\neq 0$ of the Marty-kernel $K_n(x, y)$ and $\psi_i(x)$ is corresponding sequence of characteristic functions.*

Proof. Since $K\psi_i = \lambda_i \psi_i$ implies $K^n \psi_i = \lambda_i^n \psi_i$, the functions $\psi_i(x)$ are characteristic functions of $K_n(x, y)$, belonging to the characteristic values λ_i^n. Any further characteristic function $\psi(x)$ of $K_n(x, y)$, linearly independent of $\psi_1(x), \psi_2(x), \ldots$, may be chosen so as to be H-orthogonal to all $\psi_i(x)$; hence, $(H\psi, \psi_i) = (\psi, \chi_i) = 0 \ (i = 1, 2, \ldots)$ implying $HK\psi = 0$ by Theorem 4, we have $K^2\psi = KK\psi = 0$, so that

$$\int_{\Delta} K_n(x, y)\, \psi(y)\, dy = K^n \psi = 0$$

for $n \geq 2$, which shows that $\psi(x)$ belongs to the characteristic value 0. The sequence $\lambda_i^n \ (i = 1, 2, \ldots)$ is therefore the sequence of all characteristic values $\neq 0$ of $K_n(x, y)$.

It follows from this theorem that the Theorems 1—7 hold for the integral equation with kernel $K_n(x, y)$, replacing everywhere λ_i by λ_i^n.

Mathematics. — *On the theory of linear integral equations.* IVa. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1946.)

This paper is an immediate sequel to the paper IV, bearing the same title, to which we refer for the notations.

§ 3. *Expansion Theorems for the kernel.*

Under certain conditions for the kernels $H(x, y)$ and $K(x, y)$ there may be a value of n for which the series $\sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)}$ converges in mean to a function $S_n(x, y) \in L_2^{(2m)}$. It is reasonable to suppose that there exists a connection between the kernels $K_n(x, y)$ and $S_n(x, y)$ and in the following theorem we shall prove this to be the case.

Theorem 9.

If
$$S_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)},$$

then
$$S_n(x, y) = K_n(x, y) - p_n(x, y),$$

where
$$q_n(x, y) = \int_\Delta H(x, z)\, p_n(z, y)\, dz = 0$$

for almost every point $(x, y) \in \Delta \times \Delta$.

Furthermore

$$K_{n+p}(x, y) \backsim \sum_i \lambda_i^{n+p} \psi_i(x) \overline{\chi_i(y)} \quad \text{for } p \geqq 1.$$

Proof. *From*

$$S_n(x, y) \backsim \sum \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \quad \text{follows}$$

$$A(x, y) = \int_\Delta H(x, z)\, S_n(z, y)\, dz \backsim \sum \lambda_i^n \overline{\chi_i(y)} \int_\Delta H(x, z)\, \psi_i(z)\, dz$$

or

$$A(x, y) \backsim \sum \lambda_i^n \chi_i(x) \overline{\chi_i(y)},$$

from which we deduce that $A(x, y) = \overline{A(y, x)}$ almost everywhere in $\Delta \times \Delta$. Defining the completely continuous linear transformation S_n by

$$S_n f = \int_\Delta S_n(x, y)\, f(y)\, dy,$$

this transformation is therefore symmetrisable, since $A = HS_n$ is self-adjoint. For every value of p and any arbitrary $f(x) \in L_2$ we have

$$(S_n \psi_p, f) = \int_{\Delta \times \Delta} S_n(x, y) \overline{f(x)} \psi_p(y)\, dx\, dy = (S_n(x, y), f(x) \overline{\psi_p(y)})_{2m} =$$

$$\lim_{k \to \infty} \left(\sum_{i=1}^{k} \lambda_i^n \psi_i(x) \overline{\chi_i(y)}, f(x) \overline{\psi_p(y)} \right)_{2m} =$$

$$\lim_{k \to \infty} \sum_{i=1}^{k} \lambda_i^n \int_{\Delta} \psi_i(x) \overline{f(x)}\, dx \cdot \int_{\Delta} \psi_p(y) \overline{\chi_i(y)}\, dy =$$

$$\lambda_p^n \int_{\Delta} \psi_p(x) \overline{f(x)}\, dx = (\lambda_p^n \psi_p, f);$$

hence $S_n \psi_p = \lambda_p^n \psi_p$, which shows that $\psi_p(x)$ is a characteristic function of $S_n(x, y)$, belonging to the characteristic value λ_p^n.

In the same way we prove that if $g(x) \in L_2$ is H-orthogonal to all functions $\psi_i(x)$, we have $S_n g = 0$. In particular $Hg = 0$ implies $S_n g = 0$. These results show that $S_n(x, y)$ is a Marty-kernel having the sequence λ_i^n $(i = 1, 2, \ldots)$ as the sequence of all its characteristic values $\neq 0$ with the corresponding characteristic functions $\psi_i(x)$. We consider now the Marty-kernel

$$p_n(x, y) = K_n(x, y) - S_n(x, y),$$

and we shall show that it has no characteristic value $\neq 0$. Indeed, all functions $\psi_i(x)$ are characteristic functions of $p_n(x, y)$, belonging to the characteristic value 0; any characteristic function $\psi(x)$ with characteristic value $\lambda \neq 0$ (so $(K^n - S_n)\psi = \lambda \psi$) would therefore be H-orthogonal to all $\psi_i(x)$ by Theorem 1. Then, however, by what we have just proved, $S_n \psi = 0$, and, by Theorem 4, $N(K\psi) = 0$, which is equivalent with $HK\psi = 0$, so that certainly $HK^n \psi = 0$, hence also

$$\lambda H\psi = H(K^n - S_n)\psi = 0 \text{ or } H\psi = 0.$$

This would imply, since both $K_n(x, y)$ and $S_n(x, y)$ are Marty-kernels, $(K^n - S_n)\psi = 0$ or $\lambda \psi = 0$, so that we should have $\psi = 0$, in contradiction with the assumption that $\psi(x)$ should be a characteristic function. Thus, the kernel $p_n(x, y)$ having no characteristic values $\neq 0$, we may conclude from Theorem 3 that

$$\| q_n(x, y) \|_{2m}^2 = \int_{\Delta \times \Delta} | q_n(x, y) |^2\, dx\, dy = 0,$$

where

$$q_n(x, y) = \int_{\Delta} H(x, z)\, p_n(z, y)\, dz,$$

hence $q_n(x, y) = 0$ for almost every point $(x, y) \in \triangle \times \triangle$. This completes the first part of the proof.

Observing that $\int_{\triangle} H(x, z)p_n(z, y)dz = 0$ implies $\int_{\triangle} K(x, z)p_n(z, y)dz = 0$,

we obtain

$$\int_{\triangle} K(x, z)\, K_n(z, y)\, dz - \int_{\triangle} K(x, z)p_n(z, y)\, dz \backsim \Sigma\lambda_i^n\, \overline{\chi_i(y)} \int_{\triangle} K(x, z)\, \psi_i(z)\, dz$$

or

$$K_{n+1}(x, y) \backsim \Sigma\lambda_i^{n+1}\, \psi_i(x)\, \overline{\chi_i(y)}.$$

The general relation

$$K_{n+p}(x, y) \backsim \Sigma\lambda_i^{n+p}\, \overline{\psi_i(x)}\, \overline{\chi_i(y)} \qquad (p \geqq 1)$$

follows now easily by induction.

Theorem 10. *Under the same assumption as in the preceding theorem, and writing*

$$r_n(x, y) = \int_{\triangle} K(x, z)\, p_n(z, y)\, dz,$$

we have

$$\int_{\triangle} K_{n+1}(x, x)\, dx - \int_{\triangle} r_n(x, x)\, dx = \sum_i \lambda_i^{n+1}, \quad \ldots\ldots \quad (5)$$

$$\int_{\triangle} K_{n+p}(x, x)\, dx = \Sigma\lambda_i^{n+p} \text{ for } p \geqq 2. \quad \ldots\ldots \quad (6)$$

Proof. We observe first that, although $r_n(x, y) = 0$ almost everywhere in $\triangle \times \triangle$, this does not necessarily imply that $r_n(x, x) = 0$ for almost every $x \in \triangle$, hence the term $\int_{\triangle} r_n(x, x)dx$.

We have for almost every $x \in \triangle$

$$|K_{n+1}(x, x) - r_n(x, x) - \sum_{i=1}^{p} \lambda_i^{n+1}\, \psi_i(x)\, \overline{\chi_i(x)}| =$$

$$= \left| \int_{\triangle} K\ x, z\ [K_n(z, x) - p_n(z, x) - \sum_{i=1}^{p} \lambda_i^n\, \psi_i(z)\, \overline{\chi_i(x)}]\, dz \right| \leqslant$$

$$\left(\int_{\triangle} |K(x, z)|^2\, dz \right)^{1/2} \cdot \left(\int_{\triangle} |K_n(z, x) - p_n(z, x) - \sum_{i=1}^{p} \lambda_i^n\, \psi_i(z)\, \overline{\chi_i(x)}|^2 dz \right)^{1/2};$$

hence, on account of SCHWARZ's inequality and observing that $\int_{\triangle} \psi_i(x)\overline{\chi_i(x)}dx = 1$,

$$\left| \int_{\Delta} K_{n+1}^{s}(x, x)\, dx - \int_{\Delta} r_n(x, x)\, dx - \sum_{i=1}^{p} \lambda_i^{n+1} \right| =$$

$$\left| \int_{\Delta} \left\{ K_{n+1}(x, x) - r_n(x, x) - \sum_{i=1}^{p} \lambda_i^{n+1} \psi_i(x)\, \overline{\chi_i(x)} \right\} dx \right| \leqslant$$

$$\left(\int_{\Delta \times \Delta} |K(x, z)|^2\, dx\, dz \right)^{1/2} \cdot \left(\int_{\Delta \times \Delta} |K_n(z, x) - p_n(z, x) \right.$$

$$\left. - \sum_{i=1}^{p} \lambda_i^n\, \psi_i(z)\, \overline{\chi_i(x)} \right|^2 dx\, dz \Bigg)^{1/2}.$$

Since the last factor on the right tends to 0 as $p \to \infty$, we find (5). The relation (6) is proved in the same way, using that

$$K_{n+p-1}(x, y) \backsim \sum_i \lambda_i^{n+p-1}\, \psi_i(x)\, \overline{\chi_i(y)}.$$

The last two theorems may be applied to Marty-kernels $K(x, y)$ for which the corresponding transformation K is of the form $K = AH$, where A is self-adjoint (In I, Theorem 14 we have shown that every transformation of the form $K = AH$ is symmetrisable relative to H, and that $Hf = 0$ implies $Kf = 0$). Examples are:

1°. $K(x, y) = A(x)H(x, y)$, where $A(x)$ is measurable, bounded and real,

2°. $K(x, y) = \int_{\Delta} A(x, z)H(z, y)dz$, where $A(x, y) \in L_2^{(2m)}$ is Hermitean.

Theorem 11. *If $K = AH$, where A is bounded and self-adjoint, the series $\Sigma \lambda_i^2 \psi_i(x)\overline{\chi_i}(y)$ converges in mean; hence, by the preceding theorems,*

$$K_2(x, y) - p_2(x, y) \backsim \Sigma \lambda_i^2\, \psi_i(x)\, \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\int_{\Delta} H(x, z)\, p_2(z, y)\, dz = 0$$

almost everywhere in $\Delta \times \Delta$;

$$K_n(x, y) \backsim \Sigma \lambda_i^n\, \psi_i(x)\, \overline{\chi_i(y)} \qquad (n \geqslant 3); \quad . \quad . \quad . \quad . \quad (8)$$

$$\int_{\Delta} K_3(x, x)\, dx - \int_{\Delta} r_2(x, x)\, dx = \Sigma \lambda_i^3, \quad . \quad . \quad . \quad . \quad (9)$$

where

$$r_2(x, x) = \int_{\Delta} K(x, z)\, p_2(z, x)\, dz;$$

$$\int_{\Delta} K_n(x, x)\, dx = \Sigma \lambda_i^n \qquad (n \geqslant 4). \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Proof. Since the functions $\Psi_i(x) = H^{1/2}\psi_i$ are orthonormal, we have, for arbitrary complex numbers a_i,

$$\| \sum_{i=p}^{q} a_i \Psi_i(y) \|^2 = \sum_{i=p}^{q} |a_i|^2;$$

hence

$$\| \sum_{i=p}^{q} a_i \chi_i(y) \|^2 = \| H^{1/2} \sum_{i=p}^{q} a_i \Psi_i(y) \|^2 \leqslant \| H^{1/2} \|^2 \| \sum_{i=p}^{q} a_i \Psi_i(y) \|^2 = \| H^{1/2} \|^2 \sum_{i=p}^{q} |a_i|^2,$$

or, taking $a_i = a_i(x) = \lambda_i^2 \overline{\psi_i(x)}$, and integrating over x,

$$\int_{\Delta \times \Delta} | \sum_{i=p}^{q} \lambda_i^2 \psi_i(x) \overline{\chi_i(y)} |^2 \, dx \, dy \leqslant \| H^{1/2} \|^2 \sum_{i=p}^{q} \lambda_i^4 \int_{\Delta} |\psi_i(x)|^2 \, dx. \quad (11)$$

Now we have seen in the proof of Theorem 7 that

$$\Sigma \lambda_i^2 \| H\psi_i \|^2 = \Sigma \lambda_i^2 \int_{\Delta} |\chi_i(x)|^2 \, dx$$

converges; hence, since

$$\lambda_i^2 \| \psi_i \|^2 = \| \lambda_i \psi_i \|^2 = \| AH\psi_i \|^2 \leqslant \| A \|^2 \| H\psi_i \|^2,$$

we see that also

$$\Sigma \lambda_i^4 \int_{\Delta} |\psi_i(x)|^2 \, dx = \Sigma \lambda \| \psi_i \|^2$$

converges. It follows therefore from (11) that

$$\lim_{p, q \to \infty} \int_{\Delta \times \Delta} | \sum_{i=p}^{q} \lambda_i^2 \psi_i(x) \overline{\chi_i(y)} |^2 \, dx \, dy = 0,$$

which shows the convergence in mean of $\Sigma \lambda_i^2 \psi_i(x) \overline{\chi_i(y)}$.

In the case that the positive, self-adjoint transformation H has the property that the uniquely determined positive, self-adjoint transformation $H^{1/2}$ is of the form

$$H^{1/2} f = \int_{\Delta} H_{1/2}(x, y) f(y) \, dy, \quad (12)$$

where $H_{1/2}(x, y) \in L_2^{(2m)}$, and if also the transformation $Q = AH^{1/2}$ is expressible as

$$Qf = A H^{1/2} f = \int_{\Delta} Q(x, y) f(y) \, dy, \quad (13)$$

where $Q(x, y) \in L_2^{(2m)}$, we may prove more. Before doing this we mention that it is not difficult to prove that (12) holds if and only if $\Sigma \mu_i$ converges.

where $\mu_i (i = 1, 2, \ldots)$ is the sequence of characteristic values $\neq 0$ of $H(x, y)$ (compare II, Theorem 10).

Theorem 12. *If $K = AH$, where A is bounded and self-adjoint, and if the conditions (12) and (13) are satisfied, the series $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ converges in mean, hence, by Theorem 9,*

$$K(x, y) - p(x, y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where

$$\int_{\Delta} H(x, z)\, p(z, y)\, dz = 0$$

almost everywhere in $\Delta \times \Delta$;

$$K_n(x, y) \backsim \Sigma \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqq 2). \quad . \quad . \quad . \quad . \quad (15)$$

Furthermore

$$r(x, x) = \int_{\Delta} K(x, z)\, p(z, x\ \, dz = 0$$

for almost every $x \in \Delta$, so that by Theorem 10

$$\int_{\Delta} K_n(x, x)\, dx = \Sigma \lambda_i^n \qquad (n \geqq 2). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Proof. In the same way as we have found (11), we may find

$$\int_{\Delta \times \Delta} \Big| \sum_{i=p}^{q} \lambda_i \psi_i(x) \overline{\chi_i(y)} \Big|^2 dx\, dy \leqq \| H^{1/2} \|^2 \sum_{i=p}^{q} \lambda_i^2 \int_{\Delta} |\psi_i(x)|^2 dx. \quad . \quad (17)$$

Now we have, for almost every $x \in \Delta$,

$$\lambda_i \psi_i(x) = A H^{1/2} H^{1/2} \psi_i = A H^{1/2} \Psi_i = \int_{\Delta} Q(x, y)\, \Psi_i(y)\, dy;$$

hence, the system $\Psi_i(y)$ being orthonormal,

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqq \int_{\Delta} |Q(x, y)|^2 dy$$

on account of BESSEL's inequality, so that

$$\Sigma \lambda_i^2 \int_{\Delta} |\psi_i(x)|^2 dx \leqq \int_{\Delta \times \Delta} |Q(x, y)|^2 dx\, dy.$$

It follows therefore from (17) that

$$\lim_{p, q \to \infty} \int_{\Delta \times \Delta} \Big| \sum_{i=p}^{q} \lambda_i \psi_i(x) \overline{\chi_i(y)} \Big|^2 dx\, dy = 0.$$

which shows the convergence in mean of $\Sigma \lambda_i \psi_i(x)\, \overline{\chi_i(y)}$.

To prove the second part of the theorem, we observe that the relation

$$\int_{\Delta} H(x, z)\, p(z, y)\, dz = 0$$

implies

$$\int_{\Delta} H_{1/2}(x, z)\, p(z, y)\, dz = 0$$

for almost every point $(x, y) \epsilon \Delta \times \Delta$. On account of

$$r(x, x) = \int_{\Delta} K(x, z)\, p(z, x)\, dz = \int_{\Delta \times \Delta} Q(x, y)\, H_{1/2}(y, z)\, p(z, x)\, dy\, dz =$$

$$\int_{\Delta} Q(x, y) \left[\int_{\Delta} H_{1/2}(y, z)\, p(z, x)\, dz \right] dy,$$

we find then $r(x, x) = 0$ for almost every $x \epsilon \Delta$.

Mathematics. — *Over de oplossingen van de vergelijking* $\triangle^\nu u = 0$, *die aan zekere randvoorwaarden voldoen.* I. By O. BOTTEMA and H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1946.)

In een vroegere mededeeling [1]) heeft de tweede van ons de oplossing der vergelijking $\triangle\triangle u = 0$, waarbij, als $\triangle$ een symbool in twee veranderlijke voorstelt, in de punten van een gegeven gesloten kromme, of als $\triangle$ een symbool in drie veranderlijken voorstelt, in de punten van een gegeven gesloten oppervlak, u en $\frac{\partial u}{\partial n}$ gegeven waarden aannemen, behandeld met behulp van functies van GREEN, waarbij in het gebied binnen die kromme of binnen dat oppervlak die oplossing wordt voorgesteld door een integraal over die gegeven kromme of dat gegeven oppervlak en voor de gevallen, dat de gegeven kromme een cirkel of het gegeven oppervlak een bol is, de uitdrukking van die functies van GREEN gegeven en zoo voor u integralen afgeleid, die men kan opvatten als uitbreidingen van de uit de potentiaaltheorie bekende integraal van POISSON. Tevens werd het bewijs geleverd, dat deze integralen aan alle eischen van het gestelde vraagstuk voldoen.

Wij stellen ons thans voor, aan die uitkomsten in twee richtingen een uitbreiding te geven. In de eerste plaats willen we voor willekeurige waarden van ν de functie van GREEN H_ν, behoorende bij de vergelijking $\triangle^\nu u = 0$, zoodanig dat

$$2^{2\nu-1}(\nu-1)!^2\,\pi\,u = \int \left(u\,\frac{\partial \triangle^{\nu-1} H_\nu}{\partial n} - \frac{\partial u}{\partial n}\triangle^{\nu-1} H_\nu + \right. \\ \left. + \triangle u\,\frac{\partial \triangle^{\nu-2} H_\nu}{\partial n} \ldots \right) ds \Bigg\} \qquad (1)$$

de oplossing levert, waarbij aan de gegeven gesloten kromme $u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2} \ldots \frac{\partial^\nu u}{\partial n^\nu}$ gegeven waarden aannemen, opstellen voor het geval, dat die kromme een cirkel is. Opgemerkt zij nog, dat in de formule (1) voor het geval, dat ν een even getal is, $\nu = 2\mu$, de laatste term binnen de haakjes in de integrand in het tweede lid is $-\frac{\partial \triangle^{\mu-1} u}{\partial n}\triangle^\mu H_\nu$ en voor het geval, dat ν oneven is, $\nu = 2\mu + 1$, $\triangle^\mu u\,\frac{\partial \triangle^\mu H_\nu}{\partial n}$ en dat de formule onveranderd doorgaat voor het analoge probleem in drie afmetingen, als men de integraal in het tweede lid opvat als een integraal over het oppervlak, dat in dat

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[1]) H. BREMEKAMP, Over de oplossingen der vergelijking $\triangle\triangle u = 0$ die aan zekere randvoorwaarden voldoen. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 319 (1946).

Vgl. ook E. ALMANSI, Sull' integrazione dell' Equazione $\triangle^{2n} = 0$. Annali di Matematica, Serie IIa, Tomo II°, 1899.

geval de gegevens draagt. Voor het geval, dat dat oppervlak een bol is, willen we ook bij die opvatting van het symbool $\triangle$ de uitdrukking voor H_ν voor willekeurige waarden van ν opstellen.

In de tweede plaats willen we de analoge problemen behandelen voor het geval, dat $\triangle$ een symbool in k veranderlijken beteekent en de figuur, die de gegevens draagt, een hyperspheer is.

In al die gevallen is de functie $H_\nu(P, Q)$, waarbij P en Q willekeurige punten zijn binnen de figuur, die de gegevens draagt, door de volgende eischen gedefinieerd:

1. als $P \neq Q$, geldt $\triangle^\nu H_\nu = 0$.
2. als P gaat naar eenig punt van de figuur, die de gegevens draagt, gaan
$$H_\nu, \frac{\partial H_\nu}{\partial n}, \frac{\partial^2 H_\nu}{\partial n^2} \cdots \frac{\partial^\nu H_\nu}{\partial n^\nu}$$
naar nul,
3. als men stelt $H_\nu(P, Q) = \varphi_\nu(\varrho) + h_\nu(P, Q)$, waarbij ϱ den afstand PQ voorstelt en $u = \varphi_\nu(\varrho)$ een nog nader te bespreken oplossing der vergelijking $\triangle^\nu u = 0$ is, die niet aan een vergelijking van denzelfden vorm van lagere orde voldoet, dan heeft $h_\nu(P, Q)$ doorloopende partieele afgeleiden tot inclusief die van de orde 2ν en voldoet dus in het geheele gebied binnen de gegeven figuur aan $\triangle^\nu h_\nu = 0$.

De functie $H_\nu(P, Q)$ heeft een singulariteit voor $P = Q$, de functie wordt oneindig groot of althans haar afgeleiden van een bepaalde orde af. De aard van die singulariteit is bepaald door de functie $\varphi_\nu(\varrho)$. Als $\triangle$ het symbool in k onafhankelijk veranderlijken voorstelt, geldt voor een functie v, die alleen van ϱ afhangt $\triangle v = \frac{d^2 v}{d\varrho^2} + \frac{k-1}{\varrho}\frac{dv}{d\varrho}$, als dus $\triangle v = 0$, kunnen wij, voor $k > 2$, voor v nemen $v = \frac{C}{\varrho^{k-2}}$ en voor $k = 2$, $v = C \ln \varrho$. Hieruit leiden we af, dat we, als k oneven is, kunnen nemen $\varphi_\nu = \frac{1}{\varrho^{k-2\nu}}$ en de zelfde formule geldt, als k even is en $2\nu < k$, voor $2\nu = k$ wordt $\varphi_\nu = ln\, \varrho$, voor $2\nu > k$, $\varphi_\nu = \varrho^{2\nu-k} \ln \varrho$.

In verband met het logarithmisch karakter der singulariteit zijn verschillende problemen eenvoudiger voor een ruimte van een oneven aantal afmetingen, m.a.w. als $\triangle$ een symbool voor een oneven aantal onafhankelijk veranderlijken beteekent, dan voor een ruimte van een even aantal afmetingen. Wij zullen de functies H_3, H_4 enz. dan ook vooreerst bepalen in de ruimte van drie afmetingen voor het geval van den bol, vervolgens in het platte vlak voor het geval van den cirkel.

§ 1. De functie H_3 voor den bol (en evenzoo de verdere functies van GREEN) zullen we eenigszins probeerenderwijs opstellen. Wij steunen daarbij op de stelling [2]), dat de functie door de in de inleiding genoemde eischen

[2]) H. BREMEKAMP, Over de bepaaldheid der oplossingen van $\triangle^k u = 0$. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 222 (1945).

ondubbelzinnig bepaald is. Dat brengt mee, dat, wanneer we een functie verkregen hebben, van welke wij op de een of andere manier, b.v. eenvoudig door narekenen, kunnen vaststellen, dat zij aan die eischen voldoet, wij ook verzekerd zijn, dat dat *de* bedoelde functie van GREEN is. Verder maken wij gebruik van de eigenschap [3]), dat iedere functie u, die in zeker gebied aan de vergelijking $\triangle^3 u = 0$ voldoet, geschreven kan worden in de gedaante $u = u_0 + r^2 v_0$, waarbij in het beschouwde gebied $\triangle u_0 = 0$ en $\triangle^2 v_0 = 0$ en dat evenzoo iedere functie v_0, die in zeker gebied voldoet aan $\triangle^2 v_0 = 0$ geschreven kan worden in de gedaante $v_0 = u_1 + r^2 u_2$, waarbij in dat gebied $\triangle u_1 = \triangle u_2 = 0$; r beteekent hierbij den afstand tot een willekeurig punt; wij zullen deze stelling telkens toepassen voor het geval, dat voor dat punt de oorsprong van coördinaten wordt gekozen en dien leggen we in het middelpunt O van den bol. Ook geldt het omgekeerde van die stelling en daarop zullen we ons voor het bewijs, dat de opgestelde functie aan de eischen voldoet, moeten beroepen. Wij kunnen de stelling, die wij dan toepassen, als volgt formuleeren: „als $\triangle u_0 = \triangle u_1 = \triangle u_2 = 0$, en $u = u_0 + r^2 u_1 + r^4 u_2$, dan is $\triangle^3 u = 0$". Om nu de functie $H_3(P, Q)$ uit te drukken, noemen we $OP = r$, $OQ = a$, $\angle POQ = \vartheta$. Den straal van den bol noemen wij R. Wij bepalen het punt Q op de lijn OQ gelegen zoodanig, dat $OQ_1 = a_1 = \frac{R^2}{a}$. Den afstand PQ noemen we ϱ, den afstand PQ_1, ϱ_1. Voor punten P op den bol is, zooals welbekend, $a\varrho_1 = R\varrho$. Als oplossing van $\triangle u_2 = 0$, geldend binnen den bol, komt dan in de eerste plaats in aanmerking de functie $u_2 = \frac{1}{\varrho}$, als oplossing van $\triangle^2 v_1 = 0$, $v_1 = \varrho_1$, als oplossing van $\triangle^3 v = 0$, $v = \varrho_1^3$.

De bekende reciprociteitseigenschap der functie van GREEN $H_3(P, Q) = =: H_3(Q, P)$, die men bewijst met behulp van de formule (1), voert, als wij H_3 uitdrukken in r, a en ϑ, tot $H_3(r, a) = H_3(a, r)$. Ook daardoor zullen wij ons bij het opstellen der functie H_3 laten leiden. Wij merken daartoe nog op, dat ϱ in a en r symmetrisch is, want $\varrho^2 = a^2 + r^2 - 2ar\cos\vartheta$ en evenzoo $a\varrho_1$, want $\varrho_1^2 = a_1^2 + r^2 - 2a_1 r\cos\vartheta$ dus $a^2\varrho_1^2 = R^4 + a^2 r^2 - -2R^2 ar\cos\vartheta$.

De singulariteit der functie H_3 wordt gegeven door $\varphi_3 = \varrho^3$. Wij tellen hierbij vooreerst op $-\frac{a^3\varrho_1^3}{R^3}$, die binnen den bol voldoet aan $\triangle^3 u = 0$, en die maakt, dat $H_{3,1} = \varrho^3 - \frac{a^3\varrho_1^3}{R^3}$ aan den bol nul wordt. Vervolgens tellen wij een term op, die aan $\triangle^3 u = 0$ voldoet en aan den bol nul wordt en waarmee we willen bereiken, dat aan de voorwaarde $\frac{\partial H_3}{\partial r} = 0$ aan den bol voldaan wordt. Dat de bij te voegen term aan den bol nul wordt, bereiken we door dien een factor $R^2 - r^2$ te geven; om aan de symmetrie-eigenschap te blijven voldoen voegen we dan ook een factor $R^2 - a^2$ toe. De zoo

[3]) H. BREMEKAMP, Eigenschappen der oplossingen van $\triangle^k u = 0$. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 229 (1945).

ontstaande term zal aan $\triangle^3 u = 0$ voldoen, ook nog als we vermenigvuldigen met een factor, die aan $\triangle^2 u = 0$ voldoet, waarvoor we kiezen $a\varrho_1$, waardoor ook de symmetrie in a en r gehandhaafd blijft; om een homogenen veelterm te krijgen, voegen we nog een factor R^{-3} toe. De functie

$$H_{3,2} = \varrho^3 - \frac{a^3\varrho_1^3}{R^3} + C_1 \frac{(R^2-a^2)(R^2-r^2)}{R^3} a\varrho_1$$

voldoet dan aan $\triangle^3 H_{3,2} = 0$, aan den bol is $H_{3,2} = 0$ en wij trachten de constante C_1 zoo te bepalen, dat daar ook $\frac{\partial H_{3,2}}{\partial r} = 0$. Nu is

$$\varrho\frac{\partial\varrho}{\partial r} = r - a\cos\vartheta, \qquad \varrho\frac{\partial^2\varrho}{\partial r^2} + \left(\frac{\partial\varrho}{\partial r}\right)^2 = 1, \text{ dus}$$

$$\varrho^3\frac{\partial^2\varrho}{\partial r^2} = \varrho^2 - \varrho^2\left(\frac{\partial\varrho}{\partial r}\right)^2 = \varrho^2 - (r - a\cos\vartheta)^2 = a^2\sin^2\vartheta,$$

$$\varrho_1\frac{\partial\varrho_1}{\partial r} = r - a_1\cos\vartheta = \frac{1}{a}(ar - R^2\cos\vartheta), \qquad \varrho_1^3\frac{\partial^2\varrho_1}{\partial r^2} = \frac{R^4}{a^2}\sin^2\vartheta.$$

De voorwaarde $\frac{\partial H_{3,2}}{\partial r} = 0$ aan den bol geeft nu

$$3\varrho(R - a\cos\vartheta) - \frac{3a^2}{R^2}\varrho_1(a - R\cos\vartheta) = 2C_1\frac{(R^2-a^2)}{R^2}a\varrho_1, \qquad (2)$$

dus $C_1 = \frac{3}{2}$.

Om nu te bereiken, dat $\frac{\partial^2 H_3}{\partial r^2} = 0$ aan den bol, voegen we op gronden, geheel overeenkomende met die in de vorige alinea aangevoerd, nog een term toe $C_2\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^3}\frac{1}{a\varrho_1}$.

Wij vinden dan aan den bol

$$\left.\begin{aligned} &3\varrho^2\frac{\partial^2\varrho}{\partial r^2} + 6\varrho\left(\frac{\partial\varrho}{\partial r}\right)^2 - \frac{3a^3}{R^3}\varrho_1^2\frac{\partial^2\varrho_1}{\partial r^2} - \frac{6a^3}{R^3}\varrho_1\left(\frac{\partial\varrho_1}{\partial r}\right)^2 - \cdots \\ &- \frac{4C_1(R^2-a^2)}{R^2}a\frac{\partial\varrho_1}{\partial r} - \frac{2C_1(R^2-a^2)}{R^3}a\varrho_1 + 8C_2\frac{(R^2-a^2)^2}{R}\frac{1}{a\varrho_1} = 0, \end{aligned}\right\} \qquad (3)$$

waaruit wij vinden $C_2 = -\frac{3}{8}$.

De gezochte functie van GREEN wordt dus

$$H_3 = \varrho^3 - \frac{a^3}{R^3}\varrho_1^3 + \frac{3}{2}\frac{(R^2-a^2)(R^2-r^2)}{R^3}a\varrho_1 - \frac{3}{8}\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^3}\frac{1}{a\varrho_1}. \qquad (4)$$

Wanneer het alleen er om te doen is, de getallenwaarden der coëfficiënten C_1 en C_2 te vinden, kunnen wij volstaan met de vergelijking (3). Daarin bepalen we vooreerst C_1 zóó, dat het eerste lid deelbaar wordt door $(R^2 - a^2)^2$, waarna C_2 volgt evenals in het voorgaande. Voor het bewijs,

dat de zoo geconstrueerde functie aan alle eischen voldoet, moet dan echter toch nog geverifiëerd worden, dat aan den bol ook $\frac{\partial H_3}{\partial r} = 0$.

Wij zullen van de laatste opmerking gebruik maken bij het opstellen van de functie H_4 voor den bol. Op de zelfde manier als in het vorige komen we tot

$$H_4 = \varrho^5 - \frac{a^5}{R^5}\varrho_1^5 + C_1\frac{(R^2-a^2)(R^2-r^2)}{R^5}a^3\varrho_1^3 +$$

$$C_2\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^5}a\varrho_1 + C_3\frac{(R^2-a^2)^3(R^2-r^2)^3}{R^5}\frac{1}{a\varrho_1}.$$

Voor het berekenen van $\frac{\partial^3 H_4}{\partial r^3}$ hebben wij nog noodig de volgende uitkomsten:

$$\varrho^3\frac{\partial^2\varrho}{\partial r^2} = a^2\sin^2\vartheta, \qquad \varrho^3\frac{\partial^3\varrho}{\partial r^3} + 3\varrho^2\frac{\partial\varrho}{\partial r}\frac{\partial^2\varrho}{\partial r^2} = 0,$$

dus

$$\varrho^5\frac{\partial^3\varrho}{\partial r^3} = -3\varrho\frac{\partial\varrho}{\partial r}\varrho^3\frac{\partial^2\varrho}{\partial r^2}.$$

Dus aan den bol

$$\varrho^5\frac{\partial^3\varrho}{\partial r^3} = -3(R-a\cos\vartheta)a^2\sin^2\vartheta.$$

en

$$\varrho_1^5\frac{\partial^3\varrho_1}{\partial r^3} = -3\frac{R^5}{a^3}(a-R\cos\vartheta)\sin^2\vartheta.$$

Wij vinden dan aan het boloppervlak

$$\frac{\partial^3 H_4}{\partial r^3} = 45\varrho^3\frac{\partial\varrho}{\partial r}\frac{\partial^2\varrho}{\partial r^2} + 60\varrho^2\left(\frac{\partial\varrho}{\partial r}\right)^3 - 45\frac{a^5}{R^5}\varrho_1^3\frac{\partial\varrho_1}{\partial r}\frac{\partial^2\varrho_1}{\partial r^2} - 60\frac{a^5}{R^5}\varrho_1^2\left(\frac{\partial\varrho_1}{\partial r}\right)^3$$

$$-18C_1\frac{R^2-a^2}{R^4}a^3\left\{\varrho_1^2\frac{\partial^2\varrho_1}{\partial r^2} + 2\varrho_1\left(\frac{\partial\varrho_1}{\partial r}\right)^2\right\} - 18C_1\frac{R^2-a^2}{R^5}a^3\varrho_1^2\frac{\partial\varrho_1}{\partial r}$$

$$+24C_2\frac{(R^2-a^2)^2}{R^3}a\frac{\partial\varrho_1}{\partial r} + 24C_2\frac{(R^2-a^2)^2}{R^4}a\varrho_1 - 48C_3\frac{(R^2-a^2)^3}{R^2}\frac{1}{a\varrho_1} = 0.$$

Dat geeft

$$\frac{15(R^2-a^2)}{aR^2\varrho_1}\{4R^4 + a^2R^2(7+9\cos^2\vartheta) + 4a^4 - 12R^3a\cos\vartheta - 12Ra^3\cos\vartheta\}$$

$$-18C_1\frac{R^2-a^2}{aR^2\varrho_1}\{a^2R^2(2+3\cos^2\vartheta) + 3a^4 - R^3a\cos\vartheta - 7Ra^3\cos\vartheta\}$$

$$+24C_2\frac{(R^2-a^2)^2}{aR^2\varrho_1}(R^2-3aR\cos\vartheta + 2a^2) - 48C_3\frac{(R^2-a^2)^3}{aR^2\varrho_1} = 0.$$

Uit het feit, dat de som van de eerste twee termen door $(R^2 - a^2)^2$ deelbaar moet zijn, vindt men $C_1 = \frac{5}{2}$, vervolgens $C_2 = -\frac{15}{8}$ en $C_3 = \frac{5}{16}$ dus

$$H_4 = \varrho^5 - \frac{a^5}{R^5}\varrho_1^5 + \frac{5}{2}\frac{(R^2-a^2)(R^2-r^2)}{R^5}a^3\varrho_1^3 - \left.\begin{aligned}&\\ &\end{aligned}\right\}$$
$$\frac{15}{8}\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^5}a\varrho_1 + \frac{5}{16}\frac{(R^2-a^2)^3(R^2-r^2)^3}{R^5}\frac{1}{a\varrho_1}. \left.\begin{aligned}&\\ &\end{aligned}\right\} \quad (5)$$

Deze functie voldoet inderdaad aan alle gestelde eischen.

§ 2. Algemeen zal men voor den bol vinden

$$H_\nu = \varrho^{2\nu-3} - \frac{a^{2\nu-3}}{R^{2\nu-3}}\varrho_1^{2\nu-3} + C_1\frac{(R^2-a^2)(R^2-r^2)}{R^{2\nu-3}}a^{2\nu-5}\varrho_1^{2\nu-5} + \left.\begin{aligned}&\\ &\end{aligned}\right\}$$
$$+C_2\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^{2\nu-3}}a^{2\nu-7}\varrho_1^{2\nu-7} + \ldots + C_{\nu-2}\frac{(R^2-a^2)^{\nu-2}(R^2-r^2)^{\nu-2}}{R^{2\nu-3}}a\varrho_1 + \left.\begin{aligned}&\\ &\end{aligned}\right\} \quad (6)$$
$$+ C_{\nu-1}\frac{(R^2-a^2)^{\nu-1}(R^2-r^2)^{\nu-1}}{R^{2\nu-3}}\frac{1}{a\varrho_1}, \left.\begin{aligned}&\\ &\end{aligned}\right\}$$

waarbij de constanten $C_i (i = 1, \ldots \nu - 1)$ nog nader moeten worden bepaald. Staat eenmaal vast, dat de gevraagde functie inderdaad de gedaante (6) heeft, dan kunnen deze constanten als volgt worden berekend.

Nadert a tot nul, dan nadert ϱ_1 tot oneindig, met dien verstande, dat $a\varrho_1$ nadert tot de eindige limiet R^2, zooals uit de betrekking

$$a^2\varrho_1^2 = R^4 + a^2r^2 - 2R^2ar\cos\vartheta$$

dadelijk blijkt. Een uitdrukking $a^p\varrho_1^q$ nadert dus voor $a \to 0$ tot nul, als $p > q$ is.

Wij beschouwen nu $\frac{\partial^p H_\nu}{\partial r^p}$ op den bol; voor $p = 1, \ldots \nu - 1$ moet deze uitdrukking gelijk aan nul zijn. In de aldus verkregen vergelijkingen voor C_i stellen wij $a = 0$. Denken wij ons den term

$$\frac{d^p}{dr^p}(R^2-r^2)^q a^m \varrho_1^m,$$

(waarbij $2q + m = 2\nu - 3$ is) ontwikkeld volgens het theorema van LEIBNIZ, dan is hij na den limietovergang $a \to 0$ blijkbaar gelijk aan de uitdrukking $R^{2m}\frac{d^p}{dr^p}(R^2 - r^2)^q$. Hierin is dan $r = R$ te nemen. Nu is $\frac{d^p}{dr^p}(R^2 - r^2)^q$ (voor $r = R$) gelijk aan $a_{p,q}\, R^{2q-p}$, waarin $a_{p,q}$ een constante voorstelt, die blijkbaar voor $p < q$ en voor $p > 2q$ gelijk aan

nul is. Men heeft nu

$$\frac{d^p}{dr^p}(R^2-r^2)^{q+1} = (R^2-r^2)\frac{d^p}{dr^p}(R^2-r^2)^q - \\ -2pr\frac{d^{p-1}}{dr^{p-1}}(R^2-r^2)^q - p(p-1)\frac{d^{p-2}}{dr^{p-2}}(R^2-r^2)^q,$$

waaruit voor $r = R$ volgt

$$a_{p,q+1} = -2p\,a_{p-1,q} - p(p-1)\,a_{p-2,q}. \qquad (7)$$

Voor $p = q+1$ vinden wij dus

$$a_{q+1,q+1} = -2(q+1)\,a_{q,q}.$$

en daar $a_{0,0} = 1$, hebben wij

$$a_{q,q} = (-1)^q\,2^q\,q!$$

Neemt men nu in de recursieformule (7) $p = q+2$, dan vindt men een recurrente betrekking tusschen $a_{q+2,\,q+1}$ en $a_{q+1,\,q}$, die in staat stelt de constanten van de gedaante $a_{m+1,\,m}$ achtereenvolgens uitgaande van de beginwaarde $a_{2,2}$ te berekenen. Zoo voortgaande blijkt dat door (7) de getallen $a_{p,\,q}$ geheel bepaald zijn. Slagen wij er dus in een stelsel getallen te vinden, dat voor elke waarde van p en q aan (7) voldoet, terwijl $a_{0,0} = 1$ is, dan zijn daarmee de gevraagde constanten verkregen. Het blijkt nu, dat de getallen

$$a_{p,q} = (-1)^q\,p!\,2^{2q-p}\,\frac{q!}{(p-q)!\,(2q-p)!} \qquad (8)$$

aan (7) voldoen, zooals substitutie doet zien.

De randvoorwaarde $\frac{\partial^p H_\nu}{\partial r^p} = 0$, op de beschreven wijze toegepast, voert tot de volgende vergelijking voor de constanten C_i van (6):

$$(2\nu-3)(2\nu-4)\ldots(2\nu-p-2) + \sum_{i=\left[\frac{p+1}{2}\right]}^{i=p} C_i\,a_{p,i} = 0, \qquad (9)$$

waarin $\left[\frac{p}{2}\right]$ het grootste geheele getal voorstelt dat $\leqq \frac{p}{2}$ is.

Uit de vergelijkingen voor $p = 1, 2, \ldots (\nu-1)$, kan men achtereenvolgens $C_1, C_2, \ldots C_{\nu-1}$ bepalen. Men vindt

$$C_1 = \frac{2\nu-3}{2},\quad C_2 = -\frac{(2\nu-3)(2\nu-5)}{8},\quad C_3 = \frac{(2\nu-3)(2\nu-5)(2\nu-7)}{48}\ldots,$$

zoodat vermoed kan worden dat men heeft:

$$C_i = (-1)^{i+1}\frac{(2\nu-3)(2\nu-5)\ldots(2\nu-2i-1)}{2^i\,i!} = (-1)^{i+1}\binom{\nu-\frac{3}{2}}{i}. \qquad (10)$$

Dat vermoeden wordt als volgt bewezen.

Substitutie van deze uitdrukkingen in (9) geeft voor even waarden van p

$$\left.\begin{aligned}&(2\nu-3)(2\nu-4)\ldots(2\nu-p-2)=\\&=\sum_{i=\frac{p}{2}}^{i=p}2^{i-p}\,\frac{p(p-1)\ldots(p-i+1).(2\nu-3)(2\nu-5)\ldots(2\nu-2i-1)}{(2i-p)!}\end{aligned}\right\}\quad(11)$$

en wij zullen bewijzen, dat dit een identiteit is. Daartoe merken wij op, dat het rechterlid geschreven kan worden, als (wij stellen nog $p=2m$)

$$\sum_{i=m}^{i=2m}2^{i-2m}\,\frac{2m(2m-1)\ldots(2m-i+1).(2\nu-3)(2\nu-5)\ldots(2\nu-2i-1)}{(2i-2m)!}=$$

$$=\frac{1}{2m}\,2m(2m-1)\ldots(m+1).(2\nu-3)(2\nu-5)\ldots(2\nu-2m-1)$$

$$\times\sum_{r=0}^{m}2^{r}\,\frac{m!(2\nu-2m-3)\ldots(2\nu-2m-2r-1)}{(2r)!\,(m-r)!}$$

Voor de laatste som kan men schrijven, als nog $2\nu-2m-3=n$ wordt gesteld:

$$1+2\frac{m.n}{2!}+2^2\frac{m(m-1).n(n-2)}{4!}+\ldots$$

$$\ldots+2^m\,\frac{m(m-1)\ldots2.1.n(n-2)\ldots(n-2m+2)}{(2m)!}=$$

$$=1+\frac{m.\frac{n}{2}}{1.\frac{1}{2}}+\frac{m(m-1).\frac{n}{2}\left(\frac{n}{2}-1\right)}{1.2.\frac{1}{2}.\frac{3}{2}}+\ldots$$

$$\ldots+\frac{m(m-1)\ldots2.1.\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\left(\frac{n}{2}-m+1\right)}{1.2\ldots m.\frac{1}{2}.\frac{3}{2}\ldots\frac{2m-1}{2}}$$

$$=F\left(-m,-\frac{n}{2},\tfrac{1}{2},1\right).$$

Nu is volgens een bekende formule uit de theorie der hypergeometrische functie

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma).\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

voor $\gamma>\alpha+\beta$. Wij hebben dus

$$F\left(-m,-\frac{n}{2},\tfrac{1}{2},1\right)=\frac{\Gamma(\tfrac{1}{2}).\Gamma\left(m+\frac{n}{2}+\tfrac{1}{2}\right)}{\Gamma(m+\tfrac{1}{2}).\Gamma\left(\frac{n}{2}+\tfrac{1}{2}\right)}=\frac{\Gamma(\tfrac{1}{2}).\Gamma(\nu-1)}{\Gamma(m+\tfrac{1}{2}).\Gamma(\nu-m-1)}=$$

$$=\frac{(\nu-2)!}{(m-\frac{1}{2})(m-\frac{3}{2})\ldots\frac{3}{2}.\frac{1}{2}.(\nu-m-2)!}=\frac{2^m(\nu-2)(\nu-3)\ldots(\nu-m-1)}{(2m-1)(2m-3)\ldots3.1}$$

waaruit volgt:

$$\frac{1}{2^m}\,2m(2m-1)\ldots(m+1).(2\nu-3)(2\nu-5)\ldots(2\nu-2m-1)\,F\left(-m,-\frac{n}{2},\tfrac{1}{2},1\right)=$$

$$=\frac{(2\nu-3)(2\nu-5)\ldots(2\nu-2m-1).(2\nu-4)(2\nu-6)\ldots(2\nu-2m-2)(2m)!}{2^m(2m-1)(2m-3)\ldots3.1.m!}$$

$$=(2\nu-3)(2\nu-4)\ldots(2\nu-2m-2),$$

waarmee de identiteit (11) en daarmee ook de juistheid van (10) bewezen is. Voor oneven p verloopt het bewijs analoog.

Wij hebben daarmee voor de uitdrukking (6) de coëfficiënten bepaald en vinden dus voor *de vergelijking* $\Delta^\nu\varphi = 0$ *en de in* § 1 *genoemde randvoorwaarden bij den bol de volgende functie van* GREEN:

$$H_\nu = \varrho^{2\nu-3} + \frac{1}{R^{2\nu-3}}\sum_{m=0}^{m=\nu-1}(-1)^{m+1}\binom{\nu-\frac{3}{2}}{m}R^2-a^2)^m(R^2-r^2)^m(a\varrho_1)^{2\nu-2m-3}. \quad (12)$$

§ 3. Voor het geval, dat Δ een symbool in twee onafhankelijk veranderlijken is, wordt de singulariteit der functie H_3 gegeven door $\varphi_3 = \varrho^4 \ln \varrho$. Wij zullen hiermee op een wijze analoog aan die, welke wij bij den bol gevolgd hebben, de functie H_3 voor den cirkel opstellen. Daar

$$\frac{\partial}{\partial r}\varrho^4 \ln\varrho = \varrho^3(4\ln\varrho+1)\frac{\partial\varrho}{\partial r} = \varrho^2(4\ln\varrho+1)(r-a\cos\vartheta)$$

moeten wij, om aan $\frac{\partial H_3}{\partial r} = 0$ te voldoen twee termen toevoegen, n.l. $\varrho^2(A_1\ln\varrho + B_1)$, zoo komen wij door weer den zelfden gedachtengang te volgen als in § 1 tot

$$H_3 = \varrho^4 \ln\varrho\frac{R}{a} - \frac{a^4}{R^4}\varrho_1^4\ln\varrho_1 +$$

$$+\frac{(R^2-a^2)(R^2-r^2)}{R^4}a^2\varrho_1^2(A_1\ln\varrho_1+B_1)+\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^4}(A_2\ln\varrho_1+B_2).$$

Een verschil met het voorgaande ligt nog daarin, dat nu niet onmiddellijk is te zien, dat aan de reciprociteitsvoorwaarde voldaan is. Deze geeft ons nu op betrekkelijk eenvoudige wijze de coëfficiënten A_1 en A_2. Wij hebben namelijk

$$H_3(r_1,a)-H_3(a_1,r) = \varrho^4\ln\frac{r}{a} - \frac{a^4}{R^4}\varrho_1^4\ln\frac{r}{a} +$$

$$+A_1\frac{(R^2-a^2)(R^2-r^2)}{R^4}a^2\varrho_1^2\ln\frac{r}{a}+A_2\frac{(R^2-a^2)^2(R^2-r^2)^2}{R^4}\ln\frac{r}{a}$$

dus

$$0 = R^4\varrho^4 - a^4\varrho_1^4 + A_1(R^2-a^2)(R^2-r^2)a_2\varrho_1^2 + A_2(R^2-a^2)^2(R^2-r^2)^2,$$

en daar

$$R^4\varrho^4 - a^4\varrho_1^4 = (R^2\varrho^2 + a^2\varrho_1^2)(R^2\varrho^2 - a^2\varrho_1^2) = -(R^2\varrho^2 + a^2\varrho_1^2)(R^2 - r^2)(R^2 - a^2),$$

$$R^2\varrho^2 + a^2\varrho_1^2 - A_1 a^2\varrho_1^2 \equiv A_2 (R^2 - a^2)(R^2 - r^2),$$

waaruit $A_1 = 2$, $A_2 = -1$.

De coëfficiënten B_1 en B_2 berekenen wij nu met behulp van de voorwaarde, dat aan den cirkel $\frac{\partial^2 H_3}{\partial r^2} = 0$. Wij vinden aan den cirkel

$$\frac{\partial^2 H_3}{\partial r^2} = \left(4 \ln \varrho \frac{R}{a} + 1\right) a^2 \sin^2 \vartheta + \left(12 \ln \varrho \frac{R}{a} + 7\right)(R - a\cos\vartheta)^2 -$$

$$- (4 \ln \varrho_1 + 1)\, a^2 \sin^2 \vartheta - (12 \ln \varrho_1 + 7)(a - R\cos\vartheta)^2 -$$

$$- 4 \frac{R^2 - a^2}{R^3} a^2 (2A_1 \ln \varrho_1 + 2B_1 + A_1)\varrho_1 \frac{\partial \varrho_1}{\partial r} -$$

$$- 2 \frac{R^2 - a^2}{R^4} a^2 \varrho_1^2 (A_1 \ln \varrho_1 + B_1) + \frac{8(R^2 - a^2)^2}{R^2} (A_2 \ln \varrho_1 + B_2) = 0.$$

De eerste en derde term vallen hierin tegen elkaar weg. Verder vallen, als wij de voor A_1 en A_2 gevonden waarden invoeren, de termen met $\ln \varrho_1$ weg en wij houden over

$$\frac{7}{R^2}\{R^4 - a^4 - 2aR\cos\vartheta (R^2 - a^2)\} - \frac{4}{R^2}(R^2 - a^2)(a^2 - aR\cos\vartheta)(2B_1 + A_1) -$$

$$- \frac{2}{R^2}(R^2 - a^2)(R^2 + a^2 - 2aR\cos\vartheta) B_1 + \frac{8}{R^2}(R^2 - a^2)^2 B_2 = 0.$$

dus

$$7(R^2 + a^2 - 2aR\cos\vartheta) - 8B_1 (a^2 - aR\cos\vartheta) - 8(a^2 - aR\cos\vartheta) -$$

$$- 2B_1 (R^2 + a^2 - 2aR\cos\vartheta) + 8(R^2 - a^2) B_2 = 0,$$

waaruit $B_1 = \frac{1}{2}$, $B_2 = -\frac{3}{4}$.
en dus

$$H_3 = \varrho^4 \ln \varrho \frac{R}{a} - \frac{a^4}{R^4}\varrho_1^4 \ln \varrho_1 + \frac{(R^2 - a^2)(R^2 - r^2)}{2R^4} a^2 \varrho_1^2 (4 \ln \varrho_1 + 1) + \left.\right\}$$

$$+ \frac{(R^2 - a^2)^2 (R^2 - r^2)^2}{4R^4} (4 \ln \varrho_1 + 3). \left.\right\} \quad (13)$$

Men verifieert ook nu weer gemakkelijk, dat deze functie aan alle gestelde eischen voldoet.

Op dezelfde wijze vindt men

$$H_4 = \varrho^6 \ln \varrho \frac{R}{a} - \frac{a^6}{R^6}\varrho_1^6 \ln \varrho_1 + \frac{(R^2 - a^2)(R^2 - r^2)}{2R^6} a^4 \varrho_1^4 (6 \ln \varrho_1 + 1) - \left.\right\}$$

$$- \frac{(R^2 - a^2)^2 (R^2 - r^2)^2}{4R^6} a^2 \varrho_1^2 (12 \ln \varrho_1 + 5) + \frac{(R^2 - a^2)^3 (R^2 - r^2)^3}{12R^6} (12 \ln \varrho_1 + 11). \left.\right\} \quad (14)$$

en kunnen ook in de algemeene uitdrukking

$$\left.\begin{aligned} H_\nu = \varrho^{2\nu-2} \ln \varrho \frac{R}{a} - \frac{a^{2\nu-2}}{R^{2\nu-2}} \varrho_1^{2\nu-2} \ln \varrho_1 + \\ + \frac{(R^2-a^2)(R^2-r^2)}{R^{2\nu-2}} a^{2\nu-4} \varrho_1^{2\nu-4} (A_1 \ln \varrho_1 + B_1) + \\ + \frac{(R^2-a^2)^2 (R^2-r^2)^2}{R^{2\nu-2}} a^{2\nu-6} \varrho^{2\nu-6} (A_2 \ln \varrho_1 + B_2) + \ldots + \\ + \frac{(R^2-a^2)^{\nu-1} (R^2-r^2)^{\nu-1}}{R^{2\nu-2}} (A_{\nu-1} \ln \varrho_1 + B_{\nu-1}) \ldots \end{aligned}\right\} . \quad (15)$$

de coëfficiënten A en B gevonden worden.

Ook kan men hier weer dezelfde methode volgen als in § 2 voor den bol werd toegepast. Stelt men aan den cirkel $\frac{\partial H_\nu}{\partial r}$ gelijk aan nul en voert men daarna den limietovergang $a \to 0$ uit, dan ontstaat de volgende vergelijking voor de onbepaalde coëfficiënten A en B:

$$\left.\begin{aligned} \left[\frac{d^p}{dr^p} \varrho^{2\nu-2} \ln \varrho \frac{R}{a}\right]_{r=R} + \\ + R^{2\nu-p-2} \sum_{i=1}^{i=\nu-1} a_{p,i} (A_i \ln \varrho_1 + B_i) = 0 \quad (p = 1, \ldots \nu-1) . \end{aligned}\right\} . \quad (16)$$

Stelt men in deze betrekkingen den cofactor van $\ln \varrho_1$ gelijk aan nul, dan ontstaat een rij betrekkingen voor A_i. De overige termen geven voorwaarden voor B_i. De eerste luiden

$$(2\nu-2)(2\nu-3) \ldots (2\nu-p-1) + \sum_{i=1}^{\nu-1} a_{p,i} A_i = 0 \quad (17)$$

en zijn geheel analoog met (9). Men vindt zoo het resultaat

$$A_i = (-1)^{i+1} \binom{\nu-1}{i} . \quad (18)$$

De betrekkingen voor B_i luiden

$$\sum_{s=1}^{p} (-1)^{s+1} \frac{p!}{(p-s)!\, s} \frac{(2\nu-2)_i}{(2\nu-2-p+s)!} + \sum_{i=1}^{\nu-1} a_{p,i} B_i = 0$$

of

$$\sum_{s=1}^{p} \frac{(-1)^{s+1}}{s} \binom{2\nu-2}{p-s} + \sum_{i=1}^{\nu-1} \frac{a_{p,i}}{p!} B_i = 0. \qquad (p = 1, \ldots \nu-1)$$

en door toepassing der formule (8)

$$\sum_{s=1}^{p} \frac{(-1)^{s+1}}{s} \binom{2\nu-2}{p-s} + \sum_{i=1}^{\nu-1} (-1)^i 2^{2i-p} \binom{i}{p-i} B_i = 0 \quad p=1,2,\ldots\nu-1). \quad (19)$$

Om hieruit B_i te bepalen merken we vooreerst op, dat de eerste term den coëfficiënt voorstelt van x^p in de machtreeksontwikkeling van

$$(1+x)^{2\nu-2} \ln (1+x).$$

Definiëeren wij nu B_i voor $i > \nu - 1$ door de formule (19) ook voor $p > \nu - 1$ geldig te verklaren, dan volgt hieruit als $|x|$ klein genoeg is

$$(1+x)^{2\nu-2}\, ln\,(1+x) = x\,.\,2B_1 + x^2(B_1 - 2B_2) + x^3\left\{-2\binom{2}{1}B_2 + 2^3 B_3\right\} +$$

$$+ x^4\left\{-B_2 + 2^2\binom{3}{1}B_3 - 2^4 B_4\right\} + x^5\left\{2\binom{3}{1}B_3 - 2^3\binom{4}{1}B_4 + 2^5 B_5\right\} +$$

$$+ x^6\left\{B_3 - 2^2\binom{4}{2}B_4 + 2^4\binom{5}{1}B_5 - 2^6 B_6\right\} + \ldots$$

$$= B_1 x\,.(x+2) - B_2 x^2(x+2)^2 + B_3 x^3(x+2)^3 - B_4(x^4(x+2)^4 + \ldots$$

of als we stellen $x^2 + 2x = y$, dus $(x+1)^2 = y + 1$,

$$\tfrac{1}{2}(1+y)^{\nu-1}\, ln\,(1+y) = B_1 y - B_2 y^2 + B_3 y^3 - \ldots,$$

waaruit

$$B_1 = \tfrac{1}{2},\ B_2 = -\frac{2\nu-3}{2\,.\,2!},\ \ldots B_i = \tfrac{1}{2}\sum_{k=0}^{i-1}\frac{(-1)^k}{i-k}\binom{\nu-1}{k} - \frac{(-1)^i}{2}\sum_{j=1}^{i}\frac{(-1)^j}{j}\binom{\nu-1}{i-j},$$

welke formule wij slechts hebben toe te passen voor $1 \leqq i \leqq \nu - 1$.

Wij merken nog op, dat $2B_i$ den coëfficiënt voorstelt van x^i in de machtreeksontwikkeling van $(1-x)^{\nu-1}\ \ln\ (1-x)$.

Mathematics. — *Over de oplossingen van de vergelijking* $\triangle^r u = 0$, *die aan zekere randvoorwaarden voldoen.* II. By O. BOTTEMA and H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1946.)

§ 4. Door de gevonden uitdrukkingen in de formule (1) te substitueeren vindt men uitkomsten, die als analoga van de uit de potentiaaltheorie bekende integraal van POISSON zijn te beschouwen. Wij zullen dit uitvoeren voor de functie H_3 voor den cirkel. Om de functie voor te stellen, die binnen den cirkel voldoet aan $\triangle^3 u = 0$ en waarbij aan den cirkel $u = f(\varphi)$, $\frac{\partial u}{\partial n} = g(\varphi)$, $\frac{\partial^2 u}{\partial n^2} = h(\varphi)$, waarin f, g en h gegeven functies zijn van den poolhoek φ, die de plaats op den cirkel bepaalt, zullen we op de formule (1) eerst nog een kleine transformatie toepassen. Wij hebben aan den cirkel

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = h(\varphi) + \frac{1}{R} g(\varphi) + \frac{1}{R^2}\frac{d^2 f}{d\varphi^2}.$$

De formule (1) gaat daardoor voor dit geval over in

$$u = \frac{R}{16\pi}\int_0^{2\pi}\left[f(\varphi)\frac{\partial \triangle^2 H_3}{\partial r} - g(\varphi)\triangle^2 H_3 + \right.$$

$$\left. + \left\{ h(\varphi) + \frac{1}{R} g(\varphi) + \frac{1}{R^2}\frac{d^2 f}{d\varphi^2}\right\}\frac{\partial \triangle H_3}{\partial r}\right] d\varphi,$$

wat wij door partiëel integreeren van den term met $\frac{d^2 f}{d\varphi^2}$ nog herleiden tot

$$u = \frac{R}{16\pi}\int_0^{2\pi}\left[h(\varphi)\frac{\partial \triangle H_3}{\partial r} - g(\varphi)\left\{\triangle^2 H_3 - \frac{1}{R}\frac{\partial \triangle H_3}{\partial r}\right\} + \right.$$

$$\left. + f(\varphi)\left\{\frac{\partial \triangle^2 H_3}{\partial r} + \frac{1}{R^2}\frac{\partial^2 \triangle H_3}{\partial \varphi^2 \partial r}\right\}\right] d\varphi.$$

Door het invoeren der gevonden uitdrukking voor H_3 vindt men hieruit, na vrij omvangrijk rekenwerk,

$$u = \frac{(R^2-a^2)^3}{16\pi R^4}\int_0^{2\pi}\left[h(\varphi)\frac{R^2}{R^2+a^2-2Ra\cos\varphi} - g(\varphi)\frac{(5R^2+a^2-6Ra\cos\varphi)R}{(R^2+a^2-2Ra\cos\varphi)^2} + \right.$$

$$\left. + f(\varphi)\frac{8R^4-2Ra(9R^2+a^2)\cos\varphi + 12R^2a^2\cos^2\varphi}{(R^2+a^2-2Ra\cos\varphi)^3}\right] d\varphi. \qquad (20)$$

Eenvoudiger vindt men deze uitkomst langs den volgenden weg, waarbij wij echter omtrent de functies f, g en h verschillende onderstellingen invoeren, die blijkens het vorige voor de geldigheid der uitkomst niet noodig zijn. Wij zullen namelijk aannemen, dat deze functies een Fourierontwikkeling toelaten, waarbij de coëfficiënten bij toenemend rangnummer voldoende snel afnemen om de verschillende in het volgende toegepaste omzettingen te rechtvaardigen.

Wij stellen

$$u = r^4 u_2 + r^2 u_1 + u_0 = a_0 r^4 + b_0 r^2 R^2 + c_0 R^4 + \\ + \sum_1^\infty (a_{n,1} r^4 + b_{n,1} r^2 R^2 + c_{n,1} R^4) r^n \cos n\vartheta + \\ + \sum_1^\infty (a_{n,2} r^4 + b_{n,2} r^2 R^2 + c_{n,2} R^4) r^n \sin n\vartheta.$$

De voorwaarden aan den cirkel geven dan

$$f(\vartheta) = (a_0 + b_0 + c_0) R^4 + \sum_1^\infty (a_{n,1} + b_{n,1} + c_{n,1}) R^{n+4} \cos n\vartheta + \\ + \sum_1^\infty (a_{n,2} + b_{n,2} + c_{n,2}) R^{n+4} \sin n\vartheta.$$

$$g(\vartheta) = (4a_0 + 2b_0) R^3 + \sum_1^\infty \{(n+4) a_{n,1} + (n+2) b_{n,1} + n c_{n,1}\} R^{n+3} \cos n\vartheta + \\ + \sum_1^\infty \{(n+4) a_{n,2} + (n+2) b_{n,2} + n c_{n,2}\} R^{n+3} \sin n\vartheta.$$

$$h(\vartheta) = (12 a_0 + 2 b_0) R^2 + \sum_1^\infty \{(n+4)(n+3) a_{n,1} + \\ + (n+2)(n+1) b_{n,1} + n(n-1) c_{n,1}\} R^{n+2} \cos n\vartheta + \\ + \sum_1^\infty \{(n+4)(n+3) a_{n,2} + (n+2)(n+1) b_{n,2} + n(n-1) c_{n,2}\} R^{n+2} \sin n\vartheta.$$

Hieruit volgt vooreerst

$$(a_0 + b_0 + c_0) R^4 = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi)\, d\varphi, \quad (4a_0 + 2b_0) R^3 = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\, d\varphi,$$

$$(12 a_0 + 2 b_0) R^2 = \frac{1}{2\pi} \int_0^{2\pi} h(\varphi)\, d\varphi$$

waaruit

$$a_0 = \frac{1}{16\pi R^4} \int_0^{2\pi} \{R^2 h(\varphi) - R g(\varphi)\}\, d\varphi, \quad b_0 = \frac{1}{16\pi R^4} \int_0^{2\pi} \{-2R^2 h(\varphi) + 6R g(\varphi)\}\, d\varphi,$$

$$c_0 = \frac{1}{16\pi R^4} \int_0^{2\pi} \{R^2 h(\varphi) - 5R g(\varphi) + 8 f(\varphi)\}\, d\varphi.$$

Verder

$$a_{n,1} + b_{n,1} + c_{n,1} = \frac{1}{\pi R^{n+4}} \int_0^{2\pi} f(\varphi) \cos n\varphi\, d\varphi.$$

$$(n+4) a_{n,1} + (n+2) b_{n,1} + n c_{n,1} = \frac{1}{\pi R^{n+3}} \int_0^{2\pi} g(\varphi) \cos n\varphi\, d\varphi.$$

$$(n+4)(n+3)\,a_{n,1}+(n+2)(n+1)\,b_{n,1}+n(n-1)\,c_{n,1}=$$

$$=\frac{1}{\pi R^{n+2}}\int_0^{2\pi} h(\varphi)\cos n\varphi\,d\varphi,$$

waaruit

$$a_{n,1}=\frac{1}{8\pi R^{n+4}}\int_0^{2\pi}\{n(n+2)f(\varphi)-(2n+1)Rg(\varphi)+R^2h(\varphi)\}\cos n\varphi\,d\varphi,$$

$$b_{n,1}=\frac{1}{8\pi R^{n+4}}\int_0^{2\pi}\{-2n(n+4)f(\varphi)+2(2n+3)Rg(\varphi)-2R^2h(\varphi)\}\cos n\varphi\,d\varphi,$$

$$c_{n,1}=\frac{1}{8\pi R^{n+4}}\int_0^{2\pi}\{(n+2)(n+4)f(\varphi)-(2n+5)Rg(\varphi)+R^2h(\varphi)\}\cos n\varphi\,d\varphi.$$

Wij vinden een dergelijk stel formules voor $a_{n,2}$, $b_{n,2}$, $c_{n,2}$ door $\cos n\varphi$ te vervangen door $\sin n\varphi$. Substitutie van de gevonden waarden der coëfficiënten in de reeks voor u geeft, na omzetting van sommatie en integratie

$$u=\frac{1}{8\pi}\int_0^{2\pi}(\{\tfrac{1}{2}(R^2h-Rg)\frac{r^4}{R^4}+\tfrac{1}{2}(-2R^2h+6Rg)\frac{r^2}{R^2}+\tfrac{1}{2}(R^2h-5Rg+8f)\}+$$

$$+\sum_1^\infty\Big[\{R^2h-(2n+1)Rg+n(n+2)f\}\frac{r^{n+4}}{R^{n+4}}+$$

$$+\{(-2R^2h+2(2n+3)Rg-2n(n+4)f\}\frac{r^{n+2}}{R^{n+2}}$$

$$+\{R^2h-(2n+5)Rg+(n+2)(n+4)f\}\frac{r^n}{R^n}\Big]\cos n(\varphi-\vartheta))\,d\varphi$$

$$=\frac{1}{8\pi}\int_0^{2\pi}R^2h\cdot\frac{(R^2-r^2)^2}{R^4}\Big\{\tfrac{1}{2}+\sum_1^\infty\frac{r^n}{R^n}\cos n(\varphi-\vartheta)\Big\}d\varphi+$$

$$+\frac{1}{8\pi}\int_0^{2\pi}Rg\cdot\Big\{\frac{-5R^4+6R^2r^2-r^4}{2R^4}+$$

$$+\sum_1^\infty\frac{-(2n+5)R^4+2(2n+3)R^2r^2-(2n+1)r^4}{R^4}\Big\}\frac{r^n}{R^n}\cos n(\varphi-\vartheta)\,d\varphi+$$

$$+\frac{1}{8\pi}\int_0^{2\pi}f\Big\{4+\sum_1^\infty\frac{(n+2)(n+4)R^4-2n(n+4)R^2r^2+n(n+2)r^4}{R^4}$$

$$\frac{r^n}{R^n}\cos n(\varphi-\vartheta)\Big\}d\varphi.$$

Voor de herleiding van deze uitdrukking maken we gebruik van de formules

$$\sum_0^\infty \varrho^n \cos n\alpha = Re\,\frac{1}{1-\varrho\, e^{i\alpha}} = \frac{1-\varrho\cos\alpha}{1+\varrho^2-2\varrho\cos\alpha},$$

$$\tfrac{1}{2} + \sum_1^\infty \varrho^n \cos n\alpha = \frac{1-\varrho^2}{2\,(1+\varrho^2-2\,\varrho\cos\alpha)}$$

$$\sum_1^\infty n\,\varrho^n \cos n\alpha = \varrho\left\{\frac{-\varrho}{1+\varrho^2-2\,\varrho\cos\alpha} - \frac{(1-\varrho^2)\,(\varrho-\cos\alpha)}{(1+\varrho^2-2\,\varrho\cos\alpha)^2}\right\} =$$

$$= \frac{-2\,\varrho^2 + \varrho\,(1+\varrho^2)\cos\alpha}{(1+\varrho^2-2\,\varrho\cos\alpha)^2}$$

$$\sum_1^\infty n^2\,\varrho^n \cos n\alpha = \varrho\left\{\frac{-4\,\varrho + (1+3\,\varrho^2)\cos\alpha}{(1+\varrho^2-2\,\varrho\cos\alpha)^2} - \right.$$

$$\left. - \frac{4\,(\varrho-\cos\alpha)\,\{-2\,\varrho^2 + \varrho\,(1+\varrho^2)\cos\alpha\}}{(1+\varrho^2-2\,\varrho\cos\alpha)^3}\right\} =$$

$$= \frac{-4\varrho^2\,(1-\varrho^2) + \varrho\,(1-\varrho^4)\cos\alpha + 2\varrho^2\,(1-\varrho^2)\cos^2\alpha}{(1+\varrho^2-2\,\varrho\cos\alpha)^3}.$$

Voor den eersten term vinden we

$$\frac{(R^2-r^2)^3}{16\,\pi\,R^4}\int_0^{2\pi} \frac{R^2\,h\,(\varphi)}{R^2+r^2-2\,R\,r\cos\,(\varphi-\vartheta)}\,d\varphi,$$

voor den tweeden

$$\frac{1}{8\pi}\int_0^{2\pi} R\,g\,(\varphi)\left[-\frac{(5\,R^2-r^2)(R^2-r^2)}{R^4}\left\{\tfrac{1}{2} + \sum_1^\infty \frac{r^n}{R^n}\cos n\,(\varphi-\vartheta)\right\} - \right.$$

$$\left. - \frac{2\,(R^2-r^2)^2}{R^4}\sum_1^\infty n\,\frac{r^n}{R^n}\cos n\,(\varphi-\vartheta)\right]d\varphi =$$

$$\frac{1}{8\pi}\int_0^{2\pi} R\,g\,(\varphi)\left[-\frac{(5\,R^2-r^2)\,(R^2-r^2)}{2\,R^4\,\{R^2+r^2-2\,R\,r\cos\,(\varphi-\vartheta)\}} - \right.$$

$$\left. - \frac{2\,(R^2-r^2)^2\,\{-2\,R^2\,r^2 + R\,r\,(R^2+r^2)\cos\,(\varphi-\vartheta)\}}{R^4\,\{R^2+r^2-2\,R\,r\cos\,(\varphi-\vartheta)\}^2}\right]d\varphi =$$

$$\frac{(R^2-r^2)^2}{16\,\pi\,R^4}\int_0^{2\pi} R\,g\,(\varphi)\,\frac{-5\,R^4 + 4\,R^2\,r^2 + r^4 - 6\,R\,r\,(R^2-r^2)\cos\,(\varphi-\vartheta)}{\{R^2+r^2-2\,R\,r\cos\,(\varphi-\vartheta)\}^2}\,d\varphi =$$

$$= -\frac{(R^2-r^2)^3}{16\,\pi\,R^4}\int_0^{2\pi} R\,g\,(\varphi)\,\frac{5\,R^2 + r^2 - 6\,R\,r\cos\,(\varphi-\vartheta)}{\{R^2+r^2-2\,R\,r\cos\,(\varphi-\vartheta)\}^2}\,d\varphi$$

voor den derden

$$\frac{1}{8\pi}\int_0^{2\pi} f(\varphi)\left[\frac{4(R^2-r^2)}{R^2+r^2-2Rr\cos(\varphi-\vartheta)}+\right.$$

$$+\frac{2Rr(3R^2-r^2)(R^2-r^2)\{-2Rr+(R^2+r^2)\cos(\varphi-\vartheta)\}}{R^4\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^2}+$$

$$\left.+\frac{Rr(R^2-r^2)^3\{-4Rr+(R^2+r^2)\cos(\varphi-\vartheta)+2Rr\cos^2(\varphi-\vartheta)\}}{R^4\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^3}\right]d\varphi=$$

$$=\frac{(R^2-r^2)^3}{8\pi R^4}\int_0^{2\pi} f(\varphi)\left[\frac{4R^2-2Rr\cos(\varphi-\vartheta)}{\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^2}+\right.$$

$$\left.+\frac{Rr\{-4Rr+(R^2+r^2)\cos(\varphi-\vartheta)+2Rr\cos^2(\varphi-\vartheta)\}}{\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^3}\right]d\varphi=$$

$$=\frac{(R^2-r^2)^3}{8\pi R^4}\int_0^{2\pi} f(\varphi)\frac{4R^4-Rr(9R^2+r^2)\cos(\varphi-\vartheta)+6R^2r^2\cos^2(\varphi-\vartheta)}{\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^3}\,d\varphi.$$

en dus

$$u=\frac{(R^2-r^2)^3}{16\pi R^4}\int_0^{2\pi}\left[\frac{R^2h(\varphi)}{R^2+r^2-2Rr\cos(\varphi-\vartheta)}-\frac{\{5R^2+r^2-6Rr\cos(\varphi-\vartheta)\}Rg(\varphi)}{\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^2}+\right.$$

$$\left.+\frac{\{8R^4-2Rr(9R^2+r^2)\cos(\varphi-\vartheta)+12R^2r^2\cos^2(\varphi-\vartheta)\}f(\varphi)}{\{R^2+r^2-2Rr\cos(\varphi-\vartheta)\}^3}\right]d\varphi,$$

een formule, die slechts door een geringe wijziging in de notatie van (20) verschilt. Het bewijs, dat de zoo bepaalde functie aan alle eischen voldoet, kan naar het vroeger voor de overeenkomstige uitdrukking voor het geval der vergelijking $\triangle^2 u = 0$ gegevene, worden nagemaakt.

§ 5. De in de § 1, 2 en 3 gegeven beschouwingen kunnen zonder moeite worden uitgebreid tot de differentiaalvergelijking $\triangle^\nu \varphi = 0$, waarbij $\triangle$ de differentiaaloperator van LAPLACE is voor functies van k *onafhankelijke veranderlijken*. Meetkundig geformuleerd komt het probleem dan neer op het bepalen van een functie van GREEN $H_{\nu,k}$ voor een puntenpaar P, Q in een ruimte van k afmetingen, waarbij thans op een hyperspheer aan de randvoorwaarden voldaan moet zijn. Eenvoudigheidshalve beperken wij ons eerst tot het geval $\nu = 2$.

Zooals in de inleiding is uiteengezet heeft de singulariteit der functie van GREEN dan alleen een logarithmisch karakter voor $k = 2$ en voor $k = 4$. Het geval $k = 2$ hebben wij reeds behandeld, het geval $k = 4$ sluiten wij vooreerst uit. De singulariteit der functie van GREEN is dan bepaald door $\varphi_\nu = \frac{1}{\varrho^{k-4}}$ en wij kunnen, lettend op hetgeen wij voor het

geval $k = 3$ deden, trachten de gezochte functie van GREEN voor te stellen door een uitdrukking van den vorm

$$H_{2,k} = \frac{1}{\varrho^{k-4}} - \frac{R^{k-4}}{a^{k-4}} \frac{1}{\varrho_1^{k-4}} + \alpha \frac{R^{k-4}(R^2-a^2)(R^2-r^2)}{a^{k-2}\varrho_1^{k-2}}, \qquad (21)$$

waarin R, a, r, ϱ en ϱ_1 analoge beteekenis hebben als in het geval van den cirkel of van den bol, terwijl α een nog nader te bepalen constante voorstelt.

Deze functie heeft immers de geëischte singulariteit, als $P \to Q$, voldoet voor $P \neq Q$ aan $\triangle^2 H = 0$ en wordt nul, als P op de hyperspheer S met straal R om den oorsprong komt.

Voldaan moet nog worden aan de randvoorwaarde $\frac{\partial H}{\partial r} = 0$. Inderdaad blijkt na differentiatie, dat deze voorwaarde voor de geheele hyperspheer vervuld kan worden; men vindt $\alpha = -\frac{k-4}{2}$. Ons resultaat is dus:

Voor $k \neq 2$, $k \neq 4$ wordt de functie van GREEN, *behoorend bij de vergelijking $\triangle^2 \varphi = 0$, voor een hyperspheer in een ruimte van k afmetingen, voorgesteld door*

$$H_{2,k} = \frac{1}{\varrho^{k-4}} - \frac{R^{k-4}}{a^{k-4}} \cdot \frac{1}{\varrho_1^{k-4}} - \frac{k-4}{2} \frac{R^{k-4}(R^2-a^2)(R^2-r^2)}{a^{k-2}\varrho_1^{k-2}}. \qquad (22)$$

De generalisatie van de formule (1) voor een ruimte van k afmetingen $k \neq 2$, $k \neq 4$ luidt voor $\nu = 2$:

$$2(k-2)(k-4)\, d_k\, u = \int_S \left(u \frac{\partial \triangle H}{\partial n} - \frac{\partial u}{\partial n} \triangle H \right) d\sigma, \qquad (23)$$

waarbij de integratie over de hyperspheer S moet worden uitgestrekt. Daarbij is d_k een zoodanige constante, dat de inhoud van de begrenzing van een spheer met straal R in een ruimte van k afmetingen door $d_k R^{k-1}$ wordt voorgesteld; gelijk bekend is

$$d_k = \frac{k\{\Gamma(\frac{1}{2})\}^k}{\Gamma\left(\frac{k}{2}+1\right)}. \qquad (24)$$

Door substitutie van de uitkomst (22) in (23) verkrijgt men de functie u, die binnen S voldoet aan $\triangle^2 \varphi = 0$ en waarvan op S de functiewaarde en de waarde van de normale afgeleide zijn voorgeschreven.

Met behulp van de betrekking $\triangle \frac{1}{\varrho^{k-2}} = 0$, vindt men de uitkomsten

$$\triangle \frac{1}{\varrho^{k-4}} = -\frac{2(k-4)}{\varrho^{k-2}}, \quad \triangle \frac{1}{\varrho_1^{k-4}} = -2\frac{(k-4)}{\varrho_1^{k-2}},$$

$$\triangle \frac{r^2}{\varrho_1^{k-2}} = \frac{2k}{\varrho_1^{k-2}} - 4\frac{(k-2)(a r - R^2 \cos\vartheta) r}{a \varrho_1^k}$$

waaruit volgt

$$\triangle H_{2,k} = (k-4)\left\{-\frac{2}{\varrho^{k-2}} + \frac{2R^{k-4}}{a^{k-4}\varrho_1^{k-2}} + \right.$$
$$\left. + \frac{k(R^2-a^2)R^{k-4}}{a^{k-2}\varrho_1^{k-2}} - \frac{2(k-2)R^{k-4}(R^2-a^2)(ar-R^2\cos\vartheta)r}{a^{k-1}\varrho_1^k}\right\} \quad (25)$$

zoodat op S geldt

$$\triangle H_{2,k} = (k-4)(k-2)\frac{(R^2-a^2)^2}{R^2\varrho^k} \quad (26)$$

en

$$\frac{\partial \triangle H_{2,k}}{\partial r} = (k-4)(k-2)\frac{(R^2-a^2)^2}{R^3\varrho^{k+2}}\{2\varrho^2-ka(a-R\cos\vartheta)\}. \quad (27)$$

Wij krijgen dus:

De functie u, welke voldoet aan de vergelijking $\triangle^2\varphi = 0$ en op de hyperspheer S aan de randvoorwaarden $u = f$, $\frac{\partial u}{\partial n} = g$, waarbij f en g gegeven functies zijn, wordt voor $k \neq 2$ en $k \neq 4$ gegeven door

$$u = \frac{(R^2-a^2)^2}{2d_k R^3}\int_S\left[f\cdot\frac{2\varrho^2-ka(a-R\cos\vartheta)}{\varrho^{k+2}} - g\cdot\frac{R}{\varrho^k}\right]d\sigma. \quad (28)$$

Voor $k = 3$ komt deze formule overeen met een vroeger gevondene [4]). Voor $k = 2$ werd eveneens reeds eerder de oplossing afgeleid [5]), met als resultaat

$$H_{2,2} = \varrho^2 \ln\varrho\frac{R}{a} - \frac{a^2}{R^2}\varrho_1^2\ln\varrho_1 + \frac{(R^2-a^2)(R^2-r^2)}{2R^2}(2\ln\varrho_1 + 1),$$

terwijl op den cirkel

$$\triangle H = \frac{2(R^2-a^2)^2}{R^2\varrho^2} \quad \text{en} \quad \frac{\partial\triangle H}{\partial r} = \frac{4(R^2-a^2)^2}{R^2}\frac{R-a\cos\vartheta}{\varrho^4}$$

waaruit voor u een formule volgt, die ook verkregen wordt door in (28) $k = 2$ te nemen.

Er blijft nog slechts het geval $k = 4$ ter behandeling over. Daarbij wordt de singulariteit bepaald door $\varphi_2 = \ln\varrho$, en het ligt voor de hand de functie van GREEN uit te drukken als een vorm van de gedaante

$$H_{2,4} = \ln\varrho - \ln\frac{a}{R}\varrho_1 + \alpha\frac{(R^2-a^2)(R^2-r^2)}{a^2\varrho_1^2}.$$

Deze functie vertoont de gevraagde singulariteit en is op de hyperspheer $r = R$ gelijk aan nul. Het blijkt na differentiatie, dat α inderdaad zoo be-

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[4]) BREMEKAMP, l.c. formule (9).
[5]) Idem: formule (4).

paald kan worden, dat $\frac{\partial H}{\partial r}$ op S identiek nul is; men vindt $\alpha = \frac{1}{2}$, *zoodat wij voor* $k = 4$ *als functie van* GREEN *vinden*:

$$H_{2,4} = ln\, \varrho - ln\, \frac{a}{R}\, \varrho_1 + \frac{(R^2-a^2)\,(R^2-r^2)}{2\,a^2\,\varrho_1^2}. \tag{29}$$

Wij bepalen nog de analoga van de uitkomsten (26) en (27). Men vindt

$$\triangle H_{2,4} = \frac{2}{\varrho^2} - \frac{2}{\varrho_1^2} - \frac{4\,(R^2-a^2)}{a^2\,\varrho_1^2} + \frac{4\,(R^2-a^2)\,(a\,r-R^2\cos\vartheta)\,r}{a^3\,\varrho_1^4}$$

dat is dus het rechterlid van (25) voor $k = 4$, nadat men dit rechterlid door $k-4$ heeft gedeeld. In verband daarmee heeft men nu onmiddellijk op de hyperspheer S:

$$\triangle H_{2,4} = -\,2\,\frac{(R^2-a^2)^2}{R^2\,\varrho^4}$$

en

$$\frac{\partial \triangle H_{2,4}}{\partial r} = -\,2\,\frac{(R^2-a^2)^2}{R^3\,\varrho^6}\,\{2\,\varrho^2 - 4\,a\,(a-R\cos\vartheta)\}$$

terwijl ten slotte voor u gevonden wordt de formule, die men verkrijgt door in (28) voor k de waarde 4 te substitueeren.

Voor de vergelijking $\triangle^2\varphi = 0$ *is daarmee voor willekeurige waarde van* k *het door ons gestelde randwaardeprobleem volledig en expliciet opgelost.*

§ 6. De functie van GREEN $H_{\nu,k}$ voor de vergelijking $\triangle^\nu\varphi = 0$ en in de ruimte van k afmetingen zal in de eenvoudigste gevallen (k oneven of k even en grooter dan 2ν) de gedaante hebben van de in (6) (voor $k = 3$ en willekeurige ν) en in (21) (voor k willekeurig, $\neq 2$ en $\neq 4$ en $\nu = 2$) gegeven formule, nl.

$$H_{\nu,k} = \varrho^{2\nu-k} - \frac{a^{2\nu-k}}{R^{2\nu-k}}\,\varrho_1^{2\nu-k} + \sum_{m=1}^{m=\nu-1} C_m\,\frac{(R^2-a^2)^m\,(R^2-r^2)^m}{R^{2\nu-k}}\,(a\,\varrho_1)^{2\nu-k-2m}.$$

De coëfficiënten C_m kunnen daarbij op de in § 2 aangegeven wijze worden bepaald; het resultaat is

$$C_m = (-1)^{m+1}\binom{\nu-\frac{k}{2}}{m}.$$

Voor de overige gevallen ($k = 2, 4, \ldots 2\nu$) zullen in $H_{\nu,k}$ logarithmische singulariteiten optreden, zooals die in de voorbeelden $k = 2$, ν willekeurig (15) en $k = 2$, $k = 4$, $\nu = 2$ (29) zijn ontmoet.

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Second communication *). (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 30, 1946.)

§ 6. *Irreducible convex star domains in R_2.*

Theorem G: *A convex star domain K in R_2 is irreducible if and only if all parallelograms with one vertex at O and the other three vertices on the boundary C of K are of equal areas.*

Proof: Every critical lattice of K has at least six and at most eight points on C. If it has eight points on C, then K is a parallelogram, hence irreducible; in this case, the inscribed parallelograms clearly satisfy the assertion. Assume next that every critical lattice of K has just six points on C, and let P_1 be any given point on C. There exists then on C at least one pair of points P_2, P_3 such that

$$P_2 = P_1 + P_3, \quad \{P_1, P_2\} > 0;$$

i.e., $OP_1P_2P_3$ is a parallelogram with its vertices described in this order in *positive* direction. In general, only one such parallelogram exists for given P_1. If, however, C contains a line segment parallel to OP_1 and of greater length than this vector, then there are an infinity of such parallelograms, and all are of equal areas. Select one such parallelogram $OP_1P_2P_3$ and call its area $A(P_1)$. Then the lattice Λ of basis P_1, P_2 is of determinant $d(\Lambda) = A(P_1)$ and is K-admissible. As is easily seen,

$$\Delta(K) = \min_{P_1 \text{ on } C} A(P_1).$$

The assertion follows therefore from the theorems, C, D, and E.

Theorem G enables us to construct any number of irreducible convex star domains in R_2. Take any three points Q_1, Q_2, Q_3 in R_2 such that

$$Q_2 = Q_1 + Q_3, \quad \{Q_1, Q_2\} > 0.$$

Denote by T_1, T_2, T_3 the three triangles of vertices

$$Q_1, Q_1 + Q_2, Q_2 \text{ or } Q_2, Q_2 + Q_3, Q_3 \text{ or } Q_3, Q_3 - Q_1, -Q_1$$

and by A_1 and A_2 two continuous arcs of the following kind:

(a) A_1 connects Q_1 with Q_2, and A_2 connects Q_2 with Q_3.

(b) A_1 lies in T_1, and A_2 lies in T_2.

(c) Neither A_1 nor A_2 contains a line segment [4]).

(d) The region bounded by A_1, A_2, and the two line segments OQ_1 and OQ_3 is convex.

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*) Please change in the first communication:

p. 331, line 4: add "contained in, but different from, K" after H.

p. 332, line 7: Read $F(X) \geqq 0$ for all X in R_n, and $F(O) = 0$.

p. 333, lines 4, 6 and 10: Read "O" instead of "0".

p. 337, lines 8, 15: Read "O" instead of "0".

[4]) This condition is not essential. If it is dropped, then P_2 need not be a single-valued function of P_1.

To every point P_1 on A_1, there is then a unique point P_2 on A_2 such that

$$\{P_1, P_2\} = \{Q_1, Q_2\}.$$

Denote by A_3 the arc of all points $P_3 = P_2 - P_1$, where P_1 runs over A_1; as is easily shown, A_3 lies in T_3 and connects Q_3 with $-Q_1$. Denote by $-A_1$, $-A_2$, $-A_3$ the arcs symmetrical to A_1, A_2, A_3 in O, and by K the region bounded by the six arcs $A_1, A_2, A_3, -A_1, -A_2, -A_3$. If K is convex, then K is irreducible.

As an example, let

$$Q_1 = (\tfrac{1}{2}, -1), \quad Q_2 = (1, 0), \quad Q_3 = (\tfrac{1}{2}, 1),$$

and let A_1 and A_2 be the arcs Q_1Q_2 and Q_2Q_3 of the parabola

$$x = 1 - \tfrac{1}{2} y^2.$$

A simple calculation shows that A_3 is the arc $Q_3, -Q_1$ defined by

$$4x^2 = y^4 - 8y^2 + 8y = y(y-2)\,(y^2 + 2y - 4);$$

hence

$$-\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \qquad 1 \leqq y \leqq \sqrt{(5)} - 1$$

for all points on A_3. This arc is symmetrical in the y-axis. Hence its convexity is proved if we can show that

$$\frac{d^2x}{dy^2} < 0 \text{ if } 0 \leqq x \leqq \tfrac{1}{2}, 1 \leqq y \leqq \sqrt{(5)} - 1.$$

Now

$$2x \frac{dx}{dy} = y^3 - 4y + 2 = (y-1)(y^2 + y - 3) - 1,$$

and

$$y - 1 \geqq 0, \; y^2 + y - 3 \leqq (\sqrt{(5)} - 1)^2 + \sqrt{(5)} - 3)^2 - 3 = \\ = 2 - \sqrt{5} < 0, \quad 0 \leqq 2x \leqq 1,$$

hence

$$\frac{dx}{dy} \leqq -1, \qquad \left(\frac{dx}{dy}\right)^2 \geqq 1.$$

Further

$$2x \frac{d^2x}{dy^2} + 2\left(\frac{dx}{dy}\right)^2 = 3y^2 - 4,$$

whence

$$2x \frac{d^2x}{dy^2} \leqq 3(\sqrt{(5)} - 1)^2 - 4 - 2 \times 1 = 12 - 6\sqrt{5} < 0, \; \frac{d^2x}{dy^2} < 0,$$

whence the assertion. Both A_2 and A_3 have the gradient

$$\frac{dy}{dx} = -1$$

at the point Q_3. Hence the set K bounded by the six arcs $A_1, A_2, A_3, -A_1, -A_2, -A_3$ is therefore an irreducible convex star domain of determinant

$$\Delta(K) = \{Q_1, Q_2\} = 1;$$

its boundary has everywhere a continuous tangent.

Irreducible convex star domains in R_2 can also be obtained by the following construction:

Denote by $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ four continuous functions of t of period 1/6 which satisfy the identity

$$a_1(t)\, b_2(t) - a_2(t)\, b_1(t) = \Delta,$$

where Δ is a positive constant. Let then C be the closed curve in R_2 consisting of all points $P(t) = (x_1(t), x_2(t))$ where

$$x_1(t) = a_1(t)\, \cos 2\pi t + b_1(t)\, \sin 2\pi t,$$

$$x_2(t) = a_2(t)\, \cos 2\pi t + b_2(t)\, \sin 2\pi t,$$

and where t runs over any interval of unit length. It is easily verified that

$$P(t+\tfrac{1}{6}) = P(t) + P(t+\tfrac{1}{3}), \qquad \{P(t), P(t+\tfrac{1}{6})\} = \Delta.$$

Hence C forms the boundary of an irreducible convex star domain K provided it is a convex curve. In the special case that $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are constants, C is an ellipse. But there are an infinity of other permissible choices, and it is, in particular, possible to find algebraic curves C different from ellipses and forming the boundaries of irreducible convex star domains.

§ 7. *Further examples of irreducible star domains.*

In my note, Proc. Cambridge Phil. Soc., 40, part 2 (1944), 107—116, I gave the first example of a *non-convex* irreducible star domain in R_2, namely the domain K,

$$|x_1\, x_2| \leqq 1, \qquad |x_1 + x_2| \leqq \sqrt{5},$$

of determinant $\Delta(K) = \sqrt{5}$. I shall prove in a separate paper that the following non-convex star bodies in R_2 are likewise irreducible:

(1) The domain K_1,

$$|x_1| + |x_2| \leqq 1, \qquad |x_2| \leqq \max(c, 1 - \{x^2 + 1 - 2c\}^{1/2}), \qquad (0 < c < \tfrac{1}{2}),$$

of determinant $\Delta(K_1) = c$.

(2) The domain K_2,

$$x_1^2 + x_2^2 \leqq 1, \qquad |x_2| \leqq \max(\sin c, \{2 - 2\cos c - x_1^2\}^{1/2}), \qquad \left(0 < c < \frac{\pi}{6}\right),$$

of determinant $\Delta(K_2) = \sin c$.

(3) The domain K_3,

$$|x_1| \leqq |x_2| + 1, \qquad |x_2| \leqq \min(c, \{x^2 + 1 + 2c\}^{1/2} - 1), \qquad (0 < c < \tfrac{1}{2}),$$

of determinant $\Delta(K_3) = c$.

(4) The domain K_4,

$$-1 \leqq x_1 x_2 \leqq \frac{1}{c}(1-c)^2, \qquad |x_1 + x_2| \leqq \frac{1}{c} - c, \qquad \left(\frac{3-\sqrt{5}}{2} \leqq c < 1\right)$$

of determinant $\Delta(K_4) = \frac{1}{c} - c$.

There is no difficulty in constructing an infinity of other examples in R_2. On the other hand, it is much more difficult to construct irreducible star domains in R_3; I hope, however, to discuss also some examples of this kind in the paper referred to.

§ 8. *The concavity coefficient of a star body.*

Let K: $F(X) \leq 1$ be a bounded star body. Then $F(X_1 + X_2)$ is a continuous function of X_1 and X_2 on the closed bounded set $F(X_1) + F(X_2) = 1$, and so assumes a maximum value, ω_K say, on this set. Hence, by homogeneity,

$$F(X_1 + X_2) \leqq \omega_K (F(X_1) + F(X_2)) \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for any two points X_1 and X_2. We call ω_K the *concavity coefficient* of K. This coefficient is evidently an affine invariant. On putting $X_2 = O$ in (a), we see that $\omega_K \geqq 1$; the equality sign holds if and only if K is a convex body [5]). On applying (a) repeatedly, one obtains the inequality

$$F(X_1 + \ldots + X_n) \leqq \omega_K^{n^*} (F(X_1) + \ldots + F(X_n)) \quad . \quad . \quad . \quad (b)$$

where n^* denotes the integer defined by

$$2^{n^*-1} < n \leqq 2^{n^*} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

As an example, if K is the star body in R_2 defined by

$$|x_1 x_2| \leqq 1, \qquad |x_1 + x_2| \leqq \sqrt{5},$$

i.e. if $F(X)$ is the distance function

$$F(X) = \max\left(|x_1 x_2|^{1/2}, \frac{1}{\sqrt{5}} |x_1 + x_2|\right),$$

then a simple discussion gives $\omega_K = 3/2$.

Let K be of volume $V(K)$, and let Λ be any lattice of determinant $\Delta(K)$. Then by a theorem of L. J. MORDELL [6]), Λ contains at least one point $P \neq O$ such that

$$F(P) \leqq 2\omega_K \left(\frac{\Delta(K)}{V(K)}\right)^{1/n}.$$

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[5]) If K is not bounded, but of the finite type, then (a) does not hold for all points X_1, X_2, however large ω_K is taken.

[6]) Comp. Math. 1, 248—253 (1935), in particular pp. 248 and 251.

Let now Λ be a critical lattice of K. Then $F(P) \geq 1$, and so [7])

$$V(K) \leqq (2\omega_K)^n \Delta(K). \quad \ldots\ldots\ldots \quad \text{(I)}$$

Next let Λ be a K-admissible lattice of determinant $d(\Lambda)$ with n independent points $P_1, \ldots, P_n$ on the boundary C of K,

$$F(P_1) = \ldots = F(P_n) = 1. \quad \ldots\ldots\ldots \quad (d)$$

The determinant

$$D = |\{P_1, \ldots, P_n\}|$$

of these n points is a positive integral multiple

$$D = N\, d(\Lambda)$$

of $d(\Lambda)$; we call

$$N = \text{ind}\,(P_1, \ldots, P_n)$$

the index of the n lattice points $P_1, \ldots, P_n$. An upper bound for this index is obtained in the following way [8]):

A basis $R_1, \ldots, R_n$ of Λ can be chosen such that

$$R_1 = P_1,$$

$$R_g = \frac{a_{g1} P_1 + \ldots + a_{g,g-1} P_{g-1} + P_g}{a_g} \qquad (g = 2, 3, \ldots, n);$$

where the a's are integers, and

$$a_2 \geqq 1, a_3 \geqq 1, \ldots, a_n \geqq 1, a_2 a_3 \ldots a_n = N. \quad \ldots\ldots \quad (e)$$

Every point P of Λ can be written as

$$P = u_1 R_1 + \ldots + u_n R_n$$

with integral coefficients $u_1, \ldots, u_n$. On replacing the R's by the P's, this gives

$$P = v_1 P_1 + \ldots + v_n P_n,$$

where

$$v_1 = u_1 + \frac{a_{21}}{a_2} u_2 + \ldots + \frac{a_{n1}}{a_n} u_n;$$

$$v_2 = \frac{1}{a_2} u_2 + \frac{a_{32}}{a_3} u_3 + \ldots + \frac{a_{n2}}{a_n} u_n; \ldots; \; v_n = \frac{1}{a_n} u_n.$$

By a theorem of MINKOWSKI [9]), integers $u_1, \ldots, u_n$ not all zero can be chosen such that

$$|v_1| + \ldots + |v_n| \leqq \left(\frac{n!}{N}\right)^{1/n}.$$

---

[7]) By a theorem of MINKOWSKI and HLAWKA (Math. Zeitschr. 49, 285—312 (1943), in particular pp. 288—299), there is also a lower bound for $V(K)$, namely

$$V(K) \geqq 2\,\zeta(n)\,\Delta(K).$$

[8]) MINKOWSKI, Geometrie der Zahlen, 173—176 and 187—189.

[9]) Geometrie der Zahlen, p. 122. Put $p = 1$, $r = n$, $s = 0$, and use (e).

Therefore by (b) and (d),

$$1 \leqq F(P) \leqq \omega_K^{n^*} \{F(v_1 P_1) + \ldots + F(v_n P_n)\} =$$
$$= \omega_K^{n^*} (|v_1| + \ldots + |v_n|) \leqq \omega_K^{n^*} \left(\frac{n!}{N}\right)^{1/n},$$

whence

$$\text{ind}\,(P_1, P_2, \ldots, P_n) \leqq n!\, \omega_K^{n\,n^*}. \quad \ldots\ldots\ldots \quad \text{(II)}$$

Finally, let again Λ be K-admissible, let $P_1, \ldots, P_m$ be the only points of Λ on C, and put $P_0 = O$. In the basis $R_1, \ldots, R_n$ of Λ, these points can be written as

$$P_\mu = u_1^{(\mu)} R_1 + \ldots + u_n^{(\mu)} R_n \quad (\mu = 0, 1, \ldots, m)$$

with integral coefficients $u_g^{(\mu)}$. Denote by q the integer for which

$$2\,\omega_K < q \leqq 2\,\omega_K + 1.$$

Then

$$m \leqq q^n - 1 \leqq (2\,\omega_K + 1)^n - 1. \quad \ldots\ldots \quad \text{(III)}$$

For let this assertion be false, i.e. let $m \geq q^n$. Then two of the $m+1$ points $P_0, P_1, \ldots, P_m$, the points P_μ and P_ν say, satisfy the congruences

$$u_g^{(\mu)} \equiv u_g^{(\nu)} \pmod{q} \qquad (g = 1, 2, \ldots, n).$$

Hence

$$P = \frac{1}{q}(P_\mu - P_\nu)$$

is again a point of Λ, and $P \neq O$ since $\mu = \nu$. But then

$$1 \leqq F(P) \leqq \omega_K \left\{F\left(\frac{1}{q} P_\mu\right) + F\left(-\frac{1}{q} P_\nu\right)\right\} \leqq \omega_K \left(\frac{1}{q} + \frac{1}{q}\right) < 1,$$

a contradiction [10].

The two inequalities (II) and (III) apply, in particular, to the critical lattices of K. They show that these critical lattices are essentially only of a finite number of different types, depending alone on the value of the concavity coefficient ω_K.

§ 9. *Some unsolved problems.*

Special results suggest that each one of the following four problems has an affirmative answer, though I have not succeeded in obtaining proofs. We assume always that *K is a bounded irreducible star body in R_n*, that Λ is a critical lattice of K, and that $V(K)$, ind $(P_1, \ldots, P_n)$, and m have the same meaning as in the last paragraph:

Problem 3: *To decide whether, to every dimension n, there exists a positive constant a_n such that for all K,*

$$\omega_K \leqq a_n.$$

[10] Compare MINKOWSKI, Geometrie der Zahlen, 77—80.

Problem 4: *To decide whether, to every dimension n, there exists a positive constant b_n such that for all K*

$$V(K) \leqq b_n \Delta(K).$$

Problem 5: *To decide whether, to every dimension n, there exists a positive constant c_n such that for every K and for every critical lattice Λ of K,*

$$\text{ind } (P_1, \ldots, P_n) \leqq c_n.$$

Problem 6: *To decide whether, to every dimension n, there exists a positive constant d_n such that for every K and for every critical lattice Λ of K,*

$$m \leqq d_n.$$

It is clear from the last paragraph that if the first one of these four problems has an affirmative anwer, then the same is true for the three other ones; by (I), (II), (III), we may then, in fact, put

$$b_n = (2\,a_n)^n, \qquad c_n = n!\,a_n^{nn^*}, \qquad d_n = (2\,a_n + 1)^n - 1.$$

But it is, of course, possible that no a_n, but at least one of the three numbers b_n, c_n, d_n exists.

While the last problems deal with properties of given irreducible star bodies, the main existence problem, as follows, refers to reducible star bodies:

Problem 7: *To decide whether every bounded reducible star body contains at least one irreducible star body of equal determinant.*

It is highly probable that the answer is in the affirmative, and that even a continuous infinity of irreducible star bodies of the wanted kind exists; but I have not succeeded in proving this. One reason for this failure is the following fact: If H, K, K_1, K_2, ... are star bodies such that

$$H \prec K_r \prec K \qquad (r = 1, 2, 3, \ldots),$$
$$K_1 \succ K_2 \succ K_3 \succ \ldots,$$

then the star bodies K_r tend to a limiting set, namely their intersection, but this set is not necessarily a star body. Presumably, a proof will be constructive and will consist of a finite number of steps. — If Problem 7 has an affirmative answer, then only irreducible star bodies need be considered for most purposes, in so far as bounded star bodies are concerned. — The analogous problem for unbounded star bodies has probably a negative answer; but again, I have not so far succeeded in proving this.

§ 10. *A general principle.*

We consider in the following paragraphs non-trivial examples of unbounded reducible star bodies, and begin with a simple principle on star bodies with automorphisms.

Denote by $F(X)$ and $G(X)$ two distance functions in R_n, and by $\varphi(x, y)$ a distance function in R_2, satisfying the following conditions:

(a): The star body K: $F(X) \leq 1$ is unbounded and of the finite type, and admits a group Γ of automorphisms Ω.

(b): If $s > 0$ is arbitrary, then there exists an element Ω of Γ such that if $|X| \leq s$, then $G(\Omega^{-1} X) \leq 1$.

(c): $\varphi(1, 0) = 1$, and $\varphi(1, y) \geq 1$ for all real numbers y.

Hence

$$F^*(X) = \varphi(F(X), G(X))$$

is also a distance function. By (c),

$$F^*(X) \geq F(X) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

for all points X; the star body K^*: $F^*(X) \leq 1$ is therefore contained in K.

Theorem H: $\Delta(K^*) = \Delta(K)$.

Proof: We first prove a simple inequality for φ. Denote by c the maximum of $\varphi(x, y)$ *if* $|x| + |y| = 1$; this maximum exists since φ is a continuous function. By homogeneity,

$$\varphi(x, y) \leq c(|x| + |y|). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Choose further $\varepsilon > 0$ arbitrarily small, and choose $\delta > 0$ so small that

$$2c\delta \leq 1,$$

and that further

$$\varphi(1, y) \leq 1 + \varepsilon \qquad \text{if } |y| \leq 2c\delta;$$

this is possible by (c). Then

$$\varphi(x, y) = |x| \varphi\left(1, \frac{y}{x}\right) \leq 1 + \varepsilon \qquad \text{if } \frac{1}{2c} \leq |x| \leq 1, \ |y| \leq \delta,$$

since $|y/x| \leq 2c\delta$; and by (e),

$$\varphi(x, y) \leq c(|x| + |y|) \leq 1 \qquad \text{if } |x| \leq \frac{1}{2c}, \ |y| \leq \delta,$$

since $\delta \leq 1/2c$. On combining these two inequalities,

$$\varphi(x, y) \leq 1 + \varepsilon \qquad \text{if } |x| \leq 1, \ |y| \leq \delta. \quad . \quad . \quad . \quad . \quad (f)$$

The proof proceeds now as follows: Denote by r an arbitrarily large positive number, put $s = r/\delta$, and choose the automorphism $\Omega = \Omega_r$ in Γ such that

$$\text{if } |X| \leq s, \text{ then } G(\Omega_r^{-1} X) \leq 1,$$

and so, by homogeneity,

$$\text{if } |X| \leq r, \text{ then } G(\Omega_r^{-1} X) \leq \delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

The star body $F^*(\Omega_r^{-1}X) \leq 1$ is identical with $\Omega_r K$. We saw that K^* is a subset of K; hence, by the invariance of K,

$$\Omega_r K^* \text{ is contained in } K. \quad \ldots\ldots\ldots \quad (h)$$

Next, let K_r be the set of all points X satisfying

$$F(X) \leqq 1, \quad |X| \leqq r,$$

hence by (g),

$$G(\Omega_r^{-1} X) \leqq \delta.$$

Then by (f),

$$F^*(\Omega_r^{-1} X) = \varphi(F(X), G(\Omega_r^{-1} X)) \leqq 1 + \varepsilon,$$

which means that

$$K_r \text{ is contained in } (1+\varepsilon)\,\Omega_r K^*. \quad \ldots\ldots\ldots \quad (i)$$

The two relations (h) and (i) imply, by Theorem 9 of Part I, that, as $\varepsilon \to 0$ and $r \to \infty$,

$$\lim \triangle(\Omega_r K^*) = \triangle(K).$$

But, by Theorem 17 of Part I, all automorphisms Ω_r are of determinants ∓ 1. Hence by Theorem 16 of Part I,

$$\triangle(\Omega_r K^*) = \triangle(K^*),$$

whence finally

$$\triangle(K^*) = \triangle(K),$$

as asserted.

Remark: The restriction (a) that K is of the *finite* type, is essential, as the following example in R_2 shows. Take

$$F(X) = |x_1^2 x_2|^{1/3}, \quad G(X) = |x_2|,$$
$$\varphi(x, y) = \max(|x|, |y|).$$

The star domain K: $F(X) \leq 1$ admits the automorphisms

$$x_1 = t^{-1} x_1', \quad x_2 = t^2 x_2'$$

of arbitrary determinant $t \neq 0$, hence is of the infinite type. On the other hand, the star domain

$$K^*: \qquad |x_1^2 x_2| \leqq 1, \ |x_2| \leqq 1$$

is of the finite type since it is contained in the star domain

$$H: \qquad |x_1 x_2| \leqq 1$$

of determinant $\triangle(H) = \sqrt{5}$.

We see also that the more general star domain

$$K_\tau^*: \qquad |x_1^2 x_2| \leqq 1, \ |x_2| \leqq \tau$$

is of determinant

$$\triangle(K_\tau^*) = \triangle(K^*)\sqrt{\tau},$$

an expression which tends to infinity with τ.

§ 11. *Applications of the last theorem.*

Let us assume that, in the last theorem, $G(X)$ and $\varphi(x, y)$ satisfy the additional conditions

$$G(X) \neq 0 \text{ if } F(X) \neq 0,$$

and

$$\varphi(x, 0) = |x|, \quad \varphi(x, y) > |x| \text{ if } y \neq 0.$$

Then

$$F^*(X) = \varphi(F(X), G(X)) > F(X) \text{ if } F(X) \neq 0,$$

and so every point of K^*: $F^*(X) \leq 1$ is an *inner* point of K: $F(X) \leq 1$. Hence the critical lattices of K, which by $\Delta(K^*) = \Delta(K)$ are also critical lattices of K^*, have no points on the boundary of K^*. The simplest n-dimensional star domain K^* of this kind is obtained for

$$F(X) = |x_1 \ldots x_n|^{1/n},$$

$$G(X) = \{(x_1^2 + \ldots + x_{n-1}^2)\, x_1^2 \ldots x_{n-1}^2\}^{\frac{1}{2n}}, \quad \varphi(x, y) = (x^{2n} + y^{2n})^{\frac{1}{2n}}.$$

For the star body K: $F(X) \leq 1$ admits the automorphisms

$$\Omega: \qquad x_1 = t\, x_1', \ldots, x_{n-1} = t\, x_{n-1}', x_n = t^{-(n-1)}\, x_n',$$

and so, if X is restricted by a condition $|X| \leq r$, then a number $t > 0$ depending only on r can be found such that $G(\Omega^{-1} X) \leq 1$. The star body

$$K^*: \qquad x_1^2 x_2^2 \ldots x_{n-1}^2 (x_1^2 + x_2^2 + \ldots + x_n^2) \leq 1$$

has therefore critical lattices with no points on its boundary [11].

As a second example, choose

$$F(X) = \max(|x_1^{n-1} x_n|^{1/n}, \ldots, |x_{n-1}^{n-1} x_n|^{1/n}),$$

take for $G(X)$ either of the two distance functions

$$G_1(X) = \max\left(\left|\frac{x_1}{\varepsilon_1}\right|, \ldots, \left|\frac{x_{n-1}}{\varepsilon_{n-1}}\right|\right) \text{ or } G_2(X) = \left|\frac{x_n}{\varepsilon_n}\right|,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are arbitrary positive numbers, and put

$$\varphi(x, y) = \max(|x|, |y|).$$

The star body K: $F(X) \leq 1$ is of the finite type and admits the automorphisms

$$\Omega: \qquad x_1 = t\, x_1', \ldots, x_{n-1} = t\, x_{n-1}', x_n = t^{-(n-1)}\, x_n'.$$

Therefore, if X is restricted by a condition $|X| \leq r$, then numbers $t > 0$ depending only on r can be found such that $G_1(\Omega^{-1} X) \leq 1$, or such that $G_2(\Omega^{-1} X) \leq 1$. Hence, by Theorem H, both star bodies

$$K_1^*: \qquad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leq 1, \ |x_1| \leq \varepsilon_1, \ldots, \ |x_{n-1}| \leq \varepsilon_{n-1}$$

[11] Compare a similar example in Theorem 15 of Part I which was proved in a far more complicated way.

and

$$K_2^*: \quad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leqq 1, \ |x_n| \leqq \varepsilon_n$$

are of the same determinant

$$\triangle(K_1^*) = \triangle(K_2^*) = \triangle(K), \qquad = D \text{ say}$$

as

$$K: \quad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leqq 1.$$

Hence, if $\varepsilon > 0$ and $d(\Lambda) = D$, then at least one point $P \neq O$ of Λ belongs to $(1+\varepsilon) K_1^*$, and at least one such point belongs to $(1+\varepsilon) K_2^*$.

Let now $\alpha_1, \ldots, \alpha_{n-1}$ be $n-1$ real numbers at least one of which is irrational; and let $\beta_1, \ldots, \beta_{n-1}, 1$ be n real numbers which are linearly independent over the rational field. Both lattices

$$\Lambda_1: \ x_1 = u_1 - \alpha_1 u_n, \ldots, \ x_{n-1} = u_{n-1} - \alpha_{n-1} u_n, \ x_n = D u_n \qquad (u_1, \ldots, u_n = 0, \mp 1, \mp 2, \ldots),$$

$$\Lambda_2: \ x_1 = v_1, \ldots, \ x_{n-1} = v_{n-1}, \ x_n = D(\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n) \qquad (v_1, \ldots, v_n = 0, \mp 1, \mp 2, \ldots),$$

are of determinant D. Hence, however small $\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are chosen, there exist integers $u_1, \ldots, u_n$ not all zero such that

$$(A): \ |D u_n|^{\frac{1}{n-1}} \max(|u_1 - \alpha_1 u_n|, \ldots, |u_{n-1} - \alpha_{n-1} u_n|) < 1 + \varepsilon,$$
$$|u_1 - \alpha_1 u_n| \leqq \varepsilon_1, \ldots, \ |u_{n-1} - \alpha_{n-1} u_n| \leqq \varepsilon_{n-1},$$

and however small ε and ε_n are chosen, there exist integers $v_1, \ldots, v_n$ not all zero such that

$$(B): \ |D(\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n)|^{\frac{1}{n-1}} \max(|v_1|, \ldots, |v_{n-1}|) < 1 + \varepsilon,$$
$$|\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n| \leqq v_n.$$

Let now $\varepsilon_1, \ldots, \varepsilon_n$ tend to zero. Then, from the hypothesis, both $|u_n|$ and $\max(|v_1|, \ldots, |v_{n-1}|)$ tend to infinity. Hence, by (A) and (B), there exist an infinity of systems of n integers $u_1, \ldots, u_n$ such that

$$(C): \ |u_1 - \alpha_1 u_n| < \left|\frac{1+\varepsilon}{D u_n}\right|^{\frac{1}{n-1}}, \ldots, \ |u_{n-1} - \alpha_{n-1} u_n| < \left|\frac{1+\varepsilon}{D u_n}\right|^{\frac{1}{n-1}}, \ |u_n| \to \infty,$$

and an infinity of systems of n integers $v_1, \ldots, v_n$ such that

$$(D): \ |\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n| < \frac{1+\varepsilon}{D} \max(|v_1|, \ldots, |v_{n-1}|)^{-(n-1)},$$
$$\max(|v_1|, \ldots, |v_{n-1}|) \to \infty.$$

Connected with this, the following problems seem of interest:

Problem 8: *To evaluate* $D = \triangle(K)$.

Problem 9: *To decide whether the constant factors* $D^{-\frac{1}{n-1}}$ *and* D^{-1} *in* (C) *and* (D) *are the best possible ones.*

Mathematics. — *A generalization of* TCHEBYCHEF*'s inequality to polynomials in more than one variable.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 30, 1946.)

We shall prove the following generalization of TCHEBYCHEF's inequality to polynomials in an arbitrary number of variables.

If $P(u_1, \ldots, u_h)$ is a polynomial of total degree n in h variables with real coefficients, then

$$\operatorname*{Max}_{-1 \leq u_1 \leq 1, \ldots, -1 \leq u_h \leq 1} |P(u_1, \ldots, u_h)| \geqq \frac{1}{2^{n-1}} \operatorname*{Max}_{|u_1| = \ldots = |u_h| = 1} |P^*(u_1, \ldots, u_h)|,$$

where P^* denotes the sum of the terms in P whose degree is n.

We put, for real $t_1, \ldots, t_h$,

$$F(t_1, \ldots, t_h) = P\left(\frac{e^{it_1} + e^{-it_1}}{2}, \ldots, \frac{e^{it_h} + e^{-it_h}}{2}\right) =$$

$$= \sum a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)}.$$

For each term in F we have $-n \leqq k_1 + \ldots + k_h \leqq n$. Further

$$\sum_{k_1 + \ldots k_h = n} a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)} = \frac{1}{2^n} P^*(e^{it_1}, \ldots, e^{it_h}),$$

$$\sum_{k_1 + \ldots + k_h = -n} a_{k_1 \ldots k_h} e^{i(k_1 t_1 + \ldots + k_h t_h)} = \frac{1}{2^n} P^*(e^{-it_1}, \ldots, e^{-it_h}).$$

Since

$$\frac{1}{2n}(-1)^p \sum_{p=0}^{2n-1} e^{i\left[k_1\left(t_1 + p\frac{\pi}{n}\right) + \ldots + k_h\left(t_h + p\frac{\pi}{n}\right)\right]} = \begin{matrix} 1 \text{ for } k_1 + \ldots + k_h \equiv n \pmod{2n} \\ 0 \text{ for } k_1 + \ldots + k_h \not\equiv n \pmod{2n}, \end{matrix}$$

we have

$$\frac{1}{2n}(-1)^p \sum_{p=0}^{2n-1} F\left(t_1 + p\frac{\pi}{n}, \ldots, t_h + p\frac{\pi}{n}\right) =$$

$$= \frac{1}{2^n} P^*(e^{it_1}, \ldots, e^{it_h}) + \frac{1}{2^n} P^*(e^{-it_1}, \ldots, e^{-it_h}).$$

The two terms on the right take their maximum absolute values in the same points $(t_1, \ldots, t_h)$, and among these points there is one for which both are positive. It follows that

$$\operatorname{Max} |F(t_1, \ldots, t_h)| \geqq \frac{1}{2^{n-1}} \operatorname{Max} |P^*(e^{it_1}, \ldots, e^{it_h})|,$$

and since

$$\operatorname{Max} |F(t_1, \ldots, t_h)| = \operatorname*{Max}_{-1 \leqq u_1 \leqq 1, \ldots, -1 \leqq u_h \leqq 1} |P(u_1, \ldots, u_h)|,$$

the proof is complete.

There are polynomials for which there is equality. Example:

$$P(u_1, \ldots, u_h) = T_n(u_1) + \ldots + T_n(u_h),$$

where T_n denotes the n^{th} polynomial of TCHEBYCHEF.

We restricted ourselves to polynomials with real coefficients. It will be clear that it would have been sufficient if we had assumed the reality of the coefficients of P^*.

Mathematics. — *On the G-function.* III. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 30, 1946.)

Lemma 8. *If h, m, n, p and q are integers with*

$$0 \leqq n \leqq p,\ 0 \leqq m \leqq q \text{ and } h \geqq 1 \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and λ is an arbitrary integer, then

$$\Phi_{p,q}^{m,n}(h;\lambda) = \Omega^{m,n}_{q}(h+\lambda-1) - \sum_{\tau=0}^{\lambda-1} \Phi_{p,q}^{m,n}(1;\tau)\,\Omega^{0,p}_{q}(h+\lambda-\tau-1) \quad . \quad (71)$$

Proof. From the definitions 7 and 9 it follows

$$\left\{\sum_{t=0}^{h-1} \Omega^{0,p}_{q}(t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}\right\}$$

$$= \left\{\frac{\prod_{j=1}^{q}(1-xe^{2\pi i b_j})}{\prod_{j=1}^{p}(1-xe^{2\pi i a_j})} - \sum_{t=h}^{\infty} \Omega^{0,p}_{q}(t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{\prod_{j=1}^{m}(1-xe^{2\pi i b_j})}\right\}$$

$$= \frac{\prod_{j=m+1}^{q}(1-xe^{2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{2\pi i a_j})} - \left\{\sum_{t=h}^{\infty} \Omega^{0,p}_{q}(t)\,x^t\right\}\left\{\sum_{\tau=0}^{\infty} \Phi_{p,q}^{m,n}(1;\tau)\,x^{\tau}\right\}.$$

The coefficient of $x^{h+\lambda-1}$ in the expansion of the left-hand side of this relation in ascending powers of x is in view of definition 9 equal to $\Phi_{p,q}^{m,n}(h;\lambda)$. If the right-hand side is expanded in a series of ascending powers of x, the coefficient of $x^{h+\lambda-1}$ is on account of definition 7 equal to [28])

$$\Omega^{m,n}_{q}(h+\lambda-1) - \sum_{\tau=0}^{\lambda-1} \Phi_{p,q}^{m,n}(1;\tau)\,\Omega^{0,p}_{q}(h+\lambda-\tau-1) \quad . \quad . \quad (72)$$

So the lemma is established.

---

[28]) Since $\Phi_{p,q}^{m,n}(1;\tau) = 0$ for $\tau = -1, -2, -3, \ldots$ (comp. (52)), the coefficient in question has not only for $\lambda \geqq 0$ the value (72) but also for $\lambda < 0$ (comp. definition 4).

Lemma 9. *The coefficients* $\Phi_{p,q}^{q,0}(h;\lambda)$ *satisfy the relations*

$$\Phi_{p,q}^{q,0}(1-\lambda;\lambda)=1 \qquad (\lambda \leqq 0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$\Phi_{p,q}^{q,0}(h;\lambda)=-\sum_{\tau=0}^{\lambda-1}\Phi_{p,q}^{q,0}(1;\tau)\,\Omega^{0,p}{}_{q}(h+\lambda-\tau-1) \quad (h \geqq 1;\ \lambda \neq 1-h), \quad . \quad (74)$$

$$\Phi_{p,q}^{q,0}(h;\lambda)=0 \qquad (\lambda \leqq 0;\ h \geqq 1,\ h \neq 1-\lambda). \quad . \quad . \quad . \quad (75)$$

Proof. From definition 7 it follows

$$\Omega^{q,0}{}_{q}(0)=1 \text{ and } \Omega^{q,0}{}_{q}(t)=0 \qquad (t \neq 0).$$

The relations (73) and (74) are therefore special cases of (71); formula (75) is a special case of (74).

Lemma 10. *If* m, n, p *and* q *are integers with* $0 \leqq n \leqq p$ *and* $0 \leqq m \leqq q$, *and* τ *is an arbitrary integer, then*

$$\Phi_{p,q}^{m,n}(1;\tau)=\sum_{s=0}^{\tau}\Omega^{m,n}{}_{q}(s)\,\Phi_{p,q}^{q,0}(1;\tau-s). \quad . \quad . \quad . \quad . \quad (76)$$

Proof. In virtue of the definitions 9 and 7 we have

$$\frac{\prod_{j=n+1}^{p}(1-x\,e^{2\pi i a_j})}{\prod_{j=1}^{m}(1-x\,e^{2\pi i b_j})}=\left\{\frac{\prod_{j=1}^{p}(1-x\,e^{2\pi i a_j})}{\prod_{j=1}^{q}(1-x\,e^{2\pi i b_j})}\right\}\left\{\frac{\prod_{j=m+1}^{q}(1-x\,e^{2\pi i b_j})}{\prod_{j=1}^{n}(1-x\,e^{2\pi i a_j})}\right\}$$

$$=\left\{\sum_{\lambda=0}^{\infty}\Phi_{p,q}^{q,0}(1;\lambda)\,x^{\lambda}\right\}\left\{\sum_{s=0}^{\infty}\Omega^{m,n}{}_{q}(s)\,x^{s}\right\}.$$

If the right-hand side is written in the form of a series of ascending powers of x, the coefficient of x^{τ} is

$$\sum_{s=0}^{\tau}\Omega^{m,n}{}_{q}(s)\,\Phi_{p,q}^{q,0}(1;\tau-s) \quad [29]).$$

Since the coefficient of x^{τ} in the expansion of the left-hand side is equal to $\Phi_{p,q}^{m,n}(1;\tau)$ (because of definition 9), the lemma is established.

Lemma 11. *If* h, m, n, p *and* q *are integers which satisfy the inequalities* (70) *and* λ *is an arbitrary integer, then*

$$\sum_{s=0}^{\lambda-1}\Omega^{m,n}{}_{q}(s)\,\Phi_{p,q}^{q,0}(h;\lambda-s)=\Phi_{p,q}^{m,n}(h;\lambda)-\Omega^{m,n}{}_{q}(h+\lambda-1). \quad , \quad (77)$$

Proof. We first consider the case with $\lambda<0$. Then the left-hand side of (77) is zero because of $\Omega^{m,n}{}_{q}(s)=0$ for $s<0$ (comp. definition 4). Since $\Phi_{p,q}^{m,n}(1;\tau)=0$ for $\tau<0$ it follows from (71) that the right-hand side of (77) also vanishes if $\lambda<0$. Formula (77) is therefore true for $\lambda<0$.

[29]) This holds also good for negative values of τ, because of $\Omega^{m,n}{}_{q}(s)=0$ for $s<0$.

We now suppose $\lambda \geqq 0$. Then we have because of (76)

$$\sum_{\tau=0}^{\lambda-1} \Phi_{p,q}^{m,n}(1;\tau)\, \Omega_q^{0,p}(h+\lambda-\tau-1)$$

$$= \sum_{\tau=0}^{\lambda-1} \Omega_q^{0,p}(h+\lambda-\tau-1) \sum_{s=0}^{\tau} \Omega_q^{m,n}(s)\, \Phi_{p,q}^{q,0}(1;\tau-s)$$

$$= \sum_{s=0}^{\lambda-1} \Omega_q^{m,n}(s) \sum_{\tau=s}^{\lambda-1} \Phi_{p,q}^{q,0}(1;\tau-s)\, \Omega_q^{0,p}(h+\lambda-\tau-1)$$

$$= \sum_{s=0}^{\lambda-1} \Omega_q^{m,n}(s) \sum_{t=0}^{\lambda-s-1} \Phi_{p,q}^{q,0}(1;t)\, \Omega_q^{0,p}(h+\lambda-s-t-1)$$

$$= -\sum_{s=0}^{\lambda-1} \Omega_q^{m,n}(s)\, \Phi_{p,q}^{q,0}(h;\lambda-s) \qquad \text{(because of (74))}.$$

From this relation and (71) follows (77).

Lemma 12. *If σ, m, n, p and q are integers with*

$$1 \leqq \sigma \leqq p,\ 0 \leqq n \leqq p \text{ and } 0 \leqq m \leqq q, \quad . \ . \ . \ . \ . \quad (78)$$

and λ is an arbitrary integer, then

$$\sum_{s=0}^{\lambda-1} \Omega_q^{m,n}(s)\, \Theta_p^{q,0}(\sigma;\lambda-s-1) = \Theta_p^{m,n}(\sigma;\lambda-1). \quad . \ . \ . \quad (79)$$

Proof. In view of the definitions 11 and 7 we have

$$\frac{\prod_{j=n+1}^{p}(1-xe^{2\pi i a_j})}{(1-xe^{2\pi i a_\sigma})\prod_{j=1}^{m}(1-xe^{2\pi i b_j})} = \left\{\frac{\prod_{j=1}^{p}(1-xe^{2\pi i a_j})}{(1-xe^{2\pi i a_\sigma})\prod_{j=1}^{q}(1-xe^{2\pi i b_j})}\right\}\left\{\frac{\prod_{j=m+1}^{q}(1-xe^{2\pi i b_j})}{\prod_{j=1}^{n}(1-xe^{2\pi i a_j})}\right\}$$

$$= \left\{\sum_{r=0}^{\infty} \Theta_p^{q,0}(\sigma;r)\, x^r\right\}\left\{\sum_{s=0}^{\infty} \Omega_q^{m,n}(s)\, x^s\right\}.$$

If the right-hand side of this relation is expanded in ascending powers of x, the coefficient of $x^{\lambda-1}$ is

$$\sum_{s=0}^{\lambda-1} \Omega_q^{m,n}(s)\, \Theta_p^{q,0}(\sigma;\lambda-s-1).$$

The coefficient of $x^{\lambda-1}$ in the expansion of the left-hand side is on account of definition 11 equal to $\Theta_p^{m,n}(\sigma;\lambda-1)$. From this follows the lemma.

Lemma 13. *If σ, m, n, p and q are integers which satisfy the inequalities* (78) *and r is an arbitrary integer, then*

$$\Theta_p^{m,n}(\sigma;r) = \sum_{\tau=0}^{r} e^{(2r-2\tau)\pi i a_\sigma}\, \Phi_{p,q}^{m,n}(1;\tau). \quad . \ . \ . \ . \ . \quad (80)$$

Proof. In virtue of definition 9 we have

$$\frac{\prod_{j=n+1}^{p}(1-x\,e^{2\pi i a_j})}{(1-x\,e^{2\pi i a_\sigma})\prod_{j=1}^{m}(1-x\,e^{2\pi i b_j})} = \left\{\sum_{t=0}^{\infty} e^{2t\pi i a_\sigma}\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-x\,e^{2\pi i a_j})}{\prod_{j=1}^{m}(1-x\,e^{2\pi i b_j})}\right\}$$

$$= \left\{\sum_{t=0}^{\infty} e^{2t\pi i a_\sigma}\,x^t\right\}\left\{\sum_{\tau=0}^{\infty}\Phi_{p,q}^{m,n}(1;\tau)\,x^\tau\right\}$$

$$= \sum_{r=0}^{\infty} x^r \sum_{\tau=0}^{r} e^{(2r-2\tau)\pi i a_\sigma}\,\Phi_{p,q}^{m,n}(1;\tau).$$

From this relation and definition 11 follows (80).

Lemma 14. *If h, m, n, p and q are integers which satisfy the inequalities (70) and λ is an arbitrary integer, then*

$$\Psi_{p,q}^{m,n}(h;\lambda) = \sum_{\tau=0}^{\lambda-1}\Phi_{p,q}^{m,n}(1;\tau)\,\overline{\Omega}^{0,p}_{q}(h+\tau-\lambda) \quad . \quad . \quad . \quad . \quad (81)$$

Proof. On account of definition 9 we have

$$\left\{\sum_{t=1}^{h}\overline{\Omega}^{0,p}_{q}(h-t)\,x^t\right\}\left\{\frac{\prod_{j=n+1}^{p}(1-x\,e^{2\pi i a_j})}{\prod_{j=1}^{m}(1-x\,e^{2\pi i b_j})}\right\}$$

$$= \left\{\sum_{t=1}^{h}\overline{\Omega}^{0,p}_{q}(h-t)\,x^t\right\}\left\{\sum_{\tau=0}^{\infty}\Phi_{p,q}^{m,n}(1;\tau)\,x^\tau\right\}.$$

We now expand both sides of this relation in ascending powers of x. The coefficient of x^λ in the expansion of the right-hand side is

$$\sum_{\tau=0}^{\lambda-1}\Phi_{p,q}^{m,n}(1;\tau)\,\overline{\Omega}^{0,p}_{q}(h+\tau-\lambda)\ ^{30});$$

on the left the coefficient of x^λ is in view of definition 10 equal to $\Psi_{p,q}^{m,n}(h;\lambda)$. This proves the lemma [31]).

§ 8. Definitions and Lemmas.

Definition 12. *Suppose that m, n and p are integers with*

$$0 \leqq m \leqq q \quad \text{and} \quad 0 \leqq n \leqq p;$$

suppose further that λ is an arbitrary integer.

Then the coefficient $D_{p,q}^{m,n}(\lambda)$ is defined as follows:

$$D_{p,q}^{m,n}(\lambda) = (-2\pi i)^{p-q}\exp\{\pi i(\sum_{h=1}^{p}a_h - \sum_{h=1}^{q}b_h)\}\times \left.\right\} \quad . \quad . \quad . \quad (82)$$
$$\times\{B_p^{m,n}\,\Gamma_p^{m,n}(\lambda) - \overline{B}_p^{m,n}\,\overline{\Gamma}_p^{m,n}(p-m-n-\lambda)\}. \left.\right\}$$

[30]) This is also true for $\lambda > h$, since $\overline{\Omega}^{0,p}_{q}(s)$ vanishes for $s < 0$.

[31]) Formula (81) is obvious for $\lambda \leqq 0$, because of (52) and (53) (comp. definition 4). A similar remark holds for the formulae (79) and (80).

Remark 1. If

$$b_j - b_s \neq 0, \pm 1, \pm 2, \ldots (j = 1, \ldots, m; s = 1, \ldots, m; j \neq s),$$

the coefficient $D^{m,n}_{p,q}(\lambda)$ may also be written in the form

$$D^{m,n}_{p,q}(\lambda) = (-2i)^{p-q+1}\,\pi^{m+n-q} \exp\{\pi i (\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h)\} \times \left.\begin{array}{l} \\ \times \sum_{s=1}^{m} e^{(m+n-p+2\lambda)\pi i b_s} \dfrac{\prod_{j=n+1}^{p} \sin(a_j - b_s)\pi}{\prod_{\substack{j=1 \\ j\neq s}}^{m} \sin(b_j - b_s)\pi}. \end{array}\right\} \quad (83)$$

Proof of (83). Formula (83) follows from (82) and (69).

Remark 2. The coefficient $D^{m,n}_{p,q}(\lambda)$ is zero if $p - m - n < \lambda < 0$. This follows at once from (82) and (51).

Definition 13. *Suppose that h, m, n, p and q are integers with*

$$0 \leqq n \leqq p,\ 0 \leqq m \leqq q \text{ and } h \geqq 1;$$

suppose further that λ is an arbitrary integer.

Then the coefficients $R^{m,n}_{p,q}(h;\lambda)$ and $\overline{R}^{m,n}_{p,q}(h;\lambda)$ are defined as follows:

$$R^{m,n}_{p,q}(h;\lambda) = A^{m,n}_{q}\,\Phi^{m,n}_{p,q}(h;\lambda) - A^{0,p}_{q}\,\overline{B}^{m,n}_{p}\,\overline{\Psi}^{m,n}_{p,q}(h;p-m-n-\lambda+1), \quad (84)$$

$$\overline{R}^{m,n}_{p,q}(h;\lambda) = \overline{A}^{m,n}_{q}\,\overline{\Phi}^{m,n}_{p,q}(h;\lambda) - \overline{A}^{0,p}_{q}\,B^{m,n}_{p}\,\Psi^{m,n}_{p,q}(h;p-m-n-\lambda+1). \quad (85)$$

Definition 14. *Suppose that l, m, n, p and q are integers with*

$$1 \leqq l \leqq p,\ 0 \leqq n \leqq p \text{ and } 0 \leqq m \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_p$ satisfy the condition

$$a_j - a_l \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, p; j \neq l).$$

Then the coefficient $T^{m,n}_{p,q}(l;\lambda)$ is defined in the following way:

If $1 \leqq l \leqq n$, we put

$$\left.\begin{array}{l} T^{m,n}_{p,q}(l;\lambda) = -\{e^{\pi i a_l} B^{m,n}_{p}\,\Theta^{m,n}_{p}(l;\lambda-1) + e^{-\pi i a_l}\overline{B}^{m,n}_{p}\,\overline{\Theta}^{m,n}_{p}(l;p-m-n-\lambda)\}\,\triangle^{0,p}_{q}(l) \\ \qquad + e^{(m+n-p+2\lambda-1)\pi i a_l}\,\triangle^{m,n}_{q}(l). \end{array}\right\} \quad (86)$$

If $n+1 \leqq l \leqq p$, we put

$$T^{m,n}_{p,q}(l;\lambda) = -\{e^{\pi i a_l} B^{m,n}_{p}\,\Theta^{m,n}_{p}(l;\lambda-1) + e^{-\pi i a_l}\overline{B}^{m,n}_{p}\,\overline{\Theta}^{m,n}_{p}(l;p-m-n-\lambda)\}\,\triangle^{0,p}_{q}(l). \quad (87)$$

Remark. If

$$b_j - b_s \neq 0, \pm 1, \pm 2, \ldots (j = 1, \ldots, m; s = 1, \ldots, m; j \neq s),$$

the coefficient $T^{m,n}_{p,q}(l;\lambda)$ may also be written in the form

$$T^{m,n}_{p,q}(l;\lambda) = \frac{\pi^{m+n-q-1}}{\prod_{\substack{j=1 \\ j\neq l}}^{p} \sin(a_j - a_l)\pi} \sum_{s=1}^{m} e^{(m+n-p+2\lambda-1)\pi i b_s} \frac{\{\prod_{j=n+1}^{p} \sin(a_j - b_s)\pi\}\{\prod_{\substack{j=1 \\ j\neq s}}^{q} \sin(b_j - a_l)\pi\}}{\prod_{\substack{j=1 \\ j\neq s}}^{m} \sin(b_j - b_s)\pi}. \quad (88)$$

Proof of (88). If we replace b_{m+1} by a_l, the coefficients $B_p^{m+1,n}$ and $\overline{B}_p^{m+1,n}$ reduce respect. to $-2\pi i\, e^{\pi i a_l} B_p^{m,n}$ and $2\pi i\, e^{-\pi i a_l} \overline{B}_p^{m,n}$; this follows without difficulty from (47) and (48). It further appears from the definitions 8 and 11 that the coefficients $\Gamma_p^{m+1,n}(\lambda)$ and $\overline{\Gamma}_p^{m+1,n}(\lambda)$ with $b_{m+1} = a_l$ are respect. equal to $\Theta_p^{m,n}(l;\lambda)$ and $\overline{\Theta}_p^{m,n}(l;\lambda)$.

We now take formula (69) and replace therein m by $m+1$, λ by $\lambda - 1$ and put $b_{m+1} = a_l$; then we find

$$\left.\begin{aligned} &-e^{\pi i a_l} B_p^{m,n} \Theta_p^{m,n}(l;\lambda-1) - e^{-\pi i a_l} \overline{B}_p^{m,n} \overline{\Theta}_p^{m,n}(l;p-m-n-\lambda) \\ &= -\pi^{m+n-p} \sum_{s=1}^{m} e^{(m+n-p+2\lambda-1)\pi i b_s} \frac{\prod_{j=n+1}^{p} \sin(a_j - b_s)\pi}{\{\prod_{\substack{j=1 \\ j\neq s}}^{m} \sin(b_j - b_s)\pi\} \sin(a_l - b_s)\pi} \\ &\quad - \pi^{m+n-p} e^{(m+n-p+2\lambda-1)\pi i a_l} \frac{\prod_{j=n+1}^{p} \sin(a_j - a_l)\pi}{\prod_{j=1}^{m} \sin(b_j - a_l)\pi}. \end{aligned}\right\} \quad (89)$$

The last term on the right-hand side of this relation vanishes if $n+1 \leqq l \leqq p$.

Now we have by (16)

$$\Delta_q^{0,p}(l) = \pi^{p-q-1} \frac{\prod_{j=1}^{q} \sin(b_j - a_l)\pi}{\prod_{\substack{j=1 \\ j\neq l}}^{p} \sin(a_j - a_l)\pi} \quad \ldots\ldots \quad (90)$$

Hence it appears from (89) if $1 \leqq l \leqq n$

$$\begin{aligned} &-\{e^{\pi i a_l} B_p^{m,n} \Theta_p^{m,n}(l;\lambda-1) + e^{-\pi i a_l} \overline{B}_p^{m,n} \overline{\Theta}_p^{m,n}(l;p-m-n-\lambda)\} \Delta_q^{0,p}(l) \\ &= \frac{\pi^{m+n-q-1}}{\prod_{\substack{j=1 \\ j\neq l}}^{p} \sin(a_j - a_l)\pi} \sum_{s=1}^{m} e^{(m+n-p+2\lambda-1)\pi i b_s} \frac{\{\prod_{j=n+1}^{p} \sin(a_j - b_s)\pi\}\{\prod_{\substack{j=1 \\ j\neq s}}^{q} \sin(b_j - a_l)\pi\}}{\prod_{\substack{j=1 \\ j\neq s}}^{m} \sin(b_j - b_s)\pi} \\ &\quad - \pi^{m+n-q-1} e^{(m+n-p+2\lambda-1)\pi i a_l} \frac{\prod_{j=m+1}^{q} \sin(b_j - a_l)\pi}{\prod_{\substack{j=1 \\ j\neq l}}^{n} \sin(a_j - a_l)\pi}. \end{aligned}$$

The last term on the right-hand side of this relation is in view of (16) equal to $-e^{(m+n-p+2\lambda-1)\pi i a_l} \Delta_q^{m,n}(l)$. On account of (86) formula (88) is therefore established if $1 \leqq l \leqq n$.

If $n+1 \leqq l \leqq p$, it follows from (89) and (90)

$$-\{e^{\pi i a_l} B_p^{m,n} \Theta_p^{m,n}(l;\lambda-1) + e^{-\pi i a_l} \overline{B}_p^{m,n} \overline{\Theta}_p^{m,n}(l;p-m-n-\lambda)\} \triangle_q^{0,p}(l)$$

$$= \frac{\pi^{m+n-q-1}}{\prod_{\substack{j=1 \\ j\neq l}}^{p} \sin(a_j-a_l)\pi} \sum_{s=1}^{m} e^{(m+n-p+2\lambda-1)\pi i b_s} \frac{\{\prod_{j=n+1}^{p} \sin(a_j-b_s)\pi\}\{\prod_{j=1}^{q} \sin(b_j-a_l)\pi\}}{\prod_{\substack{j=1 \\ j\neq s}}^{m} \sin(b_j-b_s)\pi},$$

and this relation is because of (87) equivalent to (88).

Lemma 15.

$$R_{p,q}^{m,n}(1;\lambda) = D_{p,q}^{m,n}(\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

and

$$\overline{R}_{p,q}^{m,n}(1;p-m-n-\lambda+1) = (-1)^{p-q+1} \exp\{2\pi i(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h)\} D_{p,q}^{m,n}(\lambda-1). \quad (92)$$

Proof. From formula (50) and the definitions 9, 10 and 8 it appears

$$\Phi_{p,q}^{m,n}(1;\lambda) = \Gamma_p^{m,n}(\lambda), \qquad \overline{\Phi}_{p,q}^{m,n}(1;\lambda) = \overline{\Gamma}_p^{m,n}(\lambda)$$

and

$$\Psi_{p,q}^{m,n}(1;\lambda) = \Gamma_p^{m,n}(\lambda-1), \qquad \overline{\Psi}_{p,q}^{m,n}(1;\lambda) = \overline{\Gamma}_p^{m,n}(\lambda-1);$$

hence we have because of (84)

$$R_{p,q}^{m,n}(1;\lambda) = A_q^{m,n} \Gamma_p^{m,n}(\lambda) - A_q^{0,p} \overline{B}_p^{m,n} \overline{\Gamma}_p^{m,n}(p-m-n-\lambda). \quad . \quad (93)$$

Now it follows from (45) and (47)

$$\frac{A_q^{m,n}}{B_p^{m,n}} = A_q^{0,p} = (-2\pi i)^{p-q} \exp\{\pi i(\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h)\}.$$

Formula (93) is therefore on account of (82) equivalent to (91).
The proof of (92) is nearly similar to that of (91).

Lemma 16. *If $b_1 = \beta$ and $b_2 = -\beta$, then* [32])

$$R_{0,2}^{2,0}(1;\lambda) = \frac{\sin 2(\lambda+1)\beta\pi}{\sin 2\beta\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

and

$$R_{0,2}^{2,0}(2;\lambda) = -\frac{\sin 2\lambda\beta\pi}{\sin 2\beta\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Proof. From (91) it follows $R_{0,2}^{2,0}(1;\lambda) = D_{0,2}^{2,0}(\lambda)$, and in virtue of (83) we have if $b_1 = -b_2 = \beta$

$$D_{0,2}^{2,0}(\lambda) = \frac{1}{2i} \frac{e^{2(\lambda+1)\beta\pi i} - e^{-2(\lambda+1)\beta\pi i}}{\sin 2\beta\pi} = \frac{\sin 2(\lambda+1)\beta\pi}{\sin 2\beta\pi},$$

so that formula (94) has been established.

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[32]) If 2β is an integer, the right-hand sides of (94) and (95) must be defined by proceeding to the limit; similarly in (96) and (97).

Because of $\Omega^{0,0}_{2}(0) = 1$ it appears from the definition of the coefficient Φ

$$\Phi^{2,0}_{0,2}(2;\lambda) = \Phi^{2,0}_{0,2}(1;\lambda+1) + \Omega^{0,0}_{2}(1)\,\Phi^{2,0}_{0,2}(1;\lambda);$$

similarly we have on account of the definition of the coefficient Ψ

$$\overline{\Psi}^{2,0}_{0,2}(2;-\lambda-1) = \overline{\Psi}^{2,0}_{0,2}(1;-\lambda-2) + \Omega^{0,0}_{2}(1)\,\overline{\Psi}^{2,0}_{0,2}(1;-\lambda-1).$$

From these two relations and (84) it follows

$$R^{2,0}_{0,2}(2;\lambda) = R^{2,0}_{0,2}(1;\lambda+1) + \Omega^{0,0}_{2}(1)\,R^{2,0}_{0,2}(1;\lambda).$$

Now $\Omega^{0,0}_{2}(1)$ with $b_1 = -b_2 = \beta$ is equal to $-2\cos 2\beta\pi$; in view of (94) we find therefore

$$R^{2,0}_{0,2}(2;\lambda) = \frac{\sin 2(\lambda+2)\beta\pi}{\sin 2\beta\pi} - 2\cos 2\beta\pi\,\frac{\sin 2(\lambda+1)\beta\pi}{\sin 2\beta\pi} = -\frac{\sin 2\lambda\beta\pi}{\sin 2\beta\pi},$$

so that formula (95) has also been proved.

Lemma 17. *If $a_1 = \alpha$, $b_1 = \beta$ and $b_2 = -\beta$, then*

$$D^{2,0}_{1,2}(\lambda) = \frac{1}{\sin 2\beta\pi}\,\{\sin 2(\lambda+1)\beta\pi - e^{2\alpha\pi i}\sin 2\lambda\beta\pi\}. \quad . \quad . \quad (96)$$

Proof. The coefficient $D^{2,0}_{1,2}(\lambda)$ with $a_1 = \alpha$ and $b_1 = -b_2 = \beta$ is in virtue of (83) equal to

$$-\frac{e^{\alpha\pi i}}{\sin 2\beta\pi}\,\{e^{(2\lambda+1)\beta\pi i}\sin(\alpha-\beta)\pi - e^{-(2\lambda+1)\beta\pi i}\sin(\alpha+\beta)\pi\}$$

$$= -\frac{e^{\alpha\pi i}}{2i\sin 2\beta\pi}\,\{e^{2\lambda\beta\pi i}e^{\alpha\pi i} - e^{2(\lambda+1)\beta\pi i}e^{-\alpha\pi i} - e^{-2\lambda\beta\pi i}e^{\alpha\pi i} + e^{-2(\lambda+1)\beta\pi i}e^{-\alpha\pi i}\}$$

$$= \frac{1}{\sin 2\beta\pi}\,\{\sin 2(\lambda+1)\beta\pi - e^{2\alpha\pi i}\sin 2\lambda\beta\pi\}.$$

Lemma 18. *If $a_1 = \alpha$, $b_1 = \beta$ and $b_2 = -\beta$, then*

$$T^{2,0}_{1,2}(1;\lambda)E_{1,2}(z\,\|\,\alpha) = \frac{2\pi i z^{\alpha-1}\sin 2\lambda\beta\pi}{\Gamma(\alpha+\beta)\,\Gamma(\alpha-\beta)\sin 2\beta\pi}\,{}_2F_0\left(1+\beta-\alpha,\,1-\beta-\alpha;-\frac{1}{z}\right). \quad (97)$$

Proof. If $a_1 = \alpha$ and $b_1 = -b_2 = \beta$, then we have because of (88)

$$T^{2,0}_{1,2}(1;\lambda) = \frac{2i}{\pi\sin 2\beta\pi}\sin(\alpha+\beta)\pi\sin(\alpha-\beta)\pi\sin 2\lambda\beta\pi,$$

and because of (14)

$$E_{1,2}(z\,\|\,\alpha) = z^{\alpha-1}\,\Gamma(1+\beta-\alpha)\,\Gamma(1-\beta-\alpha)\,{}_2F_0\left(1+\beta-\alpha,\,1-\beta-\alpha;-\frac{1}{z}\right).$$

From these two relations follows (97).

§ 9. First expansion formula.

Theorem 1. Assumptions: *k, l, m, n, p and q are integers with*

$$1 \leqq l \leqq n \leqq p \leqq q, \quad 0 \leqq m \leqq k \leqq q \quad \text{and} \quad m+n \geqq k+l; \tag{98}$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions

$$a_j - b_h \neq 1, 2, 3, \ldots \; (j = 1, \ldots, n; \; h = 1, \ldots, m), \tag{1}$$

$$a_j - b_h \neq 1, 2, 3, \ldots \; (j = n-l+2, \ldots, n; \; h = m+1, \ldots, k), \tag{99}$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \; (j = 1, \ldots, n-l+1; \; t = 1, \ldots, n-l+1; \; j \neq t); \tag{100}$$

λ is an arbitrary integer which satisfies the inequality

$$0 \leqq \lambda \leqq m+n-k-l. \tag{101}$$

Assertion[33]:

$$G_{p,q}^{m,n}(z) = \sum_{t=1}^{n-l+1} e^{(m+n-k-l-2\lambda)\pi i a_t} \, \Delta_k^{m,n-l+1}(t) \times \left\{ \times G_{p,q}^{k,l,n}\left(z \, e^{(k+l-m-n+2\lambda)\pi i} \,\|\, a_t\right). \right. \tag{102}$$

On the right-hand side of this formula the products

$$\Delta_k^{m,n-l+1}(t) \, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t) \qquad (1 \leqq t \leqq n-l+1)$$

must be defined by a limiting process when

$$a_t - b_h = 1, 2, 3, \ldots \; (m+1 \leqq h \leqq k).$$

This will be further explained in the Remark at the end of this §.

Proof. We shall suppose that the conditions (98), (101), (1), (99) and (100) are satisfied; besides we assume temporarily

$$a_j - b_h \neq 1, 2, 3, \ldots \; (j = 1, \ldots, n-l+1; \; h = m+1, \ldots, k), \tag{103}$$

$$a_j - b_h \neq 0, -1, -2, \ldots \; (j = 1, \ldots, n-l+1; \; h = 1, \ldots, k), \tag{104}$$

$$b_j - b_h \neq 0, \pm 1, \pm 2, \ldots \; (j = 1, \ldots, k; \; h = 1, \ldots, k; \; j \neq h). \tag{105}$$

Now we have according to (7)

$$G_{p,q}^{k,l,n}(\zeta \,|\, a_t) = \sum_{h=1}^{k} \frac{\Gamma(1+b_h-a_t) \prod_{j=n-l+2}^{n} \Gamma(1+b_h-a_j) \prod_{\substack{j=1 \\ j \neq h}}^{k} \Gamma(b_j-b_h)}{\prod_{j=n+1}^{p} \Gamma(a_j-b_h) \prod_{\substack{j=1 \\ j \neq t}}^{n-l+1} \Gamma(a_j-b_h) \prod_{j=k+1}^{q} \Gamma(1+b_h-b_j)} \times \left\{ \times \zeta^{b_h} \, {}_pF_{q-1}(1+b_h-a_1, \ldots, 1+b_h-a_p; 1+b_h-b_1, \ldots {}^{*} \ldots, 1+b_h-b_q; (-1)^{p-k-l}\zeta). \right. \tag{106}$$

[33] The function $G_{p,q}^{k,l,n}(\zeta \| a_t)$ is defined at the end of § 1.

From this relation it appears, since λ is an integer,

$$G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n+2\lambda)\pi i} \,||\, a_t) = \sum_{h=1}^{k} \frac{E_h}{\sin(b_h - a_t)\pi},$$

where

$$E_h = -\pi \frac{\prod_{j=n-l+2}^{n} \Gamma(1+b_h-a_j) \prod_{\substack{j=1 \\ j\neq h}}^{k} \Gamma(b_j-b_h)}{\prod_{j=n+1}^{p} \Gamma(a_j-b_h) \prod_{j=1}^{n-l+1} \Gamma(a_j-b_h) \prod_{j=k+1}^{q} \Gamma(1+b_h-b_j)} z^{b_h} e^{(k+l-m-n+2\lambda)\pi i b_h} \times \left.\right\} \quad (107)$$

$$\times {}_pF_{q-1}(1+b_h-a_1, \ldots, 1+b_h-a_p; 1+b_h-b_1, \ldots^{*} \ldots, 1+b_h-b_q; (-1)^{p-m-n} z)$$

does not depend on t.

The right-hand side of (102) is therefore equal to

$$\sum_{t=1}^{n-l+1} e^{(m+n-k-l-2\lambda)\pi i a_t} \Delta_k^{m,n-l+1}(t) \sum_{h=1}^{k} \frac{E_h}{\sin(b_h-a_t)\pi}$$

$$= \sum_{h=1}^{k} E_h \sum_{t=1}^{n-l+1} e^{(m+n-k-l-2\lambda)\pi i a_t} \frac{\Delta_k^{m,n-l+1}(t)}{\sin(b_h-a_t)\pi}$$

$$= \sum_{h=1}^{m} E_h \sum_{t=1}^{n-l+1} e^{(m+n-k-l-2\lambda)\pi i a_t} \frac{\Delta_k^{m,n-l+1}(t)}{\sin(b_h-a_t)\pi} \qquad \text{(after lemma 6)}$$

$$= -\pi^{m+n-k-l} \sum_{h=1}^{m} E_h e^{(m+n-k-l-2\lambda)\pi i b_h} \frac{\prod_{j=m+1}^{k} \sin(b_j-b_h)\pi}{\prod_{j=1}^{n-l+1} \sin(a_j-b_h)\pi} \qquad \text{(after lemma 5)}$$

$$= \sum_{h=1}^{m} \frac{\prod_{j=1}^{n} \Gamma(1+b_h-a_j) \prod_{\substack{j=1 \\ j\neq h}}^{m} \Gamma(b_j-b_h)}{\prod_{j=n+1}^{p} \Gamma(a_j-b_h) \prod_{j=m+1}^{q} \Gamma(1+b_h-b_j)} z^{b_h} \times$$

$$\times {}_pF_{q-1}(1+b_h-a_1, \ldots, 1+b_h-a_p; 1+b_h-b_1, \ldots^{*} \ldots, 1+b_h-b_q; (-1)^{p-m-n} z)$$

(after (107))

and the last expression is equal to $G_{p,q}^{m,n}(z)$ (after (7)).

Now the functions

$$G_{p,q}^{k,l,n}(\zeta \,||\, a_t) \quad (t = 1, \ldots, n-l+1)$$

are never meaningless, except if

$$a_t - b_h = 1, 2, 3, \ldots \quad (h = 1, \ldots, k)$$

and if

$$a_j - b_h = 1, 2, 3, \ldots \quad (j = n-l+2, \ldots, n;\ h = 1, \ldots, k);$$

the function $G_{p,q}^{m,n}(z)$ is meaningless if

$$a_j - b_h = 1, 2, 3, \ldots \quad (j = 1, \ldots, n;\ h = 1, \ldots, m).$$

The additional conditions (104) and (105) are therefore completely non-essential and may immediately be removed by considerations of continuity. We may remove the restriction (103) if we define the products

$$\triangle_k^{m,n-l+1}(t)\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t)$$

by a limiting process when $a_t - b_h = 1, 2, 3, \ldots$ $(m+1 \leqq h \leqq k)$; this will be established in the following Remark [34]).

Remark. *On the value of the product*

$$\triangle_k^{m,n-l+1}(t)\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t) \qquad (1 \leqq t \leqq n-l+1;\ m < k)$$

when $a_t - b_h = 1, 2, 3, \ldots$ $(m+1 \leqq h \leqq k)$.

If we first divide the function $G_{p,q}^{k,l,n}(\zeta \,\|\, a_t)$ by $\prod_{j=m+1}^{k} \Gamma(1 + b_j - a_t)$ and after that make $a_t - b_h$ (where $m+1 \leqq h \leqq k$) tend to a positive integer, we obtain on account of (106)

$$\left.\begin{aligned} &\frac{1}{\prod_{j=m+1}^{k} \Gamma(1+b_j-a_t)}\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t) \\ &= \frac{\prod_{j=n-l+2}^{n} \Gamma(1+b_h-a_j) \prod_{\substack{j=1 \\ j\neq h}}^{k} \Gamma(b_j-b_h)}{\prod_{\substack{j=m+1 \\ j\neq h}}^{k} \Gamma(1+b_j-a_t) \prod_{j=n+1}^{p} \Gamma(a_j-b_h) \prod_{\substack{j=1 \\ j\neq t}}^{n-l+1} \Gamma(a_j-b_h) \prod_{j=k+1}^{q} \Gamma(1+b_h-b_j)} \times \\ &\times \zeta^{b_h}\, {}_pF_{q-1}(1+b_h-a_1, \ldots, 1+b_h-a_p;\ 1+b_h-b_1, \ldots, \overset{*}{\ldots}, 1+b_h-b_q;\ (-1)^{p-k-l}\zeta). \end{aligned}\right\} \tag{108}$$

This formula may be written in another form. Indeed, the hypergeometric function ${}_pF_{q-1}$ on the right is a polynomial of degree $a_t - b_h - 1$ in ζ and for such a polynomial the following relation holds [35])

$$\left.\begin{aligned} &\frac{\prod_{j=1}^{q-1} \Gamma(1-\beta_j)}{\prod_{j=1}^{p-1} \Gamma(1-\alpha_j)}\, {}_pF_{q-1}\left(\begin{matrix} -\nu, \alpha_1, \ldots, \alpha_{p-1}; \\ \beta_1, \ldots, \beta_{q-1};\ (-1)^{p-q-1}\zeta \end{matrix}\right) \\ &= \frac{\prod_{j=1}^{q-1} \Gamma(1-\nu-\beta_j)}{\prod_{j=1}^{p-1} \Gamma(1-\nu-\alpha_j)}\, \zeta^{\nu}\, {}_qF_{p-1}\left(\begin{matrix} -\nu, 1-\nu-\beta_1, \ldots, 1-\nu-\beta_{q-1}; \\ 1-\nu-\alpha_1, \ldots, 1-\nu-\alpha_{p-1};\ -\zeta^{-1} \end{matrix}\right); \end{aligned}\right\} \tag{109}$$

herein is ν a positive integer or zero.

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[34]) Similar considerations of continuity apply on some other points of the paper; henceforth I will not occupy myself with such considerations.

[35]) Formula (109) may be proved by comparison of the coefficients of equal powers of ζ on both sides.

Formula (108) may therefore be written in the following way

$$\left.\begin{aligned} &\frac{1}{\prod_{j=m+1}^{k}\Gamma(1+b_j-a_t)}\,G_{p,q}^{k,l,n}\left(\zeta\,\middle\|\,a_t\right)\\ &=\frac{\prod_{j=1}^{m}\Gamma(1+b_j-a_t)\prod_{j=n-l+2}^{n}\Gamma(1+b_h-a_j)\prod_{j=k+1}^{q}\Gamma(1+b_j-a_t)}{\prod_{j=k+1}^{q}\{\Gamma(b_j-b_h)\,\Gamma(1+b_h-b_j)\}}\times\\ &\times(-1)^{(1+q-k-l)(-1+a_t-b_h)}\,\zeta^{-1+a_t}\,\frac{\prod_{j=n-l+2}^{n}\Gamma(a_j-b_h)}{\prod_{\substack{j=1\\ j\neq t}}^{p}\Gamma(1+a_j-a_t)}\times\\ &\times {}_qF_{p-1}\left(\begin{matrix}1+b_1-a_t,\ldots,1+b_q-a_t;\\ 1+a_1-a_t,\ldots,\overset{*}{\ldots},1+a_p-a_t;(-1)^{k+l-q}\,\zeta^{-1}\end{matrix}\right). \end{aligned}\right\}\quad\ldots\quad(110)$$

We now put $a_t - b_h - 1 = \nu$ (where ν is zero or a positive integer); the expression

$$\frac{\prod_{j=n-l+2}^{n}\Gamma(a_j-b_h)}{\prod_{\substack{j=1\\ j\neq t}}^{p}\Gamma(1+a_j-a_t)}\,{}_qF_{p-1}\left(\begin{matrix}1+b_1-a_t,\ldots,1+b_q-a_t;\\ 1+a_1-a_t,\ldots,\overset{*}{\ldots},1+a_p-a_t;(-1)^{k+l-q}\,\zeta^{-1}\end{matrix}\right)$$

takes then the form

$$\frac{\prod_{j=n-l+2}^{n}\Gamma(1+a_j-a_t+\nu)}{\prod_{j=1}^{q}\Gamma(1+b_j-a_t)}\sum_{h=0}^{\nu}\frac{(-1)^{h(k+l-q)}\,\zeta^{-h}\prod_{j=1}^{q}\Gamma(1+b_j-a_t+h)}{h!\prod_{\substack{j=1\\ j\neq t}}^{p}\Gamma(1+a_j-a_t+h)}$$

and this expression, considered as function of the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$, has no singularities.

The expression

$$\frac{\prod_{j=k+1}^{q}\Gamma(1+b_j-a_t)}{\prod_{j=k+1}^{q}\{\Gamma(b_j-b_h)\,\Gamma(1+b_h-b_j)\}}=\pi^{k-q}\prod_{j=k+1}^{q}\{\Gamma(1+b_j-a_t)\sin(b_j-b_h)\pi\}$$

too has no poles, since $a_t - b_h$ is an integer.

Hence, if we *merely* suppose that the conditions (1) and (99) are satisfied, then the right-hand side of (110) is significant if *one* or *more* of the differences $a_t - b_{m+1}$, $a_t - b_{m+2}$, ..., $a_t - b_k$ is equal to a positive integer.

The function

$$\frac{1}{\prod_{j=m+1}^{k} \Gamma(1+b_j-a_t)}\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t)$$

may therefore, when $a_t - b_h = 1, 2, 3, \ldots \quad (m+1 \leqq h \leqq k)$, be defined by a limiting process.

Now it follows from (17) that

$$\triangle_k^{m,n-l+1}(t)\, G_{p,q}^{k,l,n}(\zeta \,|\, a_t) = \frac{(-1)^{m+n-k-l} \prod_{j=1,\, j\neq t}^{n-l+1} \{\Gamma(a_t-a_j)\,\Gamma(1+a_j-a_t)\}}{\prod_{j=m+1}^{k} \{\Gamma(a_t-b_j)\,\Gamma(1+b_j-a_t)\}}\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t).$$

Hence, if $a_t - b_h = 1, 2, 3, \ldots \quad (m+1 \leqq h \leqq k)$, the product

$$\triangle_k^{m,n-l+1}(t)\, G_{p,q}^{k,l,n}(\zeta \,\|\, a_t)$$

may also be defined by proceeding to the limit.

Mathematics. — *Primitief-symmetrische projectieve invarianten.* III. By P. G. MOLENAAR. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 30, 1946.)

§ 9. Thans willen we het voorgaande toepassen op de binaire gemengde vorm

$$F = a_x^3\,\alpha_\xi^3 = (a_0\,x_1^3 + 3\,a_1\,x_1^2\,x_2 + 3\,a_2\,x_1\,x_2^2 + a_3\,x_2^3)\,(\alpha_0\,\xi_1^3 + 3\,\alpha_1\,\xi_1^2\,\xi_2 + 3\,\alpha_2\,\xi_1\,\xi_2^2 + \alpha_3\,\xi_2^3) =$$

$$= A_{00}\,x_1^3\,\xi_1^3 + 3\,A_{01}\,x_1^3\,\xi_1^2\,\xi_2 + \ldots + 9\,A_{11}\,x_1^2\,x_2\,\xi_1^2\,\xi_2 + \ldots$$

De isomeren van

$$L = (ab)^2\,(cd)^2\,(ad)\,(bc) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

zijn volgens § 4 (11) lineair uit te drukken in

$$L_{(1)}\;L_{(2)}\;L_{(3)}_{1}\;L_{(3)}_{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Die van

$$A = (\alpha\beta)^2\,(\gamma\delta)^2\,(\alpha\delta)\,(\beta\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

zijn lineair uit te drukken in

$$A_{(1)}\;A_{(2)}\;A_{(3)}_{1}\;A_{(3)}_{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Door contractie van de invarianten (2) en (4) ontstaan 16 invarianten van de vier gemengde vormen

$$F_1 = a_x^3\,\alpha_\xi^3 \qquad F_2 = b_x^3\,\beta_\xi^3 \qquad F_3 = c_x^3\,\gamma_\xi^3 \qquad F_4 = d_x^3\,\delta_\xi^3.$$

Elk dezer 16 invarianten geeft aanleiding tot een productvoorstelling $\Gamma_i \times \Gamma_k$. Deze moeten volledig gereduceerd worden. In de theorie der substitutiegroepen wordt hiervoor afgeleid [9])

$$\left.\begin{array}{llll} \Gamma_1 \times \Gamma_1 = \Gamma_1 & \Gamma_2 \times \Gamma_2 = \Gamma_1 & \Gamma_3 \times \Gamma_3 = \Gamma_1 + \Gamma_2 + \Gamma_3 & \\ \Gamma_1 \times \Gamma_2 = \Gamma_2 & \Gamma_2 \times \Gamma_3 = \Gamma_3 & & \\ \Gamma_1 \times \Gamma_3 = \Gamma_3 & & & \end{array}\right\} \quad . \quad . \quad (5)$$

Alleen $[L_{(3)}_{i}\,A_{(3)}_{k}]$ is dus niet primitief-symmetrisch. Door contractie vindt men dus onmiddellijk de symmetrische invarianten

$$[L_{(1)}\,A_{(1)}] = I_1 \qquad [L_{(2)}\,A_{(2)}] = I_2. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

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[9]) Zie B. L. v. d. WAERDEN, Moderne Algebra II, pag. 196, of
B. L. v. d. WAERDEN, Die Gruppentheoretische Methode in der Quantenmechanik, pag. 57, J. Springer, Berlin (1932).

de anti-symmetrische invarianten

$$[L^{(1)}\, A^{(2)}] = I_3 \qquad [L^{(2)}\, A^{(1)}] = I_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

en de cyclisch-symmetrische invarianten

$$\left.\begin{array}{ll} [L^{(1)}\, A^{(3)}_1] = I_5 & [L^{(3)}_1\, A^{(1)}] = I_6 \\ [L^{(1)}\, A^{(3)}_2] = I_7 & [L^{(3)}_2\, A^{(1)}] = I_8 \\ [L^{(2)}\, A^{(3)}_1] = I_9 & [L^{(3)}_1\, A^{(2)}] = I_{10} \\ [L^{(2)}\, A^{(3)}_2] = I_{11} & [L^{(3)}_2\, A^{(2)}] = I_{12} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

De niet primitief-symmetrische invarianten

$$[L^{(3)}_1\, A^{(3)}_1] \qquad [L^{(3)}_1\, A^{(3)}_2] \qquad [L^{(3)}_2\, A^{(3)}_1] \qquad [L^{(3)}_2\, A^{(3)}_2] \quad . \quad . \quad . \quad (9)$$

geven volgens § 8 (6) aanleiding tot een 4-dimensionale productvoorstelling $\Gamma_3 \times \Gamma_3$

$$E' = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \quad A' = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \quad \ldots . \quad F' = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{Vmatrix}$$

en deze wordt door transformatie met de matrix

$$M = \begin{Vmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

volledig gereduceerd tot

$$E'' = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \quad A'' = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix} \quad \ldots . \quad F'' = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{Vmatrix}$$

Dit is volgens § 4(9) de gereduceerde voorstelling

$$\Gamma_1 + \Gamma_2 + \Gamma_3.$$

Met behulp van de coëfficiënten uit de matrix (10) vindt men volgens § 8 (10) de symmetrische invariant

$$I_{13} = P^{(1)}_1 = \tfrac{1}{3}\,(2\,[L^{(3)}_1\, A^{(3)}_1] + [L^{(3)}_1\, A^{(3)}_2] + [L^{(3)}_2\, A^{(3)}_1] + 2\,[L^{(3)}_2\, A^{(3)}_2]) \quad . \quad (11)$$

de anti-symmetrische invariant

$$I_{14} = P^{(2)}_1 = \tfrac{1}{2}\,([L^{(3)}_1\, A^{(3)}_2] - [L^{(3)}_2\, A^{(3)}_1]) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

en twee cyclisch-symmetrische invarianten

$$\left.\begin{aligned} I_{15} &= P^{(3)}_{1} = \tfrac{1}{3}\left(-[L^{(3)}_{1} A^{(3)}_{1}] + [L^{(3)}_{1} A^{(3)}_{2}] + [L^{(3)}_{2} A^{(3)}_{1}] + 2\,[L^{(3)}_{2} A^{(3)}_{2}]\right) \\ I_{16} &= P^{(3)}_{2} = \tfrac{1}{3}\left(2\,[L^{(3)}_{1} A^{(3)}_{1}] + [L^{(3)}_{1} A^{(3)}_{2}] + [L^{(3)}_{2} A^{(3)}_{1}] - [L^{(3)}_{2} A^{(3)}_{2}]\right) \end{aligned}\right\}. \quad (13)$$

Verder is

$$M^{-1} = \begin{Vmatrix} 1 & 0 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 & 1 \\ -\frac{1}{2} & -1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{Vmatrix} \quad \ldots\ldots\ldots \quad (14)$$

zodat volgens § 8 (12)

$$\left.\begin{aligned} [L^{(3)}_{1} A^{(3)}_{1}] &= \quad I_{13} \quad - I_{15} \\ [L^{(3)}_{1} A^{(3)}_{2}] &= -\tfrac{1}{2} I_{13} + I_{14} + I_{15} + I_{16} \\ [L^{(3)}_{2} A^{(3)}_{1}] &= -\tfrac{1}{2} I_{13} - I_{14} + I_{15} + I_{16} \\ [L^{(3)}_{2} A^{(3)}_{2}] &= \quad I_{13} \quad - I_{16} \end{aligned}\right\}, \quad \ldots\ldots \quad (15)$$

Tenslotte willen we deze primitief-symmetrische invarianten $I_1, \ldots\ldots, I_{16}$ afleiden, door de invarianten

$$[L^{(3)}_{1} A^{(3)}_{1}] \qquad [L^{(3)}_{1} A^{(3)}_{2}] \qquad [L^{(3)}_{2} A^{(3)}_{1}] \qquad [L^{(3)}_{2} A^{(3)}_{2}] \quad \ldots \quad (9)$$

te splitsen in primitief-symmetrische delen.

Stelt men $L^{(3)}_{1} A^{(3)}_{1} = I_E$, en noemt men overeenkomstig § 4 (10)

$$\left.\begin{aligned} \tfrac{1}{6}(I_E + I_A + I_B + I_C + I_D + I_F) &= I_s \\ \tfrac{1}{6}(I_E + I_A + I_B - I_C - I_D - I_F) &= I_a \\ \tfrac{1}{3}(I_E \quad - I_B \quad - I_D + I_F) &= I_{c1} \\ \tfrac{1}{3}(I_E - I_A \quad + I_D - I_F) &= I_{c2} \end{aligned}\right\}, \quad \ldots\ldots \quad (16)$$

dan is volgens § 4 (11)

$$I_E = I_s + I_a + I_{c1} + I_{c2} \quad \ldots\ldots\ldots \quad (17)$$

een splitsing van I_E in primitief-symmetrische delen.

Verder is volgens § 8 (13) en § 4 (9)

$$\begin{aligned} L^{(3)}_{1\,E} &= \quad L^{(3)}_{1} & L^{(3)}_{2\,E} &= \quad L^{(3)}_{2} \\ L^{(3)}_{1\,A} &= -L^{(3)}_{1} - L^{(3)}_{2} & L^{(3)}_{2\,A} &= \quad L^{(3)}_{1} \\ L^{(3)}_{1\,B} &= \quad L^{(3)}_{2} & L^{(3)}_{2\,B} &= -L^{(3)}_{1} - L^{(3)}_{2} \\ L^{(3)}_{1\,C} &= \quad L^{(3)}_{2} & L^{(3)}_{2\,C} &= \quad L^{(3)}_{1} \\ L^{(3)}_{1\,D} &= -L^{(3)}_{1} - L^{(3)}_{2} & L^{(3)}_{2\,D} &= \quad L^{(3)}_{2} \\ L^{(3)}_{1\,F} &= \quad L^{(3)}_{1} & L^{(3)}_{2\,F} &= -L^{(3)}_{1} - L^{(3)}_{2} \end{aligned}$$

dus

$$[L_1^{(3)} A_1^{(3)}]_s = \tfrac{1}{6}([L_1^{(3)} A_1^{(3)}]_E + [L_1^{(3)} A_1^{(3)}]_A + \ldots + [L_1^{(3)} A_1^{(3)}]_F) =$$
$$= \tfrac{1}{6}([L_1^{(3)} A_1^{(3)}] + [(-L_1^{(3)} - L_2^{(3)})(-A_2^{(3)} - A_2^{(3)})] + \ldots + [L_1^{(3)} A_1^{(3)}]) =$$
$$= \tfrac{1}{3}(2[L_1^{(3)} A_1^{(3)}] + [L_1^{(3)} A_2^{(3)}] + [L_2^{(3)} A_1^{(3)}] + 2[L_2^{(3)} A_2^{(3)}]).$$

Aldus voortgaande vindt men voor de 4×4 primitief-symmetrische delen van de invarianten (9):

$$[L_1^{(3)} A_1^{(3)}]_s = -2[L_1^{(3)} A_2^{(3)}]_s = -2[L_2^{(3)} A_1^{(3)}]_s = [L_2^{(3)} A_2^{(3)}]_s =$$
$$= \tfrac{1}{3}(2[L_1^{(3)} A_1^{(3)}] + [L_1^{(3)} A_2^{(3)}] + [L_2^{(3)} A_1^{(3)}] + 2[L_2^{(3)} A_2^{(3)}]) = I_{13}$$

$$[L_1^{(3)} A_1^{(3)}]_a = [L_2^{(3)} A_2^{(3)}]_a = 0;\ (L_1^{(3)} A_2^{(3)}]_a = -[L_2^{(3)} A_1^{(3)}]_a =$$
$$= \tfrac{1}{2}([L_1^{(3)} A_2^{(3)}] - [L_2^{(3)} A_1^{(3)}]) = I_{14}$$

$$-[L_1^{(3)} A_1^{(3)}]_{c1} = [L_1^{(3)} A_2^{(3)}]_{c1} = [L_2^{(3)} A_1^{(3)}]_{c1} =$$
$$= \tfrac{1}{3}(-[L_1^{(3)} A_1^{(3)}] + [L_1^{(3)} A_2^{(3)}] + [L_2^{(3)} A_1^{(3)}] + 2[L_2^{(3)} A_2^{(3)}]) = I_{15};\ [L_2^{(3)} A_2^{(3)}]_{c1} = 0$$

$$[L_1^{(3)} A_1^{(3)}]_{c2} = 0;\quad [L_1^{(3)} A_2^{(3)}]_{c2} = [L_2^{(3)} A_1^{(3)}]_{c2} = -[L_2^{(3)} A_2^{(3)}]_{c2} =$$
$$= \tfrac{1}{3}(2[L_1^{(3)} A_1^{(3)}] + [L_1^{(3)} A_2^{(3)}] + [L_2^{(3)} A_1^{(3)}] - [L_2^{(3)} A_2^{(3)}]) = I_{16}.$$

Substitueert men deze uitkomsten in (17), dan vindt men de betrekkingen (15).

§ 10. Thans willen we de symmetrische invarianten I_1, I_2 en I_{13} nader beschouwen.

I_1 is ontstaan door contractie van $L^{(1)}$ en $A^{(1)}$.

Volgens § 5 (3a) kan men $L^{(1)}$ uit de discriminant

$$R = 12a_0 a_1 a_2 a_3 - 8a_0 a_2^3 - 2a_0^2 a_3^2 - 8a_1^3 a_3 + 6a_1^2 a_2^2 \quad . \quad . \quad . \quad (1)$$

van a_x^3 verkrijgen door middel van

$$L^{(1)} = \frac{1}{24} \sum_{p,k,l,m} \frac{\partial^4 R}{\partial a_p \partial a_k \partial a_l \partial a_m} a_p b_k c_l d_m.$$

Dan is

$$I_1 = \left(\frac{1}{24}\right)^2 \left[\sum_{p,k,l,m} \frac{\partial^4 R}{\partial a_p \partial a_k \partial a_l \partial a_m} a_p b_k c_l d_m \sum_{\pi,\varkappa,\lambda,\mu} \frac{\partial^4 R^*}{\partial \alpha_\pi \partial \alpha_\varkappa \partial \alpha_\lambda \partial \alpha_\mu} \alpha_\pi \beta_\varkappa \gamma_\lambda \delta_\mu \right]$$

waarin R^* de discriminant van α_ξ^3 voorstelt.

Stelt men

$$\frac{\partial^4 R}{\partial a_p \partial a_k \partial a_l \partial a_m} = c_{pklm}, \text{ dus ook } \frac{\partial^4 R^*}{\partial \alpha_p \partial \alpha_k \partial \alpha_l \partial \alpha_m} = c_{pklm}$$

dan wordt

$$I_1 = (\tfrac{1}{24})^2 \Big(\sum_{p,k,l,m} \sum_{\pi,\varkappa,\lambda,\mu} c_{p\,k\,l\,m}\, c_{\pi\varkappa\lambda\mu}\, A_{p\pi}\, B_{k\varkappa}\, C_{l\lambda}\, D_{m\mu} \Big). \quad (2)$$

Uit

$$c_{p\,k\,l\,m} = c_{k\,p\,l\,m} = \ldots \text{ en } c_{\pi\varkappa\lambda\mu} = c_{\varkappa\pi\lambda\mu} = \ldots$$

volgt opnieuw, dat I_1 symmetrisch is in de coëfficiëntenrijen A, B, C, D. Uit (1) berekent men de van nul verschillende coëfficiënten

$$c_{0123} = 12, \quad c_{0222} = -48, \quad c_{0033} = -16, \quad c_{1113} = -48, \quad c_{1122} = 24.$$

Nummert men de 5 indexgroepen als volgt

$(0123) = 1$, $(0222) = 2$, $(0033) = 3$, $(1113) = 4$, $(1122) = 5$

dan wordt

$c_1 = 12 \quad c_2 = -48 \quad c_3 = -16 \quad c_4 = -48 \quad c_5 = 24$

terwijl (2) dan overgaat in

$$I_1 = (\tfrac{1}{24})^2 \sum_{i=1}^{5} \sum_{j=1}^{5} c_i\, c_j\, [A\,B\,C\,D]_{i,j} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Stelt i de indexgroep $(p\,k\,l\,m)$ en j de indexgroep $(\pi\varkappa\lambda\mu)$ voor, dan is

$$[A\,B\,C\,D]_{i,j} = \sum_{(p\,k\,l\,m)} \sum_{(\pi\varkappa\lambda\mu)} A_{p\pi}\, B_{k\varkappa}\, C_{l\lambda}\, D_{m\mu} \quad . \quad . \quad . \quad . \quad (4)$$

waarin $\sum_{(p\,k\,l\,m)}$ de sommatie over de variaties der indexgroep $(p\,k\,l\,m)$ aanduidt. Zo is b.v.

$$[A\,B\,C\,D]_{2,3} = \sum_{(0222)} \sum_{(0033)} A_{00}\, B_{20}\, C_{23}\, D_{23}$$

een veelterm bestaande uit $4 \times 6 = 24$ verschillende termen.

De eerste sommatie loopt over de vier variaties

(0222) (2022) (2202) (2220)

en de tweede over de zes variaties

(0033) (0303) (3003) (0330) (3030) (3300)

Beschouwt men $A_{p\pi}, B_{p\pi}, C_{p\pi}, D_{p\pi}$ als aequivalente coëfficiëntenrijen, en vervangt men $B_{p\pi}, C_{p\pi}, D_{p\pi}$ door $A_{p\pi}$, dan worden verschillende termen in (4) aan elkaar gelijk.

Zo gaat b.v. $[A\,B\,C\,D]_{2,3}$ over in

$$[A\,A\,A\,A]_{2,3} = 12\, A_{00}\, A_{20}\, A_{23}^2 + 12\, A_{03}\, A_{23}\, A_{20}^2.$$

De invariant I_2 kan men gemakkelijk uitdrukken in de coëfficiënten der grondvormen.

Immers

$$I_2 = [L^{(2)}\, \Delta^{(2)}] = \tfrac{1}{4} \left[\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix} \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{vmatrix} \right] = \tfrac{1}{4}\, [D\,\Delta].$$

Door contractie vindt men

$$I_2 = \tfrac{1}{4} \sum_{p,k,l,m} \sum_{\pi,\varkappa,\lambda,\mu} \frac{\partial^4 D}{\partial a_p \, \partial b_k \, \partial c_l \, \partial d_m} \, \frac{\delta^4 \triangle}{\partial \alpha_\pi \, \partial \beta_\varkappa \, \partial \gamma_\lambda \, \partial \delta_\mu} \, A_{p\pi} \, B_{k\varkappa} \, C_{l\lambda} \, D_{m\mu} =$$

$$= \tfrac{1}{4} \sum_{p,k,l,m} \sum_{\pi,\varkappa,\lambda,\mu} \operatorname{sign}(p\,k\,l\,m) \operatorname{sign}(\pi\,\varkappa\,\lambda\,\mu) \, A_{p\pi} \, B_{k\varkappa} \, C_{l\lambda} \, D_{m\mu}$$

waarin sign $(p\,k\,l\,m)$ het teken van de permutatie $\begin{pmatrix} 0 & 1 & 2 & 3 \\ p & k & l & m \end{pmatrix}$ is. Nu is

$$\sum_{\pi,\varkappa,\lambda,\mu} \operatorname{sign}(\pi\,\varkappa\,\lambda\,\mu) \, A_{p\pi} \, B_{k\varkappa} \, C_{l\lambda} \, D_{m\mu} = \begin{vmatrix} A_{p0} & B_{p1} & C_{p2} & D_{p3} \\ A_{k0} & B_{k1} & C_{k2} & D_{k3} \\ A_{l0} & B_{l1} & C_{l2} & D_{l3} \\ A_{m0} & B_{m1} & C_{m2} & D_{m3} \end{vmatrix} = \det(ABCD)_{pklm}$$

dus

$$I_2 = \tfrac{1}{4} \sum_{p,k,l,m} \operatorname{sign}(pklm) \det(ABCD)_{pklm} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

waarbij gesommeerd wordt over de 24 permutaties $\begin{pmatrix} 0 & 1 & 2 & 3 \\ p & k & l & m \end{pmatrix}$.

Zijn de grondvormen aequivalent, zodat $B_{p\pi}$, $C_{p\pi}$, $D_{p\pi}$ door $A_{p\pi}$ vervangen mag worden, dan vindt men

$$I_2^* = 6 \begin{vmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Dit is een vierdegraads invariant van de enkele grondvorm $F = a_x^3 \, a_\xi^3$.

De invariant

$$I_{13} = \tfrac{1}{3} \left(2 \, [L_1^{(3)} A_1^{(3)}] + [L_1^{(3)} A_2^{(3)}] + [L_2^{(3)} A_1^{(3)}] + 2 \, [L_2^{(3)} A_2^{(3)}]\right)$$

behoeft niet in de moduulbasis te worden opgenomen. Ze is uit te drukken in I_2 en in de symmetrische invariant

$$J_{ABCD} = \tfrac{1}{3} \left(J_{AB} J_{CD} + J_{BC} J_{AD} + J_{CA} J_{BD}\right) \quad . \quad . \quad . \quad . \quad (7)$$

waarin

$$\left.\begin{aligned} J_{AB} = J_{BA} = (ab)^3 (\alpha\beta)^3 = \; & A_{00} B_{33} - 3A_{10} B_{23} + 3A_{20} B_{13} - A_{30} B_{03} + \\ & - 3A_{01} B_{32} + 9A_{11} B_{22} - 9A_{21} B_{12} + 3A_{31} B_{02} + \\ & + 3A_{02} B_{31} - 9A_{12} B_{21} + 9A_{22} B_{11} - 3A_{32} B_{01} + \\ & - A_{03} B_{30} + 3A_{13} B_{20} - 3A_{23} B_{10} + A_{33} B_{00} \end{aligned}\right\} . \quad (8)$$

een symmetrische invariant van de tweede graad is.

Men vindt nu met behulp van § 5 (7) (8)

$$2\,[L_1^{(3)}\,A_1^{(3)}] = \tfrac{2}{9}\{(ac)^3\,(db)^3 - (ab)^3\,(cd)^3\}\,\{(\alpha\gamma)^3\,(\delta\beta)^3 - (\alpha\beta)^3\,(\gamma\delta)^3\} =$$
$$= \tfrac{2}{9}\{J_{AC}\,J_{DB} - (ac)^3\,(db)^3\,(\alpha\beta)^3\,(\gamma\delta)^3 - (ab)^3\,(cd)^3\,(\alpha\gamma)^3\,(\delta\beta)^3 + J_{AB}\,J_{CD}\}$$

$$[L_1^{(3)}\,A_2^{(3)}] = \tfrac{1}{9}\{(ac)^3\,(db)^3\,(a\delta)^3\,(\beta\gamma)^3 - J_{AC}\,J_{DB} - (ab)^3\,(cd)^3\,(a\delta)^3\,(\beta\gamma)^3 +$$
$$+ (ab)^3\,(cd)^3\,(\alpha\gamma)^3\,(\delta\beta)^3\}$$

$$[L_2^{(3)}\,A_1^{(3)}] = \tfrac{1}{9}\{(ad)^3\,(bc)^3\,(\alpha\gamma)^3\,(\delta\beta)^3 - (ad)^3\,(bc)^3\,(\alpha\beta)^3\,(\gamma\delta)^3 - J_{AC}\,J_{DB} +$$
$$+ (ac)^3\,(db)^3\,(\alpha\beta)^3\,(\gamma\delta)^3\}$$

$$2\,[L_2^{(3)}\,A_2^{(3)}] = \tfrac{2}{9}\{J_{AD}\,J_{BC} - (ad)^3(bc)^3(\alpha\gamma)^3(\delta\beta)^3 - (ac)^3(db)^3(a\delta)^3(\beta\gamma)^3 + J_{AC}\,J_{DB}\}$$

dus

$$I_{13} = \tfrac{1}{2^7}\{6\,J_{ABCD} - (ab)^3(cd)^3(\alpha\gamma)^3(\delta\beta)^3 - (ab)^3(cd)^3(a\delta)^3(\beta\gamma)^3 - (ac)^3(db)^3(a\delta)^3(\beta\gamma)^3 +$$
$$- (ac)^3(db)^3(\alpha\beta)^3(\gamma\delta)^3 - (ad)^3(bc)^3(\alpha\beta)^3(\gamma\delta)^3 - (ad)^3(bc)^3(\alpha\gamma)^3(\delta\beta)^3\}$$

en volgens § 5 (5)

$$I_{13} = \tfrac{1}{2^7}\{6\,J_{ABCD} - (36\,[L^{(2)}\,A^{(2)}] - J_{AB}\,J_{CD} - J_{BC}\,J_{AD} - J_{CA}\,J_{BD})\} =$$
$$= \tfrac{1}{2^7}\{6\,J_{ABCD} - (36\,I_2 - 3\,J_{ABCD})\} = \tfrac{1}{2^7}\{9\,J_{ABCD} - 36\,I_2\}$$

dus

$$I_{13} = \tfrac{1}{3}\,J_{ABCD} - \tfrac{4}{3}\,I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Zijn de grondvormen aequivalent, dan gaat J_{AB} over in

$$J_1 = 2\,(A_{00}\,A_{33} - 3\,A_{10}\,A_{23} + 3\,A_{20}\,A_{13} - A_{30}\,A_{03} - 3\,A_{01}\,A_{32} +$$
$$+ 9\,A_{11}\,A_{22} - 9\,A_{21}\,A_{12} + 3\,A_{31}\,A_{02}). \quad (10)$$

Uit J_{ABCD} ontstaat dan

$$J = J_1^2$$

zodat I_{13} overgaat in

$$I_{13}^* = \tfrac{1}{3}\,J_1^2 - \tfrac{4}{3}\,I_2^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

§ 11. Onder de i^{de} *overschuiving* van twee binaire gemengde vormen

$$F = a_x^p\,\alpha_\xi^q \quad \text{en} \quad G = b_x^r\,\beta_\xi^s$$

zal men verstaan

$$(F, G)^{(i)} = (ab)^i\,(\alpha\beta)^i\,a_x^{p-1}\,b_x^{r-1}\,\alpha_\xi^{q-1}\,\beta_\xi^{s-1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i is ten hoogste gelijk aan de kleinste der exponenten p, q, r, s. Deze overschuivingen zijn blijkbaar covarianten.

De eerste overschuiving hangt samen met de determinanten van partieel-afgeleiden. Men vindt

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x_1\,\partial \xi_1} & \frac{\partial^2 F}{\partial x_1\,\partial \xi_2} \\ \frac{\partial^2 G}{\partial x_2\,\partial \xi_1} & \frac{\partial^2 G}{\partial x_2\,\partial \xi_2} \end{vmatrix} = \begin{vmatrix} pq\,a_x^{p-1}\,\alpha_\xi^{q-1}\,a_1\,\alpha_1 & pq\,a_x^{p-1}\,\alpha_\xi^{q-1}\,a_1\,\alpha_2 \\ rs\,b_x^{r-1}\,b_\xi^{s-1}\,b_2\,\beta_1 & rs\,b_x^{r-1}\,\beta_\xi^{s-1}\,b_2\,\beta_2 \end{vmatrix} =$$
$$= pqrs\,a_x^{p-1}\,b_x^{r-1}\,\alpha_\xi^{q-1}\,\beta_\xi^{s-1}\,a_1\,b_2\,(\alpha\beta)$$

en evenzo

$$\begin{vmatrix} \frac{\partial^2 G}{\partial x_1\,\partial \xi_1} & \frac{\partial^2 G}{\partial x_1\,\partial \xi_2} \\ \frac{\partial^2 F}{\partial x_2\,\partial \xi_1} & \frac{\partial^2 F}{\partial x_2\,\partial \xi_2} \end{vmatrix} = r\,s\,p\,q\,b_x^{r-1}\,a_x^{p-1}\,\beta_\xi^{s-1}\,\alpha_\xi^{q-1}\,b_1\,a_2\,(\beta\,\alpha)$$

dus

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x_1\,\partial \xi_1} & \frac{\partial^2 F}{\partial x_1\,\partial \xi_2} \\ \frac{\partial^2 G}{\partial x_2\,\partial \xi_1} & \frac{\partial^2 G}{\partial x_2\,\partial \xi_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial^2 G}{\partial x_1\,\partial \xi_1} & \frac{\partial^2 G}{\partial x_1\,\partial \xi_2} \\ \frac{\partial^2 F}{\partial x_2\,\partial \xi_1} & \frac{\partial^2 F}{\partial x_2\,\partial \xi_2} \end{vmatrix} =$$

$$= p\,q\,r\,s\,(ab)\,(\alpha\beta)\,a_x^{p-1}\,b_x^{r-1}\,\alpha_\xi^{q-1}\,\beta_\xi^{s-1} = p\,q\,r\,s\,(F,G)^{(1)} \quad . \quad . \quad . \quad (2)$$

Naar men weet, is de discriminant R van de enkelvoudige binaire cubische vorm a_x^3 te beschouwen als de tweede overschuiving van $\triangle_x^2$ met zichzelf, waarbij $\triangle_x^2$ weer de tweede overschuiving van a_x^3 met zichzelf is. Naar analogie hiermede kan men de volgende tweede overschuivingen van de vier gemengde cubische vormen

$$F_1 = a_x^3\,\alpha_\xi^3 \qquad F_2 = b_x^3\,\beta_\xi^3 \qquad F_3 = c_x^3\,\gamma_\xi^3 \qquad F_4 = d_x^3\,\delta_\xi^3 \quad . \quad . \quad (3)$$

berekenen.

$$(F_1, F_2)^{(2)} = (ab)^2\,(\alpha\beta)^2\,a_x\,b_x\,\alpha_\xi\,\beta_\xi = p_x^2\,\pi_\xi^2 = P$$
$$(F_3, F_4)^{(2)} = (cd)^2\,(\gamma\delta)^2\,c_x\,d_x\,\gamma_\xi\,\delta_\xi = s_x^2\,\sigma_\xi^2 = S.$$

Hierin is

$$P^{ik,\,lm} = \tfrac{1}{4}\,(ab)^2\,(\alpha\beta)^2\,(a^i\,b^k + a^k\,b^i)\,(a^l\,\beta^m + a^m\,\beta^l).$$

De tweede overschuiving van P en S wordt

$$H_E = (P,S)^{(2)} = (ps)^2\,(\pi\sigma)^2 = (ab)^2\,(\alpha\beta)^2\,(as)\,(bs)\,(a\sigma)\,(\beta\sigma) =$$
$$= \tfrac{1}{4}(ab)^2(\alpha\beta)^2(cd)^2(\gamma\delta)^2\{(ac)(bd)(\alpha\gamma)(\beta\delta) + (ac)(bd)(a\delta)(\beta\gamma) +$$
$$+ (ad)\,(bc)\,(\alpha\gamma)\,(\beta\delta) + (ad)\,(bc)\,(a\delta)\,(\beta\gamma)\} =$$
$$= \tfrac{1}{4}\,(L_E + L_D)\,(\Lambda_E + \Lambda_D) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Deze invariant is niet symmetrisch in de coëfficiëntenrijen der vier grondvormen (3).

Nu kan men deze laatsten op drie manieren twee aan twee combineren, en vindt dan

$$H_E = ((F_1, F_2)^{(2)}, (F_3, F_4)^{(2)})^{(2)} = \tfrac{1}{4}\,(L_E + L_D)\,(\Lambda_E + \Lambda_D)$$
$$H_A = ((F_2, F_3)^{(2)}, (F_1, F_4)^{(2)})^{(2)} = \tfrac{1}{4}\,(L_A + L_C)\,(\Lambda_A + \Lambda_C)$$
$$H_B = ((F_3, F_1)^{(2)}, (F_2, F_4)^{(2)})^{(2)} = \tfrac{1}{4}\,(L_B + L_F)\,(\Lambda_B + \Lambda_F)$$

Blijkbaar is

$$H = \tfrac{1}{8}\,(H_E + H_A + H_B) =$$
$$= \tfrac{1}{12}\{(L_E + L_D)\,(\Lambda_E + \Lambda_D) + (L_A + L_C)(\Lambda_A + \Lambda_C) + (L_B + L_F)(\Lambda_B + \Lambda_F)\} \quad (5)$$

een *symmetrische* invariant. Deze willen we uitdrukken in de primitief-symmetrische invarianten $I_1, \ldots\ldots, I_{16}$.

Volgens § 4 (11) vindt men

$$L_E + L_D = 2\,L^{(1)} \quad + L_2^{(3)}$$
$$L_A + L_C = 2\,L^{(1)} + L_1^{(3)}$$
$$L_B + L_F = 2\,L^{(1)} - L_1^{(3)} - L_2^{(3)}$$

en dus volgens § 9 (6) (7) (8)

$$(2\,L^{(1)} + L_2^{(3)})\,(2\,A^{(1)} + A_2^{(3)}) = 4\,I_1 + 2\,I_7 + 2\,I_8 + L_2^{(3)}\,A_2^{(3)}$$

$$(2\,L^{(1)} + L_1^{(3)})\,(2\,A^{(1)} + A_1^{(3)}) = 4\,I_1 + 2\,I_5 + 2\,I_6 + L_1^{(3)}\,A_1^{(3)}$$

$$(2\,L^{(1)} - L_1^{(3)} - L_2^{(3)})\,(2\,A^{(1)} - A_1^{(3)} - A_2^{(3)}) = 4\,I_1 - 2\,I_5 - 2\,I_7 - 2\,I_6 - 2\,I_8 +$$
$$+ L_1^{(3)}\,A_1^{(3)} + L_1^{(3)}\,A_2^{(3)} + L_2^{(3)}\,A_1^{(3)} + L_2^{(3)}\,A_2^{(3)}.$$

dus volgens § 9 (11)

$$H = \tfrac{1}{12}\,(12\,I_1 + 3\,I_{13}) = I_1 + \tfrac{1}{4}\,I_{13} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Op dezelfde wijze kan men de symmetrische invariant

$$U = \tfrac{1}{6}\,(L_E\,A_E + L_A\,A_A + \ldots. + L_F\,A_F) \quad . \quad . \quad . \quad . \quad (7)$$

in de primitief-symmetrische invarianten $I_1, \ldots\ldots, I_{16}$ uitdrukken en vindt dan volgens § 4 (11)

$$U = \tfrac{1}{6}\,(6\,L^{(1)}\,A^{(1)} + 6\,L^{(2)}\,A^{(2)} + 4\,L_1^{(3)}\,A_1^{(3)} + 4\,L_2^{(3)}\,A_2^{(3)} + 2\,L_1^{(3)}\,A_2^{(3)} + 2\,L_2^{(3)}\,A_1^{(3)})$$

en volgens § 9 (6) (11)

$$U = I_1 + I_2 + I_{13}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Anatomy. — *A new entoptic phenomenon.* By H. JANSSEN. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of March 30, 1946.)

When regarding a light formed by a small circular hole in a screen, this aperture looks like a star as a result of the irregular lens-astigmatism instead of retaining the perfectly circular form as we might expect.

The immediate surroundings of the light appear to be luminous to the extent of approximately 7°. This light-zone is related to the pupil; its size varies in accordance with the constriction or dilatation of the iris. The structure of the light-zone is not homogeneous; its appearance depends to a certain extent on the intensity of the source of light. On close observation we may perceive in it a bundle of very thin rays which forms a usually horizontal propeller-formed glow, standing out against the dark background (fig. 1).

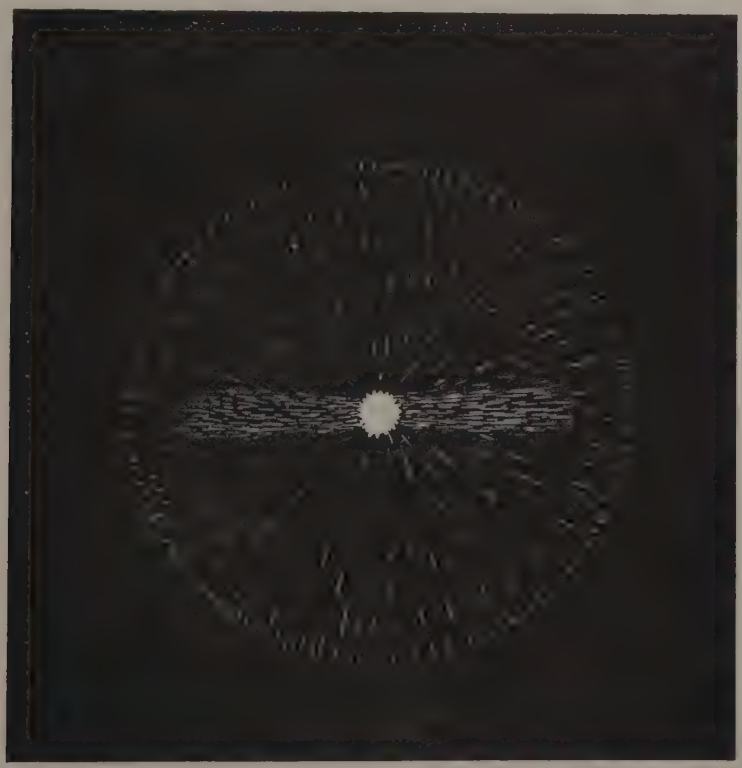

Fig. 1.

If we do not observe the "lightpropeller" — as I should like to call the phenomenon for shortness sake — or do not observe it clearly, its presence can make itself manifest if we move the head round a sagittal axis. Once having become conscious of the effect, one can observe it as a companion of any circumscript light, even of the moon.

The phenomenon can also be perceived with one eye. Some see with each of both eyes a propeller, which form an angle with each other; in that case the covering of one of the eyes is recommended for the following experiments.

However mobile the "lightpropeller" may be, after some time it will always come to rest approximately in the plane defined by our eyes and the light. From this we can conclude that the cause of the effect must be sought in our eyes and not in the medium between the light and the observer.

Two experiments will bring us closer to this cause.

First experiment: We move the head round a sagittal axis, starting for instance with the head on the left shoulder (fig. 2).

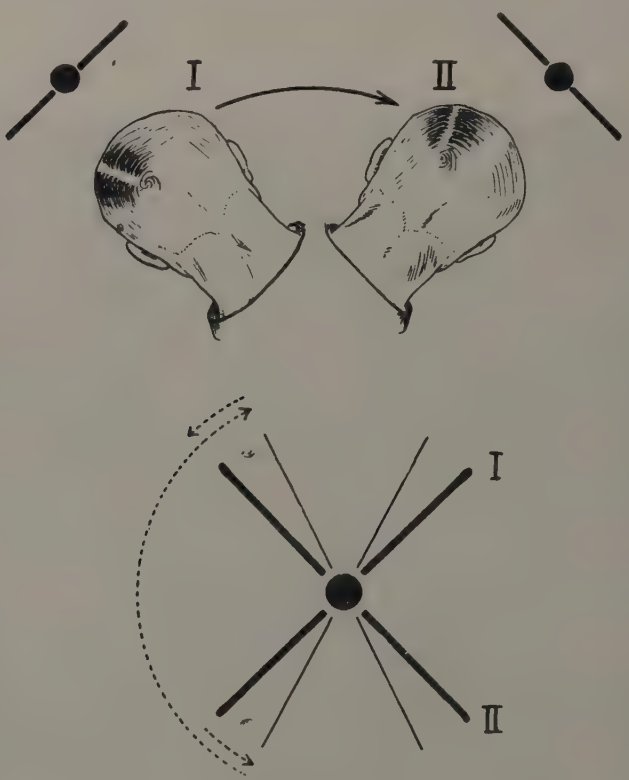

Fig. 2.

The "lightpropeller" lies parallel to the line joining the eyes, viz. from the left bottom-side to the right top-side. Upon moving the head round a sagittal axis, so that it rests upon the right shoulder (always keeping our

eyes fixed on the light), we find that after a few seconds the "lightpropeller" again lies parallel to the line joining the eyes, viz. from the left topside to the right bottom-side.

Supposing that the line joining the eyes has been moved over an angle of 90°, it appears that the "lightpropeller" did more than merely turn from its first position to the final one. At first it turned roughly 15° into the opposite direction, then changing its direction it turns 120°, thus going 15° too far, and finally corrects this excess by reversing 15° (fig. 2).

The first phase occurs during the time in which we move the head quickly from left to right. After the head has taken up its new position we observe the further movements of the "lightpropeller", which take place in a few seconds.

The starlike extensions of the light — caused by the irregular lens-astigmatism — have turned together with our head, without the somewhat slower balancing process of the "lightpropeller". The propeller thus moving independently of the starlike impression cannot find its origin in the lens.

Second experiment: We keep the head upright without moving it.

The light in front of us again appears to have the "lightpropeller". This lies horizontally. Now we direct our eyes to a point at the left hand side of the light, at an angle of say 45°. Then suddenly looking again at the light we observe a digression of the "lightpropeller" of approximately 80° in a negative direction. In the course of 5—7 seconds it slowly turns back again over these 80° in a positive direction (fig. 3).

Approaching the light from the right we find the opposite taking place.

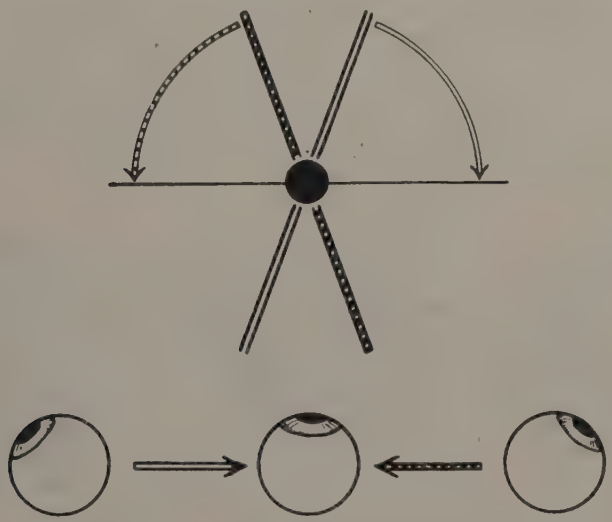

Fig. 3.

* * *

The first experiment proved that the cause of the phenomenon was not to be found in the lens.

The fact that in the second experiment the "lightpropeller" turned over an angle of approximately 80° excludes the cornea as well, since it is out of the question that the cornea, and consequently the eye itself, makes such a rotatory movement.

Nor can the liquid of the anterior chamber of the eye be held responsible for the occurrence of the propeller-phenomenon. In normal circumstances this liquid is homogeneous and has the same refraction in all directions.

As it is unlikely that the retina does anything more than play a purely perceptive part in the "propeller-effect", the only possible conclusion we can draw is, that the cause of the phenomenon must be sought in the corpus vitreum.

That it is not caused by other influences, such as irritation of the semicircular canals or deformation of the eyeball by the exterior eyemuscles, appears from the fact that the "propeller" remains stable if we keep the eyes intent upon the light and make yes- and no-movements.

We may take it for granted from the slow swinging of the "lightpropeller" that the corpus vitreum — wholly or partly — follows the movements of the eye only with a certain inertia. A supposition that may be confirmed by such "muscae volitantes" that drift slowly to a point of rest after a movement of the eyes.

I am unable to give a physical explanation of the phenomenon. The structure of the corpus vitreum might indicate from which angle the explanation of the propeller-phenomenon should be sought.

The corpus vitreum — a gel — consists of a stroma of fine fibrils forming a spongelike network with meshes filled with clear liquid. After the first few years of life the stroma begins to sag. This can be demonstrated by the lowering of the hyaloid canal of CLOQUET (fig. 4) as is shown by the investigations of IDA MANN (J. of Anatomy **62**, 290—296, 1928).

Fig. 4. Relations of CLOQUET's canal, *A.* at birth, *B.* in the adult with the eye at rest, *C.* in the adult after sudden movement upwards. (After MANN.)

In my opinion the propeller-phenomenon is closely related to the stroma of the corpus vitreum. By sudden upward movements of the eye the stroma

floats up and CLOQUET's canal assumes for a second or so its foetal position (fig. 4). It is important to note that for these few seconds the "lightpropeller" cannot be observed.

It is worth considering whether the eye-movements round a vertical axis make the bottom part of the corpus vitreum sway too far, as a result of its greater density. This might explain the behaviour of the propeller in the second experiment.

The mobility of the corpus vitreum would thus be stressed by the wide swinging movement of the propeller. This behaviour might be of interest to oculists, who might wonder whether rapid movements of the eye provoke a tear and a detachment of the retina in a predisposed eye.

Medicine. — *On the probable mode of action of thiourea and thiouracil on the thyroid function.* By P. FORMIJNE. (Communicated by Prof. A. DE KLEIJN.)

(Communicated at the meeting of March 30, 1946.)

Preliminary communication.

It has been shown by many investigators that the function of the thyroid can be profoundly influenced by the administration of thiourea, thiouracil and other related compounds.

The molecular combination of thiourea is present in all these compounds. They are very effective for the treatment of GRAVES' disease, as has first been shown by ASTWOOD [1]). In these cases a profound lowering of the basal metabolism, even to subnormal values, can be observed. Regarding the mode of action, different hypotheses have been suggested without actual proof. Some investigators have suggested the possibility, that the synthesis in vivo of thyroxin in the thyroid is disturbed by these compounds.

It has been shown, especially by ABELIN [2]), that the addition of iodine to proteins can produce substances with the biological action of thyroxin. LUDWIG and VON MUTZENBECHER [3]) have shown that under special conditions the treatment of protein with iodine can lead to highly active substances from which pure thyroxin can be isolated.

In the present investigation the action of thiourea on the in vitro formation of thyroxin was analysed.

If iodine is added to a solution of protein containing thiourea, it can be seen that the iodine disappears much more rapidly in this solution than in a control solution without thiourea. If these solutions are hydrolysed with bariumhydroxyd, the acid-unsoluble fraction, which contains the thyroxin, can be isolated. The thyroxin-content of this fraction can be determined by the butylalcohol-extraction method of LELAND and FOSTER [4]). It was found, that a marked amount of thyroxin-iodine was present in the control solution, while in the solution with thiourea only traces of thyroxin-iodine could be detected.

Description of experiment.

To 10 gms soluble casein were added 3 gms of sodium bicarbonate, 400 cc destilled water and 3 gms of thiourea (Solution *A*). In the control

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[1]) ASTWOOD, J. Am. Med. Assoc. **122**, 78 (1943).
[2]) ABELIN, Klin. Woschrift **13**, 940 (1934).
[3]) LUDWIG und VON MUTZENBECHER, Zeitschrift f. Physiol. Chemie **258**, 195 (1939).
[4]) LELAND and FOSTER, J. Biol. Chem. **95**, 165 (1932).

solution 10 gms of soluble casein, 3 gms of sodium bicarbonate were solved in 400 cc destillated water and 3 gms of urea added. (Sol. *B*)

Hereupon 20 cc of Lugol-solution, containing 1 gm of iodine, were slowly added to each of these solutions. Afterwards the iodated protein was isolated and hydrolysed with 60 gms $(Ba(OH)_2$ and 120 cc of water during 18 hours.

The acid-unsoluble fraction was isolated, solved in normal NaOH solution and shaken with an equal volume of butylalcohol.

The butylalcohol was separated, shaken again with an equal volume of normal NaOH and again separated. The last washing with normal NaOH was again shaken with half a volume of butylalcohol and the butylalcohol-extracts were combined; each of these butylalcohol-extracts contained the thyroxin of solution *A* and *B*.

The iodine content of these solutions was determined. The thyroxin-iodine of control solution *B* (with urea) was 32.26 mg, the thyroxin-iodine of solution *A* (with thiourea) was 3.1 mg.

These figures demonstrate that the formation of thyroxin by iodinisation of protein in vitro can be inhibited by thiourea. It was found, that thiouracil had a similar action.

If iodine in the form of Lugol-solution is added to a solution of urea nothing seems to happen. If Lugol-solution is added to a solution of thiourea the iodine is bound immediately and the yellow colour disappears rapidly. The quantitative relations of this reaction and the mode of binding of the iodine by thiourea are being investigated further.

A simple explanation for the action of thiourea can now be given: The thyroidcell has the property of liberating free iodine, which combines with protein to iodised compounds which contain thyroxin. In the presence of thiourea this reaction cannot take place because the liberated iodine is caught away by the thiourea and cannot combine with the protein.

It is therefore possible that the physiological activity of thiourea and related compounds is determined by two factors: The capacity for the rapid binding of free iodine and the property of entering into the thyroidcells.

Applied Mechanics. — *The convergence of a specialized iterative process in use in structural analysis.* By C. B. BIEZENO and O. BOTTEMA.

(Communicated at the meeting of April 27, 1946.)

1. *Introduction.* In recent years so-called "relaxation"-methods have been developed which have completely altered the treatment of a great number of technically important structures, such as redundant girders, redundant frame works, trusses of "Vierendeel"-construction etc. [1])

All of them are iterative methods, the convergence of which — if at all considered — is generally discussed from the energetic point of view [2]). The iterative method, under consideration here, will be investigated in a different way. It restricts itself to such two-dimensional frameworks the joints of which are only liable to rotations, and for the rest are fixed in space. The joints of the frame work may be denoted by A_i $(i = 1, 2 \ldots n)$; the length of the member joining the points A_i and A_j may be indicated by $l_{ij} = l_{ji}$ and it may be prearranged by definition that $l_{ij} = 0$ if the points A_i and A_j are unconnected and that — in agreement with this definition — $l_{ii} = 0$.

If the structure is loaded it will be brought into its "natural" state of stress and all joints A_i will rotate. If the rotations are prevented by suitable moments M_i applied to the joints A_i, the construction will be said to be "locked". It is supposed that the state of stress can easily be calculated for this fully "locked" structure. Then the problem to be solved consists in calculating all stresses which occur if the structure is "unlocked". It is this unlocking of the structure which is effectuated in degrees and in an iterative way. The locking moments, which prevail after the k'th iteration will be designated by $M_i(k)$. The procedure of iteration once being fixed, it is our aim to show that all locking moments $M_i(k)$ $(i = 1, 2 \ldots n)$ tend to zero if k tends to infinity. Obviously it must be required for the sake of practical applicability that this procedure of iteration be such that the change in state of stress, implicated by one single iteration, can easily be calculated.

2. *The iterative method.* Let the iteration be carried out k times, so that the locking moments are $M_i(k)$ $(i = 1, 2 \ldots n)$ and let the joint A_i alone be unlocked. Then A_i must be loaded with $-M_i(k)$. All joints connected with A_i will be influenced by this unloading in a way which depends upon the distribution of $-M_i(k)$ over all members by which A_i

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[1]) These methods originate from an article by HARDY CROSS. Analysis of Continuous Frames by Distributing Fixed-End Moments. Proc. Amer. Soc. C.E. 96, 1—156 (1932), and have been developed mainly by R. V. Southwell and his disciples; comp. R. V. Southwell, Relaxation Methods in Engineering Science; a treatise on approximate computation, Oxford University Press 1940.

[2]) See f.i. G. TEMPLE, The general theory of relaxation-methods applied to linear systems. Proc. Royal Soc. of London, Vol. CLXIX. 1939.

is connected with other joints A_j. This distribution can be described by attributing to every member A_iA_j a certain "elastic weight" α_{ij} which will be specified later on. The portion of $-M_i(k)$ for which the member A_iA_j stands, will then be represented by

$$-\frac{\alpha_{ij}}{\sum_h \alpha_{ih}} M_i(k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(where the summation over h may be extended over all values $h = 1, 2 \ldots n$, if only α_{ih} is put equal to zero for every h for which $l_{ih} = 0$).

The "extra" locking-moment, caused by the unlocking of A_i, depends solely on the elastic properties of the member A_iA_j and can be calculated by introducing the so-called "coefficient of transition" β_{ij} (which again will be specified later on); it then becomes

$$-\frac{\beta_{ij}\,\alpha_{ij}}{\sum_h \alpha_{ih}} M_i(k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the liquidation of the moments $M_i(k)$ is supposed to take place *simultaneously* for all joints A_i, it is evident that at the point A_i the moment $M_i(k)$ vanished, but is replaced by a moment $M_i(k+1)$ represented by

$$M_i(k+1) = -\sum_j \frac{\beta_{ji}\,\alpha_{ji}}{\sum_h \alpha_{jh}} M_j(k) \qquad (1 = 1, 2 \ldots n) \quad . \quad . \quad . \quad (3)$$

The system (3) is the analytical representation of the iterative method proposed in this paper.

By putting

$$N_i(k) = \frac{M_i(k)}{\sum_h \alpha_{ih}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

it is reduced to the system:

$$N_i(k+1) \cdot \sum_j \alpha_{ij} = -\sum_j \beta_{ji}\,\alpha_{ji}\,N_j(k) \qquad (i = 1, 2 \ldots n) \quad . \quad . \quad (5)$$

3. *The coefficients α_{ij} and β_{ij}.* To calculate the coefficients α_{ij} and β_{ij}, introduced in the preceding section, we want to know what happens if the structure is loaded by a moment M_i at A_i, whilst all other joints A_j $(j \neq i)$ are locked. To this end we consider the member A_{ij} (fig. 1).

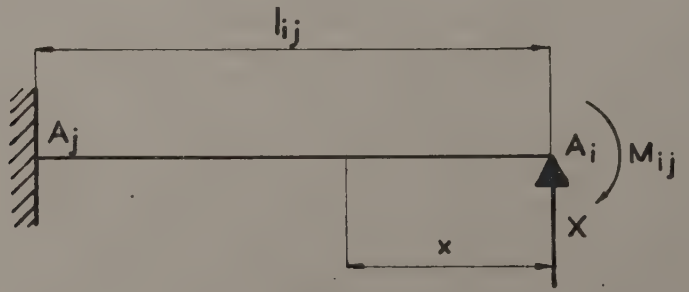

Fig. 1.

clamped at A_j, freely supported at A_i, and loaded by its share M_{ij} of M_i.

If, for abbreviation, we put

$$F_{ij}=\int_{A_i}^{A_j}\frac{dx}{EI},\quad S_{ij}=\int_{A_j}^{A_j}\frac{x\,dx}{EI};\quad I_{ij}=\int_{A_i}^{A_j}\frac{x^2\,dx}{EI};\quad \alpha_{ij}=\frac{I_{ij}}{F_{ij}\,I_{ij}-S_{ij}^2}\ ^{3}) \qquad (6)$$

(where EI represents the local bending rigidity of the member), the reaction X is determined by the equation

$$M_{ij}\,S_{ij}-X\,I_{ij}=0$$

(expressing that the deflection at A_i is zero), and the rotation φ_{ij} at A_i by

$$\varphi_{ij}=M_{ij}\,F_i-X\,S_{ij}=M_{ij}\,\frac{F_{ij}\,I_{ij}-S_{ij}^2}{I_{ij}}=M_{ij}:\alpha_{ij}.$$

Obviously all members l_{ih} by which A_i is connected with other joints A_h, will suffer the same rotation φ_i if A_i is loaded by the moment M_i (whilst all other joints remain fixed) and therefore we have

$$\frac{M_{ij}}{\alpha_{ij}}=\frac{M_{ik}}{\alpha_{ik}}=\ldots=\frac{\sum_h M_{ih}}{\sum_h \alpha_{ih}}=\frac{M_i}{\sum_h \alpha_{ih}}$$

from which it follows that

$$M_{ij}=\frac{\alpha_{ij}}{\sum_h \alpha_{ih}}\,M_i.$$

The quantity

$$\alpha_{ij}=\frac{I_{ij}}{F_{ij}\,I_{ij}-S_{ij}^2} \qquad \ldots\ldots\ldots \qquad (7)$$

defined by (6) therefore is identical with the quantity α_{ij} introduced in sect. 2. As to β_{ij}, the equilibrium of the member A_iA_j (comp. fig. 1) requires a clamping moment at A_j of the magnitude

$$X\,l_{ij}-M_{ij}=\frac{l_{ij}\,S_{ij}-I_{ij}}{I_{ij}}$$

so that

$$\beta_{ij}=\frac{l_{ij}\,S_{ij}-I_{ij}}{I_{ij}} \qquad \ldots\ldots\ldots \qquad (8)$$

It follows from (6) that $\alpha_{ij}>0$, $\beta_{ij}>0$, and furthermore that

$$F_{ji}=F_{ij};\quad S_{ji}=l_{ij}\,F_{ij}-S_{ij}\quad I_{ji}=l_{ij}^2\,F_{ij}-2\,l_{ij}\,S_{ij}+I_{ij}. \qquad (9)$$

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[3]) It may be stated that the coefficients α_{ij} have nothing to do with the influence coefficients of MAXWELL, which ordinarily are denoted in the same way, and to which reference will be made in sect. 7.

Consequently

$$F_{ij}\, I_{ij} - S_{ij}^2 = F_{ji}\, I_{ji} - S_{ji}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$l_{ij}\, S_{ij} - I_{ij} = l_{ji}\, S_{ji} - I_{ji} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

With

$$\frac{1}{\gamma_{ij}} = F_{ij}\, I_{ij} - S_{ij}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$\delta_{ij} = l_{ij}\, S_{ij} - I_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

we therefore have

$$\gamma_{ij} = \gamma_{ji}; \quad \delta_{ij} = \delta_{ji} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\alpha_{ij} = \gamma_{ij}\, I_{ij}; \quad \beta_{ij} = \frac{\delta_{ij}}{I_{ij}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whereas — with due regard to (9), (12) and (13)

$$I_{ij}\, I_{ij} - \delta_{ij}^2 = \frac{1}{\gamma_{ij}} > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

4. *The solution of the equations* (5). With the new notations and with

$$B_i = \sum_j \gamma_{ij}\, I_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the equations (5) become

$$B_i\, N_i(k+1) + \sum_j \delta_{ij}\, \gamma_{ij}\, N_j(k) = 0 \quad (1 = 1, 2 \ldots n) \quad . \quad . \quad (18)$$

In the usual way we put

$$N_i(k) = C_i\, \mu^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

by which the system (18) is replaced by the n homogeneous linear equations in C_i

$$\mu\, B_i\, C_i + \sum_j \delta_{ij}\, \gamma_{ij}\, C_j = 0 \quad (1 = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad (20)$$

admitting a solution, different from zero, only if:

$$D_n(\mu) \equiv \begin{vmatrix} B_1\,\mu & \delta_{12}\,\gamma_{12} & \delta_{13}\,\gamma_{13} \ldots & \delta_{1n}\,\gamma_{1n} \\ \delta_{21}\,\gamma_{12} & B_2\,\mu & \delta_{23}\,\gamma_{23} \ldots & \delta_{2n}\,\gamma_{2n} \\ \vdots & & & \\ \delta_{n1}\,\gamma_{n1} & \ldots\ldots & & B_n\,\mu \end{vmatrix} = 0 \quad . \quad . \quad (21)$$

The determinant $D_n(\mu)$ is symmetrical, and therefore all roots $(\mu_1, \mu_2 \ldots \mu_n)$ of (21) are real, if not necessarily different. Should a p-fold root exist, then the system (20) is reduced to a system of $(n - p)$ linearly independent

equations which admit a system of p linearly independent solutions. In accordance herewith the general solution of (18) is given by

$$N_i(k) = \sum_s a_s C_{is} \mu_s^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

a_s $(s = 1, 2 \ldots n)$ denoting integration-constants computable from the initial values $M_i(0)$ $(i = 1, 2 \ldots n)$. Equations (4) and (22) provide us with the formal means to evaluate the locking moments $M_i(k)$, required after the k'th iteration. It must, however, be emphasized that the proper aim of the iteration method lies in the avoidance of this calculation, and that the practical computation of the successive sets of quantities $M_i(k)$ takes place by making use of the "coefficients of distribution" α_{ij}, and the "coefficients of transition" β_{ij}.

Nevertheless eq. 22 provides us with the criterium of convergence of the iteration methods, which obviously consists in the condition that

$$|\mu_s| < 1 \,.\, (s = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

5. *The convergence of the iteration process.* It will now be shown that the condition (23) is always fulfilled. To this end we draw our attention to one single root μ and the corresponding (non zero)-solution $C_{is}(i=1, 2 \ldots n)$. If, for simplicity's sake C_{is} is replaced by $Q_i(i=1, 2 \ldots n)$ the following relations hold:

$$\mu_s B_i Q_i + \Sigma \delta_{ij} \gamma_{ij} Q_j = 0 \qquad (i = 1, 2 \ldots n) \quad . \quad . \quad . \quad (24)$$

from which we deduce:

$$\left.\begin{aligned} \sum_i Q_i [\mu_s B_i Q_i + \sum_j \delta_{ij} \gamma_{ij} Q_i] = \mu_s \sum_i B_i Q_i^2 + \\ + \sum_{ij} {}_{i \neq j} \delta_{ij} \gamma_{ij} Q_i Q_j = 0 \end{aligned}\right\} . \quad (25)$$

On account of (18) this relation can be transformed into:

$$(\mu_s - 1) \sum_i B_i Q_i^2 + \tfrac{1}{2} \sum_{ij} \gamma_{ij} (I_{ij} Q_i^2 + 2 \delta_{ij} Q_i Q_j + I_{ji} Q_{ij}^2) = 0. \quad (26)$$

The quadratic function

$$I_{ij} Q_i^2 + 2 \delta_{ij} Q_i Q_j + I_{ji} Q_j^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

being positive-definite (comp. 16), and consequently the second sum at the lefthand side of (26) being essentially positive, it is obvious that $(\mu_s - 1)$ is negative and therefore

$$\mu_s < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

At the other hand (25) can also be written as:

$$(\mu_s + 1) \sum_i B_i Q_i^2 - \tfrac{1}{2} \sum_{ij} \gamma_{ij} (I_{ij} Q_i^2 - 2 \delta_{ij} Q_i Q_j + I_{ji} Q_j^2) = 0 \,. \quad (29)$$

from which analogously it follows, that

$$\mu_s > -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

6. *The upper limit of* μ_s. Narrower limits can be given for μ_s by introducing the quantity

$$\beta_{ij}\,\beta_{ji} = \frac{\delta_{ij}^2}{I_{ij}\,I_{ji}} = \frac{\delta_{ij}^2}{\delta_{ij}^2 + \dfrac{1}{\gamma_{ij}}} < 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(comp. (8) and (16)).

The greatest of the numbers $\sqrt{\beta_{ij}\beta_{ji}}$ may be denoted by G, so that

$$\sqrt{\beta_{ij}\,\beta_{ji}} \leqq G < 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Then it follows from (31) and (32) that

$$(1-G^2)\,\delta_{ij}^2 = \frac{G^2}{\gamma_{ij}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Strictly imitating the trend of thought developed in the preceding section we now bring (25) into the form:

$$(\mu_s - G)\sum_i B_i\,Q_i^2 + \tfrac{1}{2}\sum_{ij}(G\,I_{ij}\,Q_i^2 + 2\,\delta_{ij}\,Q_i\,Q_j + G\,I_{ji}\,Q_j^2) = 0. \quad (34)$$

The discriminant of the quadratic polynom

$$G\,I_{ij}\,Q_i^2 + 2\,\delta_{ij}\,Q_i\,Q_j + G\,I_{ji}\,Q_j^2$$

amounts to

$$\delta_{ij}^2 - G^2 I_{ij}\,I_{ji} = \delta_{ij}^2 - G^2\left(\delta_{ij}^2 + \frac{1}{\gamma_{ij}}\right) = (1-G^2)\,\delta_{ij}^2\,\frac{G^2}{\gamma_{ij}}$$

which certainly is not negative (comp. 33).

It follows (comp. sect. 4) that

$$\mu_s \leqq G\,(= \max \sqrt{\beta_{ij}\,\beta_{ji}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and, — in replacing in equation (34) G by $-G$ — analogously, that

$$\mu_s \geqq -G\,(= -\max \sqrt{\beta_{ij}\,\beta_{ji}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If all members A_iA_j of the structure are prismatic, all coefficients β_{ij} are equal to $\frac{1}{2}$. In this case

$$\mu_s \leqq \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and a rapid convergence of the iteration process is guaranteed. From the fact, that constructions can be indicated for which one of the characteristic roots is equal to $\frac{1}{2}$ (f.i. a continuous beam supported in n points, having in every span a constant factor of rigidity EI) it follows that the limit G in (35) and (36) can be reached.

7. *Second proof. The conjugated influence coefficients of* MAXWELL.

A second method of attack of the problem is furnished by the so called "conjugated" influence numbers of MAXWELL. Just as the ordinary numbers

α_{ij} of MAXWELL are suited to express the displacements in prescribed directions in those points of an elastic body for which the α_{ij} are defined, and in which the loading forces may act in the prescribed directions, the conjugated numbers a_{ij} provide the obvious means to calculate in a formal way, the force required to act in prescribed directions in those points of an elastic body for which the a_{ij} are defined and in which prescribed displacements in the given directions occur (forces to be taken in the general sense of forces or moments, directions in the general sense of directions or axes of rotation). Taking in view a number of points n, the number a_{ij} is defined as the force (the moment) required to act in the point i in the prescribed direction (about the prescribed axis) if the point j is subjected to a unit displacement in the prescribed direction (or about the prescribed axis) connected with the point j and if all other points are locked (as to prevent a displacement in the prescribed direction in those points) [4].

With respect to the problem under consideration we define a_{ij} as the moment required to act externally at the joint i, when the j'th point is subject to a unit-rotation (caused by a suitably chosen moment) and if all other joints are locked.

Let in the natural state of loading, the rotations occurring in the joints $1, 2 \ldots n$ be designated by $\varphi_1, \varphi_2 \ldots \varphi_n$, then the locking moments required to lock *all* joints simultaneously, are represented by

$$\left.\begin{aligned} M_1 &= -[a_{11}\varphi_1 + a_{12}\varphi_2 + \ldots a_{1n}\varphi_n] \\ M_2 &= -[a_{21}\varphi_1 + a_{22}\varphi_2 + \ldots a_{2n}\varphi_n] \\ &\vdots \\ M_n &= -[a_{n1}\varphi_1 + a_{n2}\varphi_2 + \ldots a_{nn}\varphi_n] \end{aligned}\right\} \quad \ldots\ldots \quad (38)$$

(To fix our thoughts it may be supposed that the clockwise rotation — both in respect to displacement and moment — is positive.)

From the mechanical point of view the iteration method developed in the preceding sections, consisted in that every joint in its turn is unlocked with the intention to liquidate the prevailing external moment to an amount which is prescribed by the foregoing iteration. Thereafter the joint in question is re-locked and it must be emphasized that, when the k'th iteration has been performed, with the purpose to liquidate the external moments required after the $(k-1)$th iteration, there still will be to liquidate new external moments due to the fact that he unlocking of a certain joint (i) has a secondary effect upon all those joints with which the joint i is materially (and directly) related.

It is obvious, that the successive unlocking of $M_1, M_2 \ldots M_n$ gives rise to the following positive rotations at the joints (i):

$$-M_1 : a_1 ; -M_2 : a_2 ; \ldots \quad \ldots\ldots \quad (39)$$

[4]) Between the ordinary coefficients α_{ij} and the conjugated numbers a_{ij} simple relations do exist. Comp. f.i. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik (Springer, Berlin 1939, p. 95).

and therefore the angles $\varphi_1^1, \varphi_2^1 \ldots \varphi_n^1$ through which the joints have to be rotated if it is required to bring the construction from the natural state of load (in which the joints $1, 2 \ldots n$ are supposed to have rotated $\varphi_1, \varphi_2 \ldots \varphi_n$ in the positive direction) into the state of deformation acquired after the first liquidation of moments, amount to:

$$\varphi_1^1 = \varphi_1 + \frac{M_1}{a_1}; \quad \varphi_2^1 = \varphi_2 + \frac{M_2}{a_{22}}; \quad \ldots \varphi_n^1 = \varphi_n + \frac{M_n}{a_{nn}} \ldots \quad (40)$$

The external locking moments required to effectuate this forced deformation could be calculated with the aid of (38) if only in these equations $\varphi_1 \varphi_2 \ldots \varphi_n$ are replaced by $\varphi_1^1, \varphi_2^1 \ldots \varphi_n^1$. The external locking moments required to effectuate this forced deformation could be calculated with the aid of (38) if only in these equations $\varphi_1, \varphi_2 \ldots \varphi_n$ are replaced by $\varphi_1^1, \varphi_2^1 \ldots \varphi_n^1$. We refrain from this calculation but ask for such "characteristic" sets φ_1 for which the iterationproces would lead to the exceptional result, that the iterated values φ_1^1 and the original values φ_i would be proportional.

This requirement gives rise to the condition

$$\frac{\varphi_1 + M_1 : a_{11}}{\varphi_1} = \frac{\varphi_2 + M_2 : a_{22}}{\varphi_2} = \ldots = \frac{\varphi_n + M_n : a_{nn}}{\varphi_n} = \mu \ldots (41)$$

which in connection with (38) leads to the system of homogeneous linear equations

$$\left.\begin{aligned} \mu a_{11} \varphi_1 + a_{11} \varphi_2 + \ldots a_{1n} \varphi_n &= 0 \\ a_{21} \varphi_1 + \mu a_{22} \varphi_2 + \ldots a_{2n} \varphi_n &= 0 \\ \vdots & \\ a_{n1} \varphi_1 + a_{n2} \varphi_2 + \ldots \mu a_{nn} \varphi_n &= 0 \end{aligned}\right\} \ldots \ldots \quad (42)$$

This system has solutions different from zero only if:

$$D_n(\mu) = \begin{vmatrix} \mu a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & \mu a_{22} \ldots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \ldots & \mu a_{nn} \end{vmatrix} = 0 \ldots \ldots \quad (43)$$

In consequence of the fact that $a_{ij} = a_{ji}$, the roots of this equation are real and it may be stated without further proof (comp. sect. 3) that there exist n linearly independent characteristic solutions $\varphi_1^k, \varphi_2^k \ldots \varphi_n^k$ $(k = 1, 2 \ldots n)$ which, — if iterated — give rise to $\mu_k \varphi_1^k, \mu_k \varphi_2^k \ldots \mu_k \varphi_n^k$. It can easily be proved that an arbitrary set of values $\varphi_1, \varphi_2 \ldots \varphi_n$ as occurring in the natural loaded state of our structure, can always be decomposed into multiples of these n characteristic functions so that:

$$\varphi_i = C_1 \varphi_i^1 + C_2 \varphi_i^2 + \ldots C_n \varphi_i^n; \qquad (i = 1, 2 \ldots n) \ldots \quad (44)$$

and therefore it is readily seen that the iteration process, as applied to

$\varphi_1, \varphi_2 \ldots \varphi_n$ will converge and only will converge if all roots μ_k of (43) satisfy the condition

$$-1 < \mu_k < 1 \ldots\ldots\ldots\ldots \quad (45)$$

After a well-known theorem a necessary and sufficient condition to fulfill this requirement is provided by the series

$$\Delta_n(\mu) \quad \Delta_{n-1}(\mu) \ldots \Delta_1(\mu) \quad K \ldots\ldots\ldots \quad (46)$$

(K representing any positive constant number), which constitutes a so-called series of "Sturm"-functions for equation (43).

The loss in variations occurring when in this series two numbers μ_1 and μ_2 are substituted ($\mu_1 < \mu_2$) is equal to the number of real roots μ_k lying between μ_1 and μ_2. By choosing $\mu_1 = -1$, and $\mu_2 = +1$, this loss (after 45) must be equal to n, and consequently it remains to be proved that (46) has n variations for $\mu = -1$, and no variations at all for $\mu = +1$. The latter condition is obviously fulfilled, as after a well known property of the influence coefficients every determinant

$$D_n = \begin{vmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \ldots & a_{nn} \end{vmatrix} \ldots\ldots\ldots \quad (47)$$

is essentially positive. The other condition is more intricate. It requires the proof of the unequality

$$D'_n = \begin{vmatrix} -a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & -a_{22} \ldots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \ldots & -a_{nn} \end{vmatrix} \gtrless 0 \text{ for } \begin{matrix} n \text{ odd} \\ n \text{ even} \end{matrix} \ldots\ldots \quad (48)$$

We will give it for two special cases.

8. *The continuous girder supported in n fixed points.* In this case the numbers a_{ij} represent the locking moments which must be introduced at the supports i ($\neq j$) if at the support j a unit-rotation is called into being. Evidently all numbers a_{ij} ($i \neq j$) for which $|i-j| > 1$ are zero. Taking this into account we find, by multiplying the odd columns and the even rows of D'_n (comp. 48) by (-1):

$$D'_n = \begin{vmatrix} -a_{11} & a_{12} & 0 \ldots 0 \\ a_{21} & -a_{22} & a_{23} \ldots 0 \\ \vdots & & \\ 0 & 0 \ldots & a_{n,n-1} - a_{nn} \end{vmatrix} = (-1)^n \begin{vmatrix} a_{11} & a_{12} & 0 & \ldots 0 \\ a_{21} & a_{22} & a_{23} \ldots & 0 \\ \vdots & & & \\ 0 & 0 \ldots & a_{n,n-1} & a_{nn} \end{vmatrix} = (-1)^n D_n \quad (49)$$

and therefore — D_n being pos. — condition (48) is fulfilled.

9. *Trusses of Vierendeelconstruction.* A similar proof holds for constructions as represented in fig. 2.

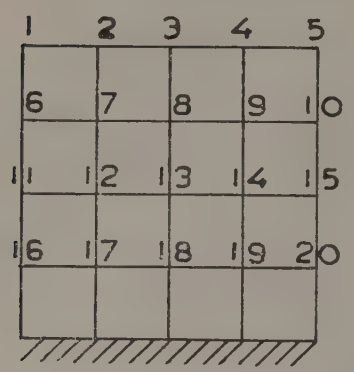


Fig. 2.

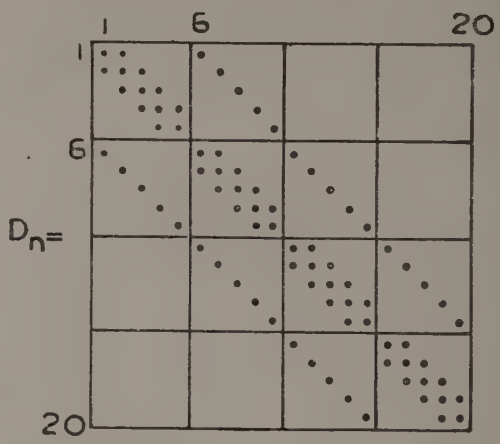


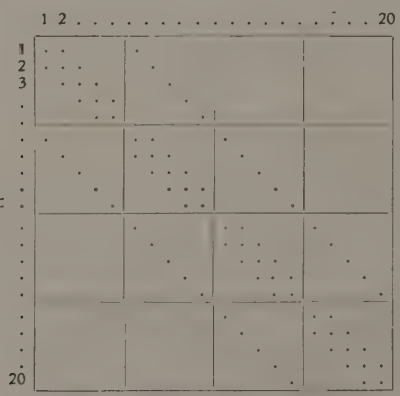
Fig. 3.

It is supposed that the different joints are only liable to rotations. Again it has to be proved that

$$D'_n = (-1)^n D_n$$

and again the proof is simplified by the fact that a great number of influence coefficients is zero; this being the consequence of the fact that only the joints $(i-1)$, i, $i+1$, $i-p$ and $i+p$ (p representing the number of joints per row) are effected by a moment acting at the joint i (all other joints being locked). The determinant D_n hereby takes the schematic form

$D_n =$



. . . . . (50)

every point denoting a non-zero coefficient. By multiplying in a suitable way the rows and columns of the corresponding determinant D'_n by a factor -1, it is easily seen that $D'_n = (-1)^n D_n$.

10. *Concluding remark.* The proof given in the preceding sections, and relating to two specified (be it) important structures, is not a general one,

and is essentially based upon the fact that a great number of coefficients a_{ij} is zero. Unsatisfactory as it may be, it seems rather troublesome to give the direct general proof of the relation (48).

It will however not have escaped to the notice of the reader that the determinants (21) and (23) are essentially the same. In fact it is readily verified that

$$\frac{\delta_{ij}\,\gamma_{ij}}{\sqrt{\beta_i\,\beta_j}} \equiv a_{ij}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The unequality (48) therefore is proved in an indirect way, and from this indirect proof a remarkable conclusion with respect to the influence numbers may be drawn.

From

$$D_n' = \begin{vmatrix} -a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & -a_{22} \ldots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \ldots & -a_{nn} \end{vmatrix}$$

it follows, by multiplying the odd rows and the even columns by -1:

$$D_n' = (-1)^n \begin{vmatrix} a_{11} & a_{12} & -a_{13} & a_{14} & -a_{15} & \ldots & \ldots \\ a_{21} & a_{22} & a_{23} & -a_{24} & a_{25} & -a_{26} & \ldots \\ -a_{31} & a_{32} & a_{33} & a_{34} & -a_{35} & +a_{36} & -a_{37} \\ \vdots & & & & & & \end{vmatrix} = (-1)^n D_n'' \quad . \quad (52)$$

The unequality (48) being adopted as true, it follows that the determinant D_n'' is essentially positive and therefore the following theorem holds:

If in a MAXWELL determinant (related to the type of structure as treated in this paper) the numbers of all diagonals of odd index (viz. all diagonals beginning with the numbers a_{13}, a_{15}, a_{17} ..., a_{31}, a_{51}, a_{71} ...) are replaced by their negative values, the determinant remains positive.

Medicine. — *About the disparity of different parts of the labyrinth.* By A. DE KLEYN.

(Communicated at the meeting of April 27, 1946.)

It is often assumed that phylogenetically older systems can resist injurious influences better than younger ones. This formula, however, lays perhaps too much stress on the phylogenetic age and it is possible that the amount of vulnerability is rather connected with the more or less important function which the various systems have to fulfill in life.

It goes without saying that even when supposing this we can always expect that generally speaking older phylogeny and greater resistance will run parallel, as systems which are found in the whole animal kingdom will mostly have to fulfill functions that are more important to life than systems or part of systems which have developed only later together with higher forms.

This does not alter the fact, however, that if in a certain case a phylogenetically younger part fulfills a function important to life, under those circumstances the phylogenetically older, but functionally less important part may be more vulnerable.

The various parts of the equilibrium organ are a fine example and here we mean by equilibrium organ not only the peripheral labyrinth but also the central connections.

In several communications [1]) DE BURLET has shown, according to me convincingly, that the three semicircular canals are not equivalent, but that the horizontal semicircular canal is phylogenetically younger and built differently from the two vertical semicircular canals.

The arguments which he advanced for his view are the following:

a. In many kinds of animals the horizontal semicircular canal arises ontogenetically later and is mostly shorter and narrower than the vertical semicircular canal. The ampulla contains an undivided crista with one planum semilunatum, and is only innervated by one nerve-branch.

The vertical semicircular canals have, on the contrary, a crista and ampulla, which consist more or less of two parts, often in the form of a septum cruciforme. On both sides of the crista there is a planum semilunatum, while two separate nerve-fibres provide for the innervation on those sides.

b. In the petromyzonts which have only two semicircular canals, the latter look quite like vertical semicircular canals. In the other groups of

[1]) a.o. DE BURLET, H. M. Ztschr. f. Anat. und Entw. Gesch., **104**, 79 (1935). Vergl. Anatomie des stato-akust. Organs, Die innere Ohrsphäre. from: Handb. d. vergl. Anatomie, 1934.

animals the third horizontal semicircular canal separates itself, as it were, from the anterior vertical semocircular canal. For this DE BURLET found, among other things, the following indications; the way in which the ends of the horizontal semicircular canal debouch in the various kinds of animals, in some even quite in the anterior vertical canal; the proportion of the raphe of the anterior vertical canal with respect to the raphe of the horizontal semicircular canal; the innervation of the crista horizontalis in connection with the course of the ramus ampullaris horizontalis and a curious teratological find in Echidna, where DE BURLET and VERSTEEGH [2]) found a fourth abortive semicircular canal, shoved in between the anterior vertical and the horizontal one, which they considered "as the result of a repeated division. The first of these divisions gives rise to the horizontal canal".

With these considerations in view we now want to examine what the experimental and clinical investigation made of man and animal, teach us.

If we consider which movements of the head man most often makes in daily life, it appears that when he sits and walks movements round the vertical axis (turning of the head to right and left) and round the bitemporal (movements forwards and backwards) occur most frequently, while movements round the fronto-occipital axis (inclination of the head to the right and left shoulder) are only seldom made. When he makes the former movement (turning of the head) it is nearly exclusively the horizontal canals, when making the two latter movements of the head it is the various vertical canals that are excited. It is not considered which combination of vertical canals on both sides we are here dealing with.

When the above-mentioned head-movements are made, at the same time from the equilibrium organs reflectorially effective eye-movements are caused and these eye-movements can be examined more in detail by turning the patients on a revolving-chair or an electrically driven turning chair. When the head is turned horizontal eye-movements occur, when it is moved forwards and backwards vertical ones and when it is inclined rotatory ones. As has already been said, in daily life inclinations with the head are only seldom made and so one may expect, if the supposition made in the beginning is right, that the last-mentioned reflectory rotatory eye-movements will be most vulnerable. Inversely the reflectory horizontal and vertical eye-movements, which are caused when — as occurs so often in daily life — the head is turned and moved forwards and backwards, would have to show a much greater resistance.

Now this is in accordance with reality and is proved true by the so-called symptom of EAGLETON [3]). As early as 1912 EAGLETON showed that in various affections of the central nervous system it frequently occurs that from the equilibrium organ horizontal, but not rotatory eye-movements can be evoked. He concluded from this that probably the central paths of the

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[2]) BURLET, H. M. DE and C. VERSTEEGH, Acta Otolaryng, 16, 516 (1931).

[3]) a.o. EAGLETON, Laryngoscope, 33, 483 (1923).

horizontal and of the vertical canals each went its separate way. All investigators agree about the frequent occurrence of this symptom. In the extensive literature opinions are rather divided about the question in how far this symptom also has a localisatory value. We shall not consider this question here, however.

AUBRY and CAUSSÉ [4]) now further pointed out that for a complete examination of the vertical canals not only the rotatory, but also the vertical eye-movements should be examined. The result of their extensive examination — and my own experience is in accordance with it — was that *if* in affections of the central nervous systems anomalies are found, they are nearly always such that by stimulating the vestibular organ horizontal and vertical, but no rotatory eye-movements can be brought about. AUBRY and CAUSSÉ found no exceptions to this. As, however, the symptom is found in so many and various pathological processes, exceptions must be expected. I myself could observe a few cases in which horizontal, but no vertical and rotatory eye-movements could be brought about or in which inversely the reflexes for the latter were intact and were absent for the horizontal movements [5]). Up to now, as far as I know, no cases have been mentioned in which the rotatory eye-movements could and the others could not be brought about [6]).

Summarizing, it appears that the vestibular rotatory eye-movements which in man are least important, are most vulnerable and that the horizontal and vertical movements, which are much more important to life, have much more resistance and this notwithstanding the fact that the vestibular horizontal eye-movements are evoked in the phylogenetically younger horizontal semicircular canals.

Another picture can be expected if one examines the vestibular eye-movements in animals with eyes placed laterally, as rabbits, guinea-pigs etc. They make no spontaneous eye-movements, so that these latter only arise through reflectory influences.

If in these animals we examine which movement of the head they perform mostly in daily life, these are again in the first place movements forwards and backwards with the head round the bitemporal axis. Here, however, from the vertical semicircular canals *no vertical* eye-movements are brought about, as in man and in animals with eyes directed forwards, but as a consequence of the fact that the eyes of these animals are placed laterally, reflectory *rotatory* eye-movements occur. The second, frequently occurring,

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[4]) AUBRY, M. et R. CAUSSÉ, Ann. d'Otol. Laryng., **1**, 1331 (1931).

[5]) KLEYN, A. DE, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 547 (1941).

[6]) Up to now few clinical investigators have made examinations of patients with affections of the central nervous system in which the horizontal as well as the vertical and rotatory vestibular eye-movements were thoroughly examined. This is due to the fact that properly speaking such examinations can only be made safely and exactly if one has at one's disposal an electrically driven turning-chair on which a stretcher has been mounted. If one then lets the table turn slowly there is no objection to examining completely even serious cases of affections of the central nervous system.

movement of the head of the normally sitting animal takes place in the horizontal plane and here the horizontal semicircular canals are excited and horizontal eye-movements are caused. On the other hand it is only seldom that a movement of the head occurs round the occipital snout axis and here the stimulation of the vertical canals expresses itself in a reflectory manner by vertical eye-movements.

In rabbits, therefore, according to the views expressed above, the greatest vulnerability should not, as in man, be expected for the vestibular rotatory, but for the vestibular vertical eye-movements. This now appears to be really the case. For in literature one meets with several communications which confirm the above-mentioned supposition.

Thus in ROTHFELD [7]), who without going further into the matter, supposes in men and animals a prevalence of the horizontal nystagmus, one finds that in rabbits, both after poisoning with alcohol and with chloroform, ether and paraldehyd, the quick component of the nystagmus „in folgender Reihenfolge verschwindet *a.* des vertikalen, *b.* des rotatorischen, *c.* des horizontalen Nystagmus. Das Wiedererscheinen des Nystagmen vollzieht sich in umgekehrter Reihenfolge".

Later the well-known experiments of VERSTEEGH [7]) about the influence of narcotics on the various labyrinthine reflexes in rabbits came to confirm these observations. MAGNUS [8]) who in his „Körperstellung" published the experiments of VERSTEEGH for the first time, pointed to the curious selective action of the narcotics with regard to the various labyrinthine reflexes and remarked here: "Von den Augendrehnystagmus ist der horizontale am resistentesten, der verticale am empfindlichsten".

In the thesis of JONGKEES [9]) about ten years later, an analogous description is found as regards the poisoning with paraldehyd.

Also when there is poisoning with veramon MAGNUS [10]), as appears from this tables, found several times: "vertical Nystagmus absent, horizontal nystagmus present".

It is not only for narcotics, for that matter, that we find the above-mentioned sequence.

If one reads over exactly the protocol of which KLEITMAN and MAGNUS [11]) inform us in their publication about poisoning with insulin, one finds several times: "Augendrehnystagmus nur horizontal". SCHOEN [12]) examined the various labyrinthine reflexes after poisoning with coffein, cardiazol and hexetone and described the stimulating action of coffein,

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[7]) ROTHFELD, J., Arbeiten Obersteiner's, **20**, 89 (1913); Pflügers Arch., **149**, 435 (1913).

[8]) MAGNUS, R., Körperstellung, Berlin, Springer, 1924, p. 652.

[9]) JONGKEES, L. B. W., Picrotoxine als tegengif tegen eenige slaapmiddelen, Thesis Utrecht 1935.

[10]) MAGNUS, R., Jnl. of Pharmacology and exp. Therp., **29**, 35 (1926).

[11]) KLEITMAN, N. und MAGNUS, R., Pflügers Arch., **205**, 148 (1924).

[12]) SCHOEN, R., Arch. f. exp. Path. und Pharm., **113**, 246, 257, 275 (1926).

camphor, cardiazol and hexetone on rabbits which had been narcotised with alcohol, paraldehyd and urethane.

Expressions as: "beim Nystagmus und Nachnystagmus wird die vertikale Komponente früher als die rotatorische und horizontale gelähmt (with coffein); "die Lähmungserscheinungen beschränkten sich dabei auf den vertikalen Nystagmus der Augen (with hexetone) and "es gelingt bei beiden Giften die Dosis so zu wählen, dass Krämpfe und Lähmungserscheinungen vermieden werden ausser der Lähmung der vertikalen Komponente des Nystagmus" (with hexetone and cardiazol) are met with again and again and speak for themselves [13]).

Let us finally point to the communication of KLEITMAN and MAGNUS [14]) about asphyxia in rabbits, here too it is said: "von den Augendrehnystagmus erwies sich der horizontale am resistentesten, der vertikale am empfindlichsten".

Summarising one finds, therefore, that in rabbits, during poisoning with very different pharmaca and during asphyxia the horizontal and rotatory vestibular eye-movements resist much more than the vertical ones. The greatest resistance one finds, therefore, also in these animals, in the reflex-system of the horizontal semicircular canals which, it is true, is phylogenetically younger, but for daily life more important, and not in the vertical labyrinthine eye-reflexes, which are phylogenetically older, but for daily life far less important.

The so-called compensatory eye-*positions* play in the life of the rabbit a still more important part than the above-mentioned eye-*movements*.

These reflexes are also regulated completely in a reflectory manner.

Compensatory eye-positions occur when the normally sitting animal brings its head from one position in space into the other. They are called compensatory because whenever there is a change of position of the head, the eyes do not change their place in space, so that the visual field of the animal always remains unchanged.

When for instance the rabbit bends its head downwards 30° from the normal position, eye-movements occur which cause the eye to roll exactly 30° backwards and to remain in that position as long as the head keeps its new position in space. In this way, the result is that notwithstanding the fact that the head bends 30° downwards, the position of the eye in space does not undergo the least change, nor, as a consequence, the visual field. This is brought about by several reflexes. During the movement from one position into the other, the reflexes of the semicircular canals described above appear, which make it possible for the eye to execute its required counter-rolling. As soon, however, as the other position of the head is

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[13]) E.g. also with alcohol + coffein: "es fehlt nur der vertikale Nystagmus"; with paraldehyd + hexetone: "Reflexe bis auf vertikalen Nystagmus vorhanden" and with paraldehyd + cardiazol: "Reflexe alle normal bis auf das Fehlen des vertikalen Augennystagmus".

[14]) KLEITMAN, N. and MAGNUS, R., Pflügers Arch., **205**, 142 (1924).

reached, the position of the eye newly obtained by this is kept by means of other reflexes coming from the utriculus. Here other reflexes also cooperate, which are caused in the neck as a consequence of the fact that the animal, in bending its head forwards, not only changes the position of the head in space, but also the position of the head with respect to the trunk. By the harmonious cooperation of these reflexes of the semicircular canals, utriculus and neck it now becomes possible for the animal to bring its head from the normal position into different others without change of the visual field [15]) through movements upwards and downwards in the sagittal plane in a region of about 100°, in the horizontal plane in one of about 40° and in the frontal plane in one of about 30°. From these figures it appears already that the changes of position in the sagittal plane, where the rotatory compensatory eye-positions occur, are most important for the animal. It could be shown experimentally that the horizontal compensatory eye-positions, which occur when the positions are changed in the horizontal plane, are regulated almost exclusively from the neck and that here labyrinthine reflexes do not clearly cooperate. As regards the two other planes, where labyrinthine reflexes do play an important part, it can be said that in daily life changes of position by turning of the head occur very seldom, while they constantly take place in the sagittal plane by bending and raising of the head.

Quite in accordance with this VERSTEEGH [16]) and JONGKEES [17]) could show that when there is poisoning with paraldehyd, the rotatory compensatory eye-positions, occurring as a consequence of bending and raising the head resist the poisoning much more than the so-called vertical compensatory eye-positions which, when there are changes of position, are brought about in connection with the turning of the head. In MAGNUS [18]) we read: "von den compensatorischen Augenstellungen sind die rotatorischen resistenter als die vertikalen".

We see another example when there is poisoning with oleum chenopodii. JONKHOFF [19]) from his experiments with rabbits had drawn the conclusion that in a certain stage of the poisoning the tonic labyrinthine reflexes are paralysed, while the labyrinthine reflexes of movement can then still be brought about. By tonic labyrinthine reflexes are meant those labyrinthine reflexes which remain unchanged as long as a certain position of the head in space is not changed. So among these are also the compensatory eye-

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[15]) KLEYN, A. DE, Proc. Kon. Akad. v. Wetensch., Amsterdam, **23**, 509 (1920); Pflügers Arch. ,**186**, 82 (1921); Arch. néerl. de Phys. de l'Homme et des Anim., **7**, 138 (1922).

MAGNUS, R. und A. DE KLEYN, Bethe's Handb. der Phys., **15** I, 55 (1930).

[16]) MAGNUS, R., Körperstellung, Berlin, Springer, 1924, p. 652.

[17]) JONGKEES, L. B. W., Pirotoxine als tegengif tegen eenige slaapmiddelen. Thesis. Utrecht 1935.

[18]) MAGNUS, R., Körperstellung, Berlin, Springer, 1924, p. 652.

[19]) JONKHOFF, D. J., De invloed van eenige geneesmiddelen op de labyrinth-reflexen van konijnen, caviae en katten. Thesis, Utrecht 1921.

positions. This find of JONKHOFF's seems contrary to our view that the rotatory compensatory eye-positions must play such an important part in the daily life of the rabbit. It appears, however, that JONKHOFF had only determined the vertical compensatory eye-positions. So when the examination by JONKHOFF though with another purpose, was repeated [20]) and the rotatory compensatory eye-positions were also examined, these reflexes appeared to be an exception among all tonic reflexes and to be among the least vulnerable labyrinthine reflexes, when there was poisoning with oleum chenopodii.

Finally, observations of other labyrinthine reflexes might be mentioned, which might support our views. These, however, being too small in number as yet, will not be considered for the present.

Summary.

In connection with the much varying influence which several reflex-systems show under the influence of pathological processes and poisonings, the supposition was pronounced, with reference to clinical and experimental observations of men and animals, that those reflex-systems are most resistent which have to play an important part in daily life and not those which are phylogenetically oldest.

This was made clearer by referring to several labyrinthine eye-reflexes of man and the rabbit. We pointed to the fact that in accordance with the above, in man the rotatory and in the rabbit the vertical labyrinthine reflexes of the semicircular canals being the least important in life, are most vulnerable, while the important horizontal eye-reflexes, though brought about in the phylogenetically younger horizontal semicircular canals, offer most resistance.

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[20]) MAGNUS, R., Pflügers arch., **198**, 428 (1923).

Medicine. — *Some remarks on the absence of the sinus maxillares and on sinusitis ossificans.* By A. DE KLEYN and B. VAN OMMEN.

(Communicated at the meeting of March 30, 1946.)

While the absence of the sinus frontales occurs rather often and is even found in 3—5 % of the cases in persons otherwise normal, the absence of the sinus maxillares only seldom occurs and the number of cases of this kind mentioned in the literature is very small.

We do not consider here various affections of the cranial bones, such as osteo-dystrophia fibrosa, PAGET's disease and leontiasis ossea, when the sinuses of the nose and especially the sinus maxillares are not seldom entirely filled up with pathological bony tissue. Nor will monstra, when the sinuses of the nose have not developed, be discussed further [1]).

The absence of the sinus maxillares, which we have in view here, is often ascribed in the literature to primary hereditary factors or to inhibitions of the pneumatisation, though, as we hope to show more in detail, it is quite doubtful whether this is right. It goes without saying that secondary inhibitions of the pneumatisation can be expected much more often with the sinus frontales than with the sinus maxillares. For the latter are already present at birth, while the sinus frontales penetrate much later into the ossa front. So injurious influences, inhibiting the pneumatisation, which come into being especially after birth, will leave their traces especially where the sinus frontales are developing.

A number of investigators have determined the size of the sinus maxillares by various methods. Thus SCHÜRCH [2]), with 118 skulls, filled the skulls with shot after having cut off the communications with the nasal cavity and after this measured the quantity of shot needed for a complete filling, in a cylinder. With his skulls, an absence of the sinus maxillares did not occur.

LOEB [3]) made plaster-casts of the sinuses and apparently did not mention an absence of the sinus maxillares either. FREER [4]) again used another method and with his patients he filled the free space in the sinus maxillares with lipiodol, at the same time making a Röntgenogram. When, with his 120 cases, he indicates that in 27 pathological cases the free contents of the cavities amounted to only 0—4 ccm, this only means, however, that

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[1]) Cf. e.g. LANG, B. Virch. Arch. **285**, 93 (1932) who described a case of proboscis lateralis and another with arhinencephalia bilateralis, where the sinuses of the nose were not present.

[2]) SCHÜRCH, O. Arch. f. Laryng., **18**, 197 (1906).

[3]) LOEB, H. ref. Mon. f. Ohr., **48**, 228 (1914).

[4]) FREER, J. Mon. f. Ohr., **66**, 173 (1932), **67**, 681 (1933); **67**, 841 (1933); **67**, 1065 (1933).

these cavities were completely or nearly completely filled up by polypous tissue, but not at all that a bony mass took up the space of the sinus maxillares.

Finally, MÜNDNICH [5]), with some 100 skulls, cut off all the vascular canals and dehiscences of the sinus maxillares, after which he filled them with water. Then the consumption of water indicated the contents of the cavities. An absence of the sinus maxillares did not occur either with the skulls investigated by him.

Nor did we find observations made by the investigators themselves in this field in the various hand- and text-books, e.g. not in those of COATES and CHEVALIER JACKSON [6]), GRÜNWALD [7]), HAJEK [8]), LEDERER [9]), LOEB [10]), ST. CLAIR THOMSON [11]) and ZARNIKO [12]). COATES and JACKSON do not even raise the question of an absence of the sinus maxillares. In the handbook of DENKER and KAHLER GRÜNWALD writes about this absence: "Dieses ist ganz überaus selten. MORGAGNI berichtet von beiderseitigem Defekt bei einem Weibe. Sonst ist nur ein Fall derart von BENJAMINS bekannt" [13]). This, however, is no longer the case now, as will appear in the following.

HAJEK, in his famous monography on the sinuses, says: "Nur selten ist die Resorption der Spongiosa so mangelhaft, dass die Kieferhöhle sich fast gar nicht entwickelt." LEDERER only communicates that "partial or complete absence of a sinus has been observed" without going further into this. LOEB indicates as "cubical capacity in cubic centimeter; "greatest 28,4 and least 4,5". In Sir St. CLAIR THOMSON's book we read: "a large adult maxillary sinus will hold 1 ounce of fluid, but the cavity may be represented by a mere chink, or more rarely be entirely absent", thus quoting only the old communication of MORGAGNI in 1779. Finally, ZARNIKO [14]) in his text-book, speaks about the absence of the sinus frontales, but not about that of the sinus maxillares. Even ONODI, who has a great knowledge of

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[5]) MÜNDNICH, K. Ztschr. f. Hals usw. **43**, 5 (1938).

[6]) COATES G. M. and JACKSON, Chev. L. The nose-throat and ear and their diseases. Saunders Philadelphia 1929.

[7]) GRÜNWALD, L. Denker–Kahler Bnd. I, Springer, Bergmann 1925.

[8]) HAJEK, M. Pathologie und Therapie der Entzündl. Erkrankungen der Nebenhöhlen der Nase. F. Deuticke, Leipzig–Wien, 1925, IV Auflage.

[9]) LEDERER, F. L. Diseases of the ear, nose and throat, Third revised Edition. Davis Company, Philadelphia 1942.

[10]) LOEB, H. W. Operative surgery of the nose, throat and ear, Vol. I, H. Kimpton, London 1914.

[11]) ST. CLAIR THOMPSON. Disease of the nose and throat. Third Edition. Cassell and comp., London 1926.

[12]) ZARNIKO, C. Die krankheiten der Nase und des Nasenrachens. IIIe Aufl., Berlin, Karger 1910.

[13]) Cf. also KOCH, J. Arch. f. Ohren, Nasen und Kehlkopfheilkunden, **125**, 174 (1930). „Völliges Fehlen der Kiefehöhle ist überaus selten. MORGAGNI und BENJAMINS berichten davon."

[14]) ZARNIKO. L.c.

the anatomy of the sinuses of the nose, only wrote in the handbook of KATZ–PRESING–BLUMENFELD [15]) that: "Die Kapazität der Kieferhöhle ist groszen Schwankungen ausgesetzt; nach BRÜHL ist die mittlere Kapazität der Kieferhöhle 10,5 cm."

These notes may suffice to show that anyhow the complete absence of the sinus maxillares occurs very rarely. Before investigating the cause of this anomaly by means of the cases published up to now, we should first like to communicate the latter briefly.

I. As early as 1779 MORGAGNI [16]) described the absence of the sinus maxillares. ZUCKERKANDL [17]) however, wrote: "doch sind die anatomischen Daten über diesen höchst merkwürdigen Casus so aphoristisch, dass man sich keine klare Anschauung über diesen vielfach citierten Fall bilden kann."

The same objection obtains more or less, however, for all the cases that have been investigated only macroscopically and anatomically and not microscopically or clinically. The macroscopic-anatomic investigation is certainly not sufficient to decide whether in a special case we are dealing with a pathological process or with a disturbance of development [18]).

Nor is the well-known case of ZUCKERKANDL an exception to this.

II. He described that on the left side of his preparation the hiatus semilunaris was absent "an seiner Stelle zieht die Nasenschleimhaut glatt vorüber und es erstreckt sich von hier aus ein über bohnengrosser, dichter, mit einigen, bis hanfkorngrossen, glashellen Cysten durchsetzter Bindegewebspropf, welcher mit der Submucosa der Nasenschleimhaut in Verbindung steht, in den Kiefer hinein. Der Kiefer besitzt keinen Sinus, da von dem eben beschriebenem Bindegewebskern abgesehen, sich statt der Höhle eine engmaschige, fetthaltige Spongiosa im Kiefer findet." Considering the whole picture with many synechiae in the nose, found at the same time, ZUCKERKANDL decided that in his case there must be disturbance of development and that several facts prove that a pathological process does not exist.

According to him against a pathological process are:

a. das normale Aussehen der Schleimhaut

b. der Mangel an Narben in der Nähe der Synechiën.

As there was no microscopic examination and in consideration of the existence of the synechiae this pronouncement seems to us very challenge-

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[15]) ONODI, O. in KATZ–PREYSING–BLUMENFELD: Handb. der spez. Chirurgie des Ohres und der oberen Luftwege. 2e Aufl. C. Kabitzch, 1922.

[16]) MORGAGNI. De Sedibus et Causis Morborum 1779. Quoted according to ZUCKERKANDL.

[17]) ZUCKERKANDL, E. Normale und pathol. Anatomie der Nasenhöhle. Wien 1882.

[18]) So this also obtains for the cases mentioned by RESCHREITER (MECHELS, SOMMERING and ROZENMULLER): RESCHREITER, G. Zur Morphologie des Sinus maxill., Stuttgart 1878.

able. ZUCKERKANDL further remarks that in favour of a disturbance of development are:

c. der Defekt der linken Highmorshöhle

d. die Einmündung der Keilbeinhöhle in das Siebbeinlabyrinth

e. das Fehlen der halbemondförmigen Spalten

f. die Mündung des Ostium maxillare dextrum ins Siebbein und schlieszlich

g. das normale Aussehen des rechten Sinus maxillaris.

These arguments, however, are not sufficient to exclude a pathological process as cause of the absence of the sinus maxillaris. The case of GRABSCHEID [19]) published in 1933, concerns a woman of 53 with a congenital anophthalmia on both sides and suffering from anosmia. She was admitted into the clinic of BRUNNER in Vienna for a peculiar form of sinus-thrombosis. As an accidental discovery *röntgenologically* an absence of the sinuses oft he nose was found. As a transition to those cases in which symptoms of inflammation undoubtedly play a part, we should first like to mention the communications of LANGE [20]) and BENJAMINS [21]).

III. LANGE, in 1912, could examine a man of 54. When working, a horse knocked against the right half of his face and nose. After this, rather vehement pains arose on this side, which expended towards the forehead, polyps in the right nasal cavity, conjunctivitis and facialis-spasms in the right half of the face. Röntgenologically one got the impression that the sinus maxillaris was filled up; puncture of this cavity appeared to be impossible. At the operation according to DESAULT–KÜSTER it is mentioned: "Es stellt sich heraus dasz die ganze Kieferhöhle von spongiösem Knochen erfüllt ist, der sich mit dem scharfen Löffel unschwer entfernen läszt; dabei erscheinen zwischen Blut und Knochenteilchen massenhafte Fetttröpchen. Dieser spongiöse Knochen wird sorgfältig ausgeräumt, bis man auf kompakte Knochenwände kommt, die anatomisch denen der Kieferhöhle entsprechen."

The microscopic picture was described as follows: Microscopically the removed contents appear to consist of a network of irregular bone-trabeculae, among which lies here and there entirely loose connective tissue largely fat-tissue, however. Neither pictures of apposition or resorption nor symptoms of inflammation can be found.

The case took a favourable course, all complaints disappeared after the operation. LANGE asked himself: "Hat es sich um die sekundäre Ausfüllung einer vorhandenen Kieferhöhle gehandelt oder ist überhaupt keine vorhanden gewesen?" He thought that the first supposition was right. Microscopically, the spongious bone removed at the operation did not resemble

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19) GRABSCHEID, E. Revue de laryng., **54**, 1184 (1933).

20) LANGE, Passow Schäfer's Beitr., **5**, 58 (1912).

21) BENJAMINS, C. E. Arch. f. Ohr. usw., **109**, 71 (1922).

normal spongious bone of the skull; it could be removed easily and by it one could "einen Raum herstellen, der einer normalen Kieferhöhle entsprach". LANGE thought that, owing to the trauma, between the mucous membrane and bone and also in the sinus maxillaris itself, a haemorrhage arose, which after organisation led to the sinus maxillaris being filled up with spongious bone. The absence of infection is in such a case a conditio sine qua non.

According to this conception, remains of the mucous membrane of the sinus maxillaris must be expected at the operation and when a microscopic examination is made. Now this was not so in the case of LANGE. The latter, however, is right in pointing out that only part of the removed spongious bone was examined microscopically and that if any mucous membrane was present at the operation, it may very well have escaped observation, owing to the haemorrhage.

In the case of BENJAMINS, to be discussed now, it is difficult to decide whether we must assume a pathological process or a disturbance of development.

IV. A woman of 60 appeared to have a squamous-celled carcinoma of the left concha inferior. When an attempt was made to remove the tumour according to the method of DENKER, when the frontal wall of the sinus maxillaris was being chiseled away, only spongious bone was found, which hurt very much when touched. Here, as BENJAMINS expressed it, "ein kleines aus festem Gewebe bestehendes Zellnest ohne Ausläufer und ohne sichtbaren Zusammenhang mit der Nasenhöhle" appeared, which he took to be a metastasis of the carcinoma. According to what will be communicated when the following cases are discussed, the question arises, however, whether this tissue has perhaps been mucous membrane of the original sinus maxillaris.

At a second operation the rather extensive carcinoma was removed. We read here that the operating doctor was struck by the fact that neither the concha media nor cells of the ethmoid were found at the operation. On the other hand, the sinus sphenoidalis was present and filled with pus. BENJAMINS says himself it is not quite impossible that there was only a seeming absence of the concha media and the ethmoid, they having been pushed upwards by the tumour. He did not think this likely, however, as the offshoots of the tumour upwards were not strongly developed. Therefore he himself thought that it was caused by a disturbance of development in the region of the infundibulum ethmoidale, so that neither the sinus maxillaris nor the frontal cavity nor the ethmoid nor the concha media had developed.

If one reads the clinical history of this patient attentively, symptoms of inflammation also appear to be present in this case [22]). This, however, is not at all without interest to explain the absence of the sinus maxillaris.

[22]) "An der Stelle des verschwundenen ersten Molaris war der Alveolarrand unter-

V. Interesting in this respect is the communication of UNDERWOOD [23]), who remarked, among other things, when studying a great number of skulls of various races: "in chronic abscesses over the roots of the cheek teeth there was an abundant formation of bone thrown up between the abscess and the sinus, causing the floor of the latter to retreat from the neighbourhood of the root trouble, and thus protecting the cavity from perforation. There were many of these cases among the skulls examined, especially in uncivilised races, where dental treatment had not been attempted". Here he describes a "very extreme case", where "in fact the deposit of new bone has almost entirely obliterated the sinus". He himself, however, points out that some thickening of the bone occurs without an affection of the teeth playing a part: "In some cases the whole of the walls of the sinus were much thickened by bony deposit. This thickening had no apparent connection with the teeth."

That, for that matter, following upon certain forms of sinusitis maxillaris it may occur that the original sinus maxillares disappear, appears from the following case, observed by GODFREY COLLINS [24]).

VI. A woman of 51 was operated upon radically on the left because of a sinusitis maxillaris. 4 months after the operation already the opening that had been made towards the nose was closed again and it soon appeared that a new opening made by means of diathermy had narrowed again. After 4 years, serious complaints made a new operation necessary. It appeared that the opening towards the nose had entirely disappeared, the sinus maxillaris was filled up with spongius bone, only in the middle there was still a small cavity covered with mucous membrane. At the end of the operation patient succumbed as a consequence of a weakness of the heart. Obduction was refused. We regret to say that a histological examination did not take place. In this case we know for certain now that the almost complete absence of the sinus maxillaris must not be ascribed to a disturbance of development, but to a process of inflammation, as the cavity was still present at the previous radical operation.

In our opinion the picture agrees very much with the sinusitis ossificans, which e.g. by SKILLERN [25]) was described as "obliterative frontal sinusitis" and which occurs with certain inflammations of the frontal cavities.

VII. FETISSOW [26]) already seems to have observed a similar process according to the report at my disposal. He thought that the narrowings of the sinus maxillaris by conditions of protracted excitation were based on

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brochen; eine livide Granulation zeigte hier, dass ein rückgängiger Entzündungsprozess anwesend war".

[23]) UNDERWOOD, A. S. Jnl. of Anatomy and Physiology, **44**, 354 (1910).

[24]) GODFREY COLLINS, E. Jnl. of Laryng., **54**, 125 (1939).

[25]) SKILLERN, S. R. Arch. Otolaryng., **23**, 267 (1936).

[26]) FETISSOW, Russisch, cf. Zentr.bl. f. H. N. Ohrenh., **12**, 35 (1928).

a fibrous hyperplasia of the mucous membrane. According to him the cases in which the whole cavity has been obliterated and replaced by spongious bone are rare. In the two cases observed by him a slowly growing swelling of the half of the face, not attended with pain, arose after toothache. The process lasted resp. 5 and 12 years. The diagnosis made: "osteosarcoma and osteoma" were not confirmed by the microscopic examination. For normal spongious bone-substance was found with connective tissue and fat-deposits in between. FETISSOW proposed to speak of a "Highmoritis ossificans" and took the at first alveolar perisostitis to be the aetiological factor.

Two years afterwards V. GANGELEN [27]) published a case that perhaps may be considered as a beginning stage of such an inflammation.

VIII. A patient of 50 had a tooth drawn in 1898. A root had remained in the upper maxilla and this was only discovered 8 years later. After this the sinus maxillaris was washed through the alveolus because of a sinusitis maxillaris. In 1928 an affection of the sinus maxillaris was established röntgenologically. In 1929 patient was operated upon, hard tissue-traits, distributed irregularly over the sinus maxillaris, were found. Both the *small* cavity and the sinus sphenoidalis were covered with rough bone on the inside. The writer never saw so much formation of bone in maxilla and sinus sphenoidalis and therefore felt obliged to speak of an osteosclerosis. Pathologically and anatomically:

1. mucous membrane with glands, rich in lymphocytes,
2. connective tissue rich in vessels,
3. bony tissue with fibrous marrow (no tumour-tissue)

were found.

This writer also looks for the explanation of the process in the chronic condition of excitation of the mucous membrane, so that the tissue-layer, serving as periost- of the submucosa began to form new bone in the sinus maxillaris, the cells of the ethmoid and sinus sphenoidalis.

IX. We cannot investigate from the data at our disposal in how far MATSUYAMA and HAZAMA [28]) observed a beginning stage of a sinusitis ossificans. In a few patients they removed at a radical operation a few free pieces of bone lying in the sinus maxillaris. Histologically spongious bone was found, which according to them had arisen metaplastically, from the periost of the sinus maxillaris, viz. following upon a chronic tooth-inflammation.

LINDBLOM [29]) gave the name of sinusitis Highmori calcificans to pro-

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[27]) V. GANGELEN. Ned. Keel-, Neus- en Oorheelk. Vereeniging, Vergadering 1 en 2, Juni 1920 te Leiden.

[28]) MATSUYAMA, Sch. und HAZAMA, F. Jap. Ztschr. f. Otorhinolar, **43**, 55 (1937).

[29]) LINDBLOM, A. F. Acta Radiologica, **10**, 210 (1929).

cesses where a partly calcified detritus mass was found in the sinus maxillaris. This mass was surrounded by granulation-tissue with metaplastic bony tissue. It seems very doubtful to us whether these cases are in any way connected with the clinical pictures discussed just now.

We also had an opportunity ourselves to observe a few cases of sinusitis ossificans.

X. A woman of 33 was sent to our policlinic by an oculist. She had an iridocyclitis for which no cause could be found and we now asked ourselves whether in our field an infectious focus was perhaps present, which might be held responsible for her eye-anomaly, especially as patient complained that she lost much pus from the nose. At transillumination and at the röntgenological examination the right sinus maxillaris and the ethmoid seemed to be darkened, the walls of the sinus maxillares were bounded unsharply, while the other sinuses did not show any anomalies.

At the operation according to CALDWELL–LUC it appeared that the maxilla, at the place where the sinus maxillaris was expected, was entirely filled up with spongious bone, which could be removed easily and without pain with the sharp spoon, until the originally smooth, but partly very hard maxillary walls appeared. The ethmoid was so hard that we did not succeed in opening it. In the upper part of the spongious bone there was a cavity of the size of a grain of rice, covered with mucous membrane. By a mistake this was not examined microscopically. The spongious bone, however, was subjected to a minute microscopic examination with the very much appreciated help of the pathological-anatomist colleague SITSEN, so experienced in the judgment of bone-diseases. The following could be established: On the surface the spongious bone is covered with new formation of bone and numerous osteoblasts. The walls of the marrowy cavities are partly considerably thickened, the bony marrow has been made more dense fibrously and contains a few infiltrations. This is the picture of an inducing inflammation: spongious bone changes slowly into compacta. Also in a second preparation by the side of the old bone numerous new formations of bone-trabeculae of non-differentiated bone, surrounded by fibrous marrowy tissue, were found.

XI. A second case of far advanced sinusitis ossificans, also attended with clear symptoms of inflammation, is the following observed by us.

A man of 32 was admitted with a swelling of the left upper eye-lid, which had been treated in vain with Priessnitz-compresses and incision. The secretion from his nose, which had existed for months, had become a little less during the last week.

At rhinoscopia anterior a septum deviation was found on the left and pus in the middle and lower part of the nasal cavity, at rhinoscopia posterior on the same side pus below the concha inferior. Röntgenologicallen on the

left both the sinus maxillaris and the ethmoid were darkened, the frontal cavities were large and clear on both sides.

At the operation according to CALDWELL–LUC no sinus maxillaris was found on the left, the upper maxilla appeared to be one bony mass. The puncture of the maxilla, for that matter, had always failed in spite of repeated attempts.

At the pathological-anatomical investigation (Dr. SITSEN was again so kind as to help us) the following could be established: Some 10 small pieces of bone, apparently broken off from spongious bone, were examined. It appeared that the bone was lamellose everywhere and showed in a few places on the surface symptoms of bone-destruction. In the irregular marrowy cavities fibrous oedematous tissue with fat marrow was found. In a few places there was on the surface old fibrillar connective tissue with a few round-cell infiltrations. Neither building of new bone nor osteoblasts were found.

XII. The following case observed by us is interesting because here several stages of a sinusitis ossificans of the sinus maxillaris were found.

An office-clerk of 42 came to our consulting-hour with complaints of nasal obstruction. There were polyps there on both sides and some mucous pus. On the röntgenogram the sinus frontales were clear, the ethmoid and sinus maxillaris darkened on both sides. The polyps coming from the ethmoid were removed and after this we tried to wash the sinus maxillaris. On the left the liquid resulting from the washing contained some pus, on the right the puncture failed, though it was done in several places.

At the operation according to CALDWELL–LUC it appeared that on the left there was a very small cavity surrounded by spongious bone, on the right the whole sinus maxillaris was absent and only spongius bone was found. To our great regret a pathological-anatomical record is wanting in the clincal history and as the pathological-anatomist of that time died (patient was operated upon in 1928) no further information could be obtained.

In 1931 patient was examined for the last time and was then in complete good health.

This case seems highly important to us, because here on the left the picture of a sinusitis maxillaris ossificans in an already far advanced stage was found. However, the purulent sinusitis could still be diagnosed, both by washing and during the operation, while on the right the process had already advanced so far that the whole sinus maxillaris had disappeared.

From the cases communicated just now it appears very clearly indeed that the cause of the absence of the sinus maxillaris must in the greater number of cases be looked for in a process of inflammation or in haemorrhages, as in the case of LANGE.

Of the remaining cases, as far as is known to us, there are only two that were examined somewhat extensively, namely the cases of ZUCKER-

KANDL and BENJAMINS. In the case of ZUCKERKANDL several anomalies, described extensively above, could be found which pointed in the direction of congenital anomalies.

On the other hand, in this case no microscopic examination was made and the various synechiae found in the nose rather point to a previous intra- or extra-uterine process of inflammation. Is it not possible that the "Bindegewebspropf" found by him in the maxilla, which was provided with some cysts, is the remains of the original sinus maxillaris? In the case of BENJAMINS at the operation an inflammation of the left processus alveolaris was found, on the other hand the absence of the concha media and the ethmoid on this left side were not established with sufficient certainty. The absence of the left sinus frontalis, observed röntgenologically, can certainly not be brought forward as an argument for the proposition that also the other anomalies must have been due to disturbances of development, as the absence is seen in too great a number with skulls otherwise normal. Accordingly, it does not seem at all impossible to us that the "Zellnest" found in the middle of the spongiosa, which BENJAMINS took to be a metastasis of the carcinoma in the nose, but which was not examined microscopically, has in reality been the remains of the original sinus maxillaris.

The interesting patient of GRABSCHEID, as far as the sinuses of the nose were concerned, could only be examined röntgenologically. Accordingly, GRABSCHEID writes: "nous mentionnons ce fait et ne pouvons pas nier la possibilité de l'existence de cavités accessoires du nez peu développées." Undoubtedly in this case combined with congenital anophthalmia and anosmia the possibility exists that the sinuses of the nose were entirely absent congenitally or were very small.

Summary.

Summarizing, the following appears from the above:

1. The absence of the sinus maxillaris only occurs very rarely, as opposed to the absence of the sinus frontalis.

2. Neither osteo-dystrophia fibrosa, disease of Paget nor leontiasis ossea are considered here. In these cases we observe not seldom absence of the maxillary sinuses, because there is a filling up with new formation of bone. In monstra, the sinuses of the nose often have not developed.

3. The other cases, in which the cavities of the sinus maxillares are completely or almost completely absent, are perhaps entirely, but certainly in by far the largest majority, due to local processes of inflammation of the maxilla and to the fact that then the sinus maxillares are filled up with spongious bone.

4. In accordance with the courses of the disease, already known for a long time in connection with the sinus frontalis, one might speak in these

cases of a sinusitis ossificans or obliterative sinusitis (SKILLERN). So the cavity, already present, is replaced entirely or almost entirely by spongious bone, as a consequence of an inflammation.

5. Up to now no cases are known in which the absence of the sinus maxillaris must be ascribed with certainty to disturbances of development or to inhibitions occurring with the pneumatisation-process, though e.g. in the case of GRABSCHEID described above, with congenital anomalies this possibility must certainly be considered seriously.

Mathematics. — *Continuous classification of all microcompact 0-dimensional spaces.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of April 27, 1946.)

1. All continuous invariants of the family of all countable sets having been classified in a previous paper ([3], chapter II; comp. also [4]), we set ourselves as chief task the classification of all continuous invariants of the family of *all microcompact 0-dimensional sets* (theorem III). To accomplish this purpose we begin by proving a theorem concerning retractions of 0-dimensional sets. The subset A of B is, as known (BORSUK [1]), by definition a *retract* of B if there exists a continuous mapping f of B on A which in A is the identity ($A = f(B)$, $f(x) = x$ for $x \subset A$). f is called a retraction of B in A. We shall now prove (theorem I) that, given an arbitrary 0-dimensional set B, a not-vacuous subset $A \subset B$ is a retract of B only if A is *closed in* B.

The notations and results of [3] are supposed to be known. *All sets* (or spaces) *considered will be separable.*

2. **Theorem I.** *A not-vacuous subset A of an 0-dimensional space N is a retract of N only if A is closed in N.*

Proof. According to a well-known theorem of SIERPINSKI [6] N is homoeomorphic with a subset of the *Cantor discontinuum* D (i.e., the set of all real numbers expressible in the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0$ or 2). We denote this homoeomorphic subset of D again by N. Be A an (in N) closed subset of N. We shall construct a retraction of N in A.

We consider an infinite system of coverings $\{I\}$ of D, consisting of lump-sets (i.e., both open and closed subsets of D). Each lump-set $I_{a_1 a_2 \ldots a_n}$ ($a_i = 0$ or 2) will consist of all points of the form

$$\sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{3^i} \qquad (b_i = 0 \text{ or } 2).$$

For arbitrary but fixed n the 2^n sets $I_{a_1 a_2 \ldots a_n}$ apparently determine a covering, the so-called n^{th} covering of D. Every monotonicly decreasing sequence of sets

$$I_{a_1} \supset I_{a_1 a_2} \supset I_{a_1 a_2 a_3} \supset \ldots \qquad (1)$$

apparently determines exactly one point $d_{a_1 a_2 a_3 \ldots}$ of D and, conversely, for every point $d \subset D$ there exists exactly one defining sequence (1). A number of sequences (1) defines exactly the points $a \subset A$. If in an I lies at least one point of A we also denote this I by I. Now consider the first

covering of D, consisting of the two sets I_0 and I_2. If both lump-sets I_0 and I_2 are at the same time sets $\hat{I}_0$ and $\hat{I}_2$, nothing happens. If, however, one of these lump-sets, e.g. I_2, is not an $\hat{I}$-set, so that $I_2 \cdot A = 0$, we map the set $I_2 \cdot N$, if it is not vacuous, onto one fixed but arbitrarily chosen point of $\hat{I}_0 \cdot A$. Then we strike out the set I_2 and also all sets $I_{2\, b_2 b_3 \dots b_n}$ ($b_i = 0$ or 2; $n = 2, 3, \dots$) in the subsequent coverings (a set of this shape can never be an $\hat{I}$-set and all (possible) points $I \cdot N$ have already been mapped for every I of that kind.

Now consider the $(n-1)^{\text{th}}$ covering of D consisting of the 2^{n-1} sets $I_{a_1 a_2 \dots a_{n-1}}$ ($a_i = 0$ or 2). A number of these I-sets is also an $\hat{I}$-set. The remaining I-sets (possibly none at all) have been struck out according to the supposed induction, while the points of N which they may contain have already, at one of the first $(n-1)$ steps, been mapped onto points of A. We now construct the mapping at the n^{th} step as follows. Consider the n^{th} covering of D by the sets $I_{a_1 a_2 \dots a_{n-1} a_n}$. Every $\hat{I}_{a_1 a_2 \dots a_{n-1}}$ is divided into two sets $I_{a_1 a_2 \dots a_{n-1} 0}$ and $I_{a_1 a_2 \dots a_{n-1} 2}$. If both of these are $\hat{I}$-sets nothing happens. If they are not, then at least one, for instance $I_{a_1 a_2 \dots a_{n-1} 0} = \hat{I}_{a_1 a_2 \dots a_{n-1} 0}$, is an $\hat{I}$-set. We map all possible points of $N \cdot I_{a_1 a_2 \dots a_{n-1} 2}$ on a fixed but arbitrarily chosen point of $A \cdot \hat{I}_{a_1 a_2 \dots a_{n-1} 0}$. This process takes place for the considered n for every $\hat{I}_{a_1 a_2 \dots a_{n-1}}$. After this all sets $I_{a_1 a_2 \dots a_n}$ which are not $\hat{I}$-sets, are struck out.

This process we think infinitely continued in the same way and we shall prove that the thus constructed mapping f, if we moreover put $f(a) = a$ for $a \subset A$, is a retraction of N in A.

First every point $n \subset N$ is indeed mapped onto one point of A. For, by definition, $f(a) = a$ for the points a of A; a point $n \subset N - A$ lies at a certain distance ε from A, as A is closed in N, so that there may be found a j^{th} covering of A by $\hat{I}$-sets for sufficiently great j such that the point n does not belong to any of the $\hat{I}$-sets of this covering, in other words is mapped on a point of A.

The mapping f further is continuous. Let a sequence of points n_i ($i = 1, 2, \dots$) of N be converging to a point $n \subset N - A$. The point n is, e.g. at the k^{th} step, mapped on a point a of A as it is belonging to an I-set which is struck out (which in other words is not an $\hat{I}$-set). But this I-set is a neighbourhood of n so that also nearly all points n_i are mapped on the point a; the mapping f therefore is continuous for such a sequence. — There remains the possibility that the sequence of points n_i converges to a point $a \subset A$. a is determined by the intersection of the lump-sets of a sequence

$$D \supset \hat{I}_{a_1} \supset \hat{I}_{a_1 a_2} \supset \hat{I}_{a_1 a_2 a_3} \supset \dots$$

For every natural number j almost all points n_i are lying in $\hat{I}_{a_1 a_2 \ldots a_j}$. Then these points n_i have not yet been mapped by previous steps, since they cannot be lying in a struck-out set I. At the $(j+1)^{\text{th}}$ and subsequent steps the points n_i are mapped on points within $\hat{I}_{a_1 a_2 \ldots a_j}$, so that also nearly all points $f(n_i)$ $(i = 1, 2, \ldots)$ are lying in $\hat{I}_{a_1 a_2 \ldots a_j}$, i.e., f is continuous.

Conversely, we still have to prove that a retract A of N is closed in N. This is, however, a matter of course.

3. We now determine all continuous invariants of the family F_c of all compact 0-dimensional sets. The countability is a not-trivial continuous invariant; further, it has been shown in [3] that the (compact) countable (and so naturally 0-dimensional) sets possess exactly the not-trivial continuous invariants $[<\lambda, l]$. These are already all (not-trivial) continuous invariants of F_c. For, be N a compact, not-countable, 0-dimensional set of F_c; then, because of the non-countability, N has a not-vacuous kernel, dense in itself. This kernel is, according to a theorem of BROUWER [2], homoeomorphic with the discontinuum D. Further, if A is an arbitrary compact 0-dimensional set, then according to the already mentioned theorem of SIERPINSKI [6] A is homoeomorphic with a closed subset of D, and therefore also with a closed subset of N; according to theorem I A may be considered as a retraction of N. So *every not-countable compact 0-dimensional set N may be mapped continuously on every compact 0-dimensional set A*, by which we have shown that there exist no other continuous invariants in F_c than the already mentioned ones.

Theorem II [1]. *The family of the compact 0-dimensional sets has, as only (not-trivial) continuous invariants, the properties of the "countability" and $[<\lambda, l]$.*

4. We shall now proceed to determine all continuous invariants of the family F_m of all microcompact 0-dimensional sets. All sets mentioned in this number will be supposed to be microcompact and 0-dimensional. We know already the existence of the following (not-trivial) continuous invariants in F_m: the properties $[<\lambda, l]$, the property $\varkappa \cdot \alpha$, i.e. "both compact and countable", the property α (countability), the property $\varkappa$ (compactness). Also the property $\varkappa + \alpha$ (a set has the property $\varkappa + \alpha$ if the set is the sum of two (disjunct) sets one of which is compact while the other is countable; one of these two sets may be vacuous) is an apparently continuous invariant. The logical dependency of these axioms is illustrated by fig. 1. We shall prove that these are the only continuous invariants of F_m.

Theorem III. *The family of microcompact 0-dimensional sets has, as*

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[1]) Mentioned without proof in [4], theorem IV.

only not-trivial continuous invariants, the properties: $[<\lambda, l]$ [2]), $\varkappa\ a$, a, $\varkappa$ and $\varkappa + a$.

By this theorem especially all (e.g. in an n-dimensional Euclidian space) *closed 0-dimensional sets have been classified continuously* [3]).

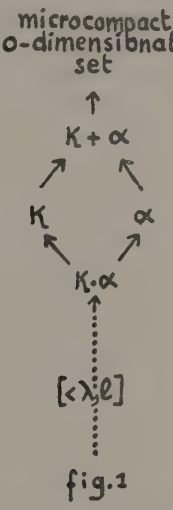

fig. 1

P r o o f. The compact sets have already been investigated (theorem II); the investigation of the countable sets has been described in [3] (theorem V), where we even proved that *every* (not only microcompact) not-compact countable set may be mapped continuously on *every* countable set.

Therefore we now first must investigate a set with property $\varkappa + a$, which does not possess the property $\varkappa$ or (and) a (for those have already been investigated); in other words, we start from a set $N = C + A$ where C is compact and not-countable, while A is countable and not-compact. We have to prove that N may be mapped continuously on every set $N' = C' + A'$ (C' compact, A' countable; C' or A' may be vacuous) with property $\varkappa + a$.

Before proceeding to the construction of the continuous mapping in question we insert a necessary remark. Every micro-compact 0-dimensional set N may, as known, be compactified to a compact 0-dimensional (separable) space $\bar{N} = N + P$ by one point P. Putting this for $N = C + A$, so $\bar{N} = C + A + P$, then apparently there may be found a neighbourhood $U = U(C/N)$ of C in N, which, like C, is compact and not-countable, so that $N - U$ is countable and not-compact, while both U and $N - U$ are closed in N. U is again denoted by C and $N - U$ by A; then $N = C + A$ has therefore been divided into two disjunct closed sets, of which C is compact and not-countable, and A is countable but not-compact.

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[2]) There are apparently $\aleph_1$ continuous invariants of this kind.

[3]) Since every such closed set is microcompact.

We now construct the continuous mapping on $N' = C' + A'$. If N' is compact, then N' is topologically equivalent with a closed subset N'' of C, since C has a discontinuum as a subset; a retraction of N in N'' (theorem I) together with the topological mapping of N'' on N' gives the required continuous mapping of N on N'.

If N' is not compact, so if A' is not compact, and moreover if C' is not vacuous, then we map C continuously on C' (which apparently is possible according to the above-written) and A on A'. Both mappings together give an apparently continuous mapping of N on N'.

If, finally, N' is not compact but C' is vacuous, then we map A on A' and all C on one arbitrary point of A'. The required continuous mapping is therefore always possible.

Secondly we have to consider an 0-dimensional microcompact set N which does not possess the property $\varkappa + \alpha$, and we must prove that every such N may be mapped on every other (0-dimensional micro-compact) set N'. Suppose N' is not compact (if N' is compact the following proof may be simplified). We compactify the micro-compact sets N and N' each by one point, P and P' respectively, to the compact 0-dimensional sets $\overline{N} = N + P$ and $\overline{N}' = N' + P'$. We determine two monotonicly decreasing systems of neighbourhoods U_i and $U_i'(i = 0, 1, 2, \ldots)$ of P and P' in $\overline{N}$ and $\overline{N}'$, consisting of lump-sets, while $U_0 = N$ and $U_0' = N'$. Since N does not possess the property $\varkappa + \alpha$ there are infinitely many values i (e.g. j_k $(k = 0, 1, 2, \ldots)$) so that the lump-set $V_{j_k} = U_{j_k} - U_{j_{k+1}}$ has a discontinuum as a subset, in other words, every V_{j_k} may be mapped continuously on the lump-set $U_k' - U_{k+1}'$ $(k = 0, 1, 2, \ldots)$. The possibly remaining sets $U_i - U_{i+1}$ $(i \neq j_k)$ we all map on one point of N'. Thus an apparently continuous mapping of N on N' is constructed.

It follows immediately from the above that there exist no other continuous invariants than the already mentioned, by which the proof has been completed.

5. *Final remark.* The above used method is not suited to the determination of the continuous invariants of the family of all 0-dimensional sets nor to the determination of the topological invariants of the family of all compact 0-dimensional sets.

On the contrary, in these cases things turn out to be very complicated. This is already shown by the following example. The 0-dimensional sets N, having a discontinuum D as a subset, play an exceptional part, since every such N may be mapped continuously on every compact 0-dimensional N' (N' is topologically equivalent with a closed subset N'' of D and N'' is a retraction of N). At the same time there exist, however, not-countable 0-dimensional sets which do not contain a discontinuum as a subset (according to a theorem of BERNSTEIN, comp. [5], p. 176); sets of this kind are called *totally imperfect*.

We now think a discontinuum D divided into two totally imperfect subsets S_1 and S_2 (this is possible) and define S_1' and S_2' as pointsets on the real axis, homoeomorphic with S_1 and S_2, and isolated in respect to each other. The sum $S' = S_1' + S_2'$ therefore is also totally imperfect. Yet it is apparently possible *to map this totally imperfect set S' one-to-one and continuously on the discontinuum D*, namely by mapping the corresponding points of S_1 and S_1', resp. S_2 and S_2', on each other.

But to what extent is it possible to map an arbitrary totally imperfect set continuously on a given (not-countable) 0-dimensional set?

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[4]) I take this opportunity to correct a disturbing error in this paper:
p. 238, second line: read $\aleph_1$ instead of $\aleph_0$.
p. 243, nr. 1.10, 11th line: read $\aleph_1$ instead of $\aleph_0$.
p. 243, nr. 1.10, 12th line: cross out the word "even".

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Third communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 27, 1946.)

§ 12. *Boundedly irreducible and reducible star bodies.*

In the case of unbounded star bodies of the finite type, the following definition seems to be of interest:

Definition C: *The unbounded star body K of the finite type is called boundedly reducible if there exists a bounded star body H such that $H \prec K$, and it is called boundedly irreducible if no bounded star body H with $H \prec K$ exists.*

Theorem J: *For every dimension n, there exists a boundedly irreducible star body K in R_n.*

Proof: We choose for K the star body

$$K^*: \qquad x_1^2 x_2^2 \dots x_{n-1}^2 (x_1^2 + x_2^2 + \dots x_n^2) \leqq 1$$

considered already in the last paragraph, and for H any bounded star body contained in K. As we saw, K^* is contained in

$$K_0: \qquad |x_1 x_2 \dots x_n| \leqq 1$$

and of the same determinant $\Delta(K^*) = \Delta(K_0)$; moreover, all boundary points of K^* are *inner* points of K_0. Hence the boundary points of H are likewise *inner* points of K_0; there exists then a constant θ with $0 < \theta < 1$ such that

$$|x_1 x_2 \dots x_n| \leqq \theta$$

for all points of H. But this implies that

$$\Delta(H) \leqq \theta \, \Delta(K_0) < \Delta(K^*),$$

and so it is not true that $H \prec K$, whence the assertion.

If K is any star body, then, as in Part I, we denote by K^t the set of all points X of K for which $|X| \leq t$.

Theorem K: *If the star body K: $F(X) \leq 1$ is boundedly irreducible, then there exists to every $t > 0$ a critical lattice Λ of K and an infinite sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \dots$, with the following properties:*

(a): *All lattices Λ_r are K^t-admissible.*

(b): $d(\Lambda_r) < \Delta(K) \qquad (r = 1, 2, 3, \dots).$

(c): *The lattices Λ_r tend to the lattice Λ.*

Proof: Denote, for $r = 1, 2, 3, \ldots$, by $\Lambda^{(r)}$ any critical lattice of K^{r+t}; all these lattices are K^t-admissible. Since K^{r+t} is a bounded subset of K, from the hypothesis,

$$d(\Lambda^{(r)}) = \triangle(K^{r+t}) < \triangle(K) \qquad (r = 1, 2, 3, \ldots).$$

Further, by the corollary to Theorem 10 of Part I,

$$\lim_{r\to\infty} d(\Lambda^{(r)}) = \lim_{r\to\infty} \triangle(K^{r+t}) = \triangle(K).$$

The lattices $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \ldots$ form therefore a *bounded* sequence, and so, by Theorem 2 of Part I, there exists an infinite subsequence

$$\Lambda_1 = \Lambda^{(k_1)}, \quad \Lambda_2 = \Lambda^{(k_2)}, \quad \Lambda_3 = \Lambda^{(k_3)}, \ldots \quad (1 \leqq k_1 < k_2 < k_3 < \ldots),$$

which converges to a limiting lattice, Λ say. It is clear that the so defined lattices Λ_r and Λ satisfy the assertions (a), (b), and (c) of the theorem; but there remains to prove that Λ is a critical lattice of K.

We show firstly that Λ is *K-admissible*. Let $P \neq O$ be any point of Λ. There is then in each lattice Λ_r a point $P_r \neq O$ such that

$$\lim_{r\to\infty} |P_r - P| = 0.$$

Further, if r is sufficiently large,

$$|P_r| < t + k_r.$$

Since Λ_r is K^{t+k_r}-admissible, this means that

$$F(P_r) \geqq 1,$$

whence by the continuity of $F(X)$,

$$F(P) = \lim_{r\to\infty} F(P_r) \geqq 1,$$

i.e. Λ is K-admissible.

Secondly, Λ is even critical since

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda_r) = \triangle(K).$$

This completes the proof.

Definition D: *Let K be an infinite star body of the finite type. Then a critical lattice Λ of K is called strongly critical if there exists a bounded star body K^* contained in K such that*

$$d(\Lambda^*) \geqq d(\Lambda)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ [12]).

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[12]) We say that Λ^* is near to Λ if there exist reduced bases

$$Y_1, Y_2, \ldots, Y_n \text{ and } Y_1^*, Y_2^*, \ldots, Y_n^*$$

of Λ and Λ^* such that all numbers

$$Y_g - Y_g^* \qquad (g = 1, 2, \ldots, n)$$

are less than a prescribed constant.

It is clear from this definition and from Theorem K that if K is boundedly irreducible, then at least one critical lattice of K is not strongly critical. Hence the following theorem follows at once:

Theorem L: *Let K be an infinite star body of the finite type, and let further every critical lattice of K be strongly critical. Then K is boundedly reducible.*

Proof Assume that, on the contrary, K is boundedly irreducible, and denote by K^* any *bounded* star body contained in K. There exists then a positive number t such that $|X| \leq t$ for every point X of K^*. If Λ is now the critical lattice of K given for this value of t by Theorem K, then Λ is clearly *not* strongly critical.

Theorem L allows in many cases to decide whether a given unbounded star body is boundedly reducible. A few such cases are discussed in the next paragraphs.

§ 13. *Examples of boundedly reducible star domains in R_2.*

In his work on binary cubic forms [13], L. J. MORDELL showed that the two star domains

K_1: $$|x_1 x_2 (x_1 + x_2)| \leqslant 1$$

and

K_2: $$|x_1^3 + x_2^3| \leqslant 1$$

are of determinants

$$\Delta(K_1) = \sqrt[3]{7} \text{ and } \Delta(K_2) = \sqrt[6]{\tfrac{2^3}{2^7}}.$$

It is of interest that his proof gave, incidentally, the result that both star domains are *boundedly reducible*; they were the first non-trivial examples of this kind. I later gave an even simpler example,

K_3: $$|x_1 x_2| \leqslant 1, \qquad \text{with } \Delta(K_3) = \sqrt{5},$$

of a boundedly reducible star domain, and made some applications of this property of K_3 [14].

By means of Theorem L, independent proofs that K_1, K_2, and K_3 are boundedly reducible, may be easily obtained. To this purpose, one uses considerations analogous to those in the next paragraphs.

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[13] Since his latest proof has not yet appeared, I refer to two articles Journal Lond. Math. **18**, 201—210 and 210—217 (1943), where the two affine-equivalent regions

$$|x_1^3 + x_1^2 x_2 - 2 x_1 x_2^2 - x_2^3| \leqslant 1 \text{ and } |x_1^3 - x_1 x_2^2 - x_2^3| \leqslant 1$$

are considered.

[14] Proc. Cambr. Phil. Soc. **40**, 108—116, 116—120 (1943), and Journ. Lond. Math. Soc. **18**, 233—238 (1943).

§ 14. *The star body* $|x_1 x_2 x_3| \leq 1$ *in* R_3.

By a theorem of H. DAVENPORT [15]), the star body

K: $\qquad |x_1 x_2 x_3| \leqq 1$

is of determinant

$$\triangle(K) = 7.$$

Let

$$\theta = 2\cos\frac{2\pi}{7}, \quad \varphi = 2\cos\frac{4\pi}{7}, \quad \psi = 2\cos\frac{6\pi}{7}$$

be the three roots of

$$t^3 + t^2 - 2t - 1 = 0.$$

Then

$$\Lambda_0: x_1 = \theta u_1 + \varphi u_2 + \psi u_3, \; x_2 = \varphi u_1 + \psi u_2 + \theta u_3, \; x_3 = \psi u_1 + \theta u_2 + \varphi u_3, \qquad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots)$$

is a critical lattice of K, and every other critical lattice of K is of the form $\Lambda = \Omega \Lambda_0$ where Ω is one of the automorphisms

Ω: $\qquad x_1 = t_1 x'_\alpha, \quad x_2 = t_2 x'_\beta, \quad x_3 = t_3 x'_\gamma$

of K; here t_1, t_2, t_3 are real numbers satisfying

$$t_1 t_2 t_3 = \mp 1,$$

and α, β, γ is any permutation of 1, 2, 3.

Theorem M: *The star body* K: $|x_1 x_2 x_3| \leq 1$ *in* R_3 *is boundedly reducible.*

Proof: It suffices to show that Λ_0 *is a strongly critical lattice of* K because, by affine invariance, the same is then true for all critical lattices of K, and so the assertion follows immediately from Theorem L.

By definition, the lattice Λ_0 is strongly critical if there exists a bounded star body $K^* < K$ such that

$$d(\Lambda^*) \geqq d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ_0. Such a lattice Λ^* near to Λ_0 contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 obtained for $u_1 = 1$, $u_2 = 0$, $u_3 = 0$. There exists then an automorphism

Ω^*: $\qquad x_1 = t_1^* x_1^*, \quad x_2 = t_2^* x_2^*, \quad x_3 = t_3^* x_3^* \qquad (t_1^* t_2^* t_3^* = 1)$

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[15]) Proc. Lond. Math. Soc. **44**, 412—431 (1938).

of K which changes P^* into a point $\Omega^* P^*$ collinear with O and P_0:

$$t_1^* \theta^* : t_2^* \varphi^* : t_3^* \psi^* = \theta : \varphi : \psi.$$

Hence, by affine invariance, it suffices to show that

$$d(\Lambda^*) \geqq d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* which is (i) sufficiently near to Λ_0, and which (ii) contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 such that O, P_0, P^* are collinear.

Now every lattice Λ^* near to Λ_0 can be written in the form

$$\Lambda^* : x_1 = \theta v_1 + \varphi v_2 + \psi v_3, \quad x_2 = \varphi v_1 + \psi v_2 + \theta v_3, \quad x_3 = \psi v_1 + \theta v_2 + \varphi v_3$$

with

$$\left.\begin{aligned} v_1 &= u_1 + (a_{11} u_1 + u_{12} u_2 + u_{13} u_3), \\ v_2 &= u_2 + (a_{21} u_1 + u_{22} u_2 + u_{23} u_3), \\ v_3 &= u_3 + (a_{31} u_1 + a_{32} u_3 + a_{33} u_3), \end{aligned}\right\} (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$

where the coefficients a_{hk} are real numbers such that

$$a = \max_{h,k=1,2,3} |a_{hk}|$$

is less than any given constant. The point P^* of Λ^* corresponding to P_0 is

$$P^* = \big((1+a_{11})\theta + a_{21}\varphi + a_{31}\psi, (1+a_{11})\varphi + a_{21}\psi + a_{31}\theta, (1+a_{11})\psi + a_{21}\theta + a_{31}\varphi\big)$$

and is collinear with O and P_0 if and only if

(a): $$a_{21} = a_{31} = 0,$$

because the three points

$$P_0 = (\theta, \varphi, \psi), \quad P_1 = (\varphi, \psi, \theta), \quad P_2 = (\psi, \theta, \varphi)$$

are linearly independent. We consider from now on only lattices Λ^* for which the condition (a) is satisfied.

Put for shortness,

$$S(U) = (\theta u_1 + \varphi u_2 + \psi u_3)(\varphi u_1 + \psi u_2 + \theta u_3)(\psi u_1 + \theta u_2 + \varphi u_3) =$$
$$= (u_1^3 + u_2^3 + u_3^3) - 4(u_2 u_3^2 + u_3 u_1^2 + u_1 u_2^2) + 3(u_2^2 u_3 + u_3^2 u_1 + u_1^2 u_2) - u_1 u_2 u_3,$$

so that

$$x_1 x_2 x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. Similarly

$$x_1 x_2 x_3 = S(V)$$

for the point of A^* belonging to $V = (v_1, v_2, v_3)$, or, on replacing V by its value in U,

$$x_1 x_2 x_3 = S(U) + T(U);$$

here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^2) + (B_1 u_2 u_3^2 + B_2 u_3 u_1^2 + B_3 u_1 u_2^2) + \\ + (C_1 u_2^2 u_3 + C_2 u_3^2 u_1 + C_3 u_1^2 u_2) + D u_1 u_2 u_3,$$

with the coefficients

$$\begin{aligned}
A_1 &= \cdot 3 a_{11} && && && + O(a^2),\\
A_2 &= && 3 a_{22} && - 4 a_{12} && + 3 a_{32} && + O(a^2),\\
A_3 &= && && 3 a_{33} && - 4 a_{23} && + 3 a_{13} + O(a^2),\\
B_1 &= && - 4 a_{22} && - 8 a_{33} && + 3 a_{12} && + 6 a_{23} && + 3 a_{32} - a_{13} + O(a^2),\\
B_2 &= - 8 a_{11} && && - 4 a_{33} && + 3 a_{23} && + 3 a_{13} + O(a^2),\\
B_3 &= - 4 a_{11} && - 8 a_{22} && + 6 a_{12} && - a_{32} && + O(a^2),\\
C_1 &= && 6 a_{22} && + 3 a_{33} && - a_{12} && + 3 a_{23} && - 8 a_{32} - 4 a_{13} + O(a^2),\\
C_2 &= 3 a_{11} && && + 6 a_{33} && - a_{23} && - 8 a_{13} + O(a^2),\\
C_3 &= 6 a_{11} && + 3 a_{22} && + 3 a_{12} && - 4 a_{32} && + O(a^2),\\
D &= - a_{11} && - a_{22} && - a_{33} && - 8 a_{12} && - 8 a_{23} && + 6 a_{32} && + 6 a_{13} + O(a^2),
\end{aligned}$$

where, in all cases, the O-term consists of the products of two or three of the a_{hk}. These formulae imply, in particular, that when a tends to zero, then the maximum

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|)$$

satisfies the inequality

$$A = O(a).$$

On solving for the coefficients a_{hk}, we find further that

$$\begin{aligned}
3 a_{11} &= A_1 && && && && && + O(a^2),\\
105 a_{22} &= - 70 A_1 - 15 A_2 + 30 A_3 + 18 B_1 - 12 B_2 - 27 B_3 - 6 D + O(a^2),\\
105 a_{33} &= 65 A_1 + 45 A_2 && && - 18 B_1 + 12 B_2 + 27 B_3 - 9 D + O(a^2),\\
35 a_{12} &= - 25 A_1 - 5 A_2 + 15 A_3 + 9 B_1 - 6 B_2 - 6 B_3 - 3 D + O(a^2),\\
35 a_{23} &= 35 A_1 + 15 A_2 - 5 A_3 - 6 B_1 + 9 B_2 + 9 B_3 - 3 D + O(a^2),\\
35 a_{32} &= - 10 A_1 + 10 A_2 + 10 A_3 + 6 B_1 - 4 B_2 + B_3 - 2 D + O(a^2),\\
35 a_{13} &= 25 A_1 + 5 A_2 + 5 A_3 - 2 B_1 + 8 B_2 + 3 B_3 - D + O(a^2),
\end{aligned}$$

and we also obtain the three identities,

$$\begin{aligned}
5 C_1 &= 5 A_1 - 5 A_2 - 10 A_3 - 7 B_1 + 3 B_2 - 2 B_3 - D + O(a^2),\\
5 C_2 &= - 10 A_1 + 5 A_2 - 5 A_3 - 2 B_1 - 7 B_2 + 3 B_3 - D + O(a^2),\\
5 C_3 &= - 5 A_1 - 10 A_2 + 5 A_3 + 3 B_1 - 2 B_2 - 7 B_3 - D + O(a^2),
\end{aligned}$$

and the inequality

$$a = O(A), \qquad O(a^2) = O(A^2).$$

So far, the star body K^* has not yet been defined; nor have we yet used that Λ^* is K^*-admissible. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqslant 3, \quad |u_2| \leqslant 3, \quad |u_3| \leqslant 3,$$

belong to K^t. Then the ten points of Λ_0 given by

$$U = (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (0, 1, 1),\ (1, 0, 1),\ (1, 1, 0),$$
$$(0, -1, -2),\ (-2, 0, -1),\ (-1, -2, 0),\ (-1, -1, -1),$$

satisfy the equation,

$$S(U) = 1.$$

The points of Λ^* belonging to the same U cannot be *inner* of $K^* = K^t$ since Λ^* is K^*-admissible. The numbers

$$\alpha_1, \alpha_2, \alpha_3, \quad \beta_1, \beta_2, \beta_3, \quad \gamma_1, \gamma_2, \gamma_3, \quad \delta$$

defined by

$$T(1, 0, 0) = \alpha_1,\ T(0, 1, 1) = \beta_1,\ T(0, -1, -2) = \gamma_1,$$
$$T(0, 1, 0) = \alpha_2,\ T(1, 0, 1) = \beta_2,\ T(-2, 0, -1) = \gamma_2,\ T(-1, -1, -1) = \delta,$$
$$T(0, 0, 1) = \alpha_3,\ T(1, 1, 0) = \beta_3,\ T(-1, -2, 0) = \gamma_3,$$

are therefore non-negative because

$$x_1 x_2 x_3 = S(U) + T(U) = 1 + T(U) \geqslant 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{aligned}
\alpha_1 &= A_1, \\
\alpha_2 &= A_2, \\
\alpha_3 &= A_3, \\
\beta_1 &= A_2 + A_3 + B_1 + C_1, \\
\beta_2 &= A_1 + A_3 + B_2 + C_2, \\
\beta_3 &= A_1 + A_2 + B_3 + C_3, \\
\gamma_1 &= -A_2 - 8A_3 - 4B_1 - 2C_1, \\
\gamma_2 &= -8A_1 - A_3 - 4B_2 - 2C_2, \\
\gamma_3 &= -A_1 - 8A_2 - 4B_3 - 2C_3, \\
\delta &= -A_1 - A_2 - A_3 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3 - D,
\end{aligned}$$

and conversely,

$$
\begin{aligned}
A_1 &= \quad \alpha_1, \\
A_2 &= \qquad \alpha_2, \\
A_3 &= \qquad\quad \alpha_3, \\
B_1 &= \quad \tfrac{1}{2}\alpha_2 - 3\alpha_3 - \beta_1 \qquad - \tfrac{1}{2}\gamma_1, \\
B_2 &= -3\alpha_1 \quad + \tfrac{1}{2}\alpha_3 \quad - \beta_2 \qquad - \tfrac{1}{2}\gamma_2, \\
B_3 &= \tfrac{1}{2}\alpha_1 - 3\alpha_2 \qquad - \beta_3 \qquad - \tfrac{1}{2}\gamma_3, \\
C_1 &= \quad -\tfrac{3}{2}\alpha_2 + 2\alpha_3 + 2\beta_1 \qquad + \tfrac{1}{2}\gamma_1, \\
C_2 &= 2\alpha_1 \quad - \tfrac{3}{2}\alpha_3 \quad + 2\beta_2 \qquad + \tfrac{1}{2}\gamma_2, \\
C_3 &= -\tfrac{3}{2}\alpha_1 + 2\alpha_2 \qquad + 2\beta_3 \qquad + \tfrac{1}{2}\gamma_3, \\
D &= \quad \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 - \beta_3 \qquad\qquad - \delta.
\end{aligned}
$$

From these formulae, we deduce that identically,

$$
\begin{aligned}
& \gamma_1 = \quad 2\alpha_1 + \alpha_2 - 2\alpha_3 \qquad + 2\beta_2 \qquad + 2\delta + O(a^2), \\
(b): \quad & \gamma_2 = -2\alpha_1 + 2\alpha_2 + \alpha_3 \qquad + 2\beta_3 + 2\delta + O(a^2), \\
& \gamma_3 = \quad \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\beta_1 \qquad + 2\delta + O(a^2).
\end{aligned}
$$

If further

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then, by these formulae, all three numbers a, A, α are of the same order,

$$\alpha = O(a) = O(A), \quad O(a^2) = O(\alpha^2), \quad O(A^2) = O(\alpha^2).$$

The proof of the theorem proceeds now as follows:

The lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0) \begin{vmatrix} 1 + a_{11} & a_{12} & a_{13} \\ 0 & 1 + a_{22} & a_{23} \\ 0 & a_{32} & 1 + a_{33} \end{vmatrix} = d(\Lambda_0)(1 + \sigma),$$

where

$$
\begin{aligned}
& \sigma = a_{11} + a_{22} + a_{33} + O(a^2), \\
(c): \quad & = \tfrac{2}{7}(A_1 + A_2 + A_3) - \tfrac{1}{7} D + O(A^2), \\
& = \tfrac{1}{7}(\alpha_1 + \alpha_2 + \alpha_3) + \tfrac{1}{7}(\beta_1 + \beta_2 + \beta_3) + \tfrac{1}{7}\delta + O(\alpha^2).
\end{aligned}
$$

Assume now that Λ^* is so near to Λ_0 that a, hence also A and α, are sufficiently small. Then

either

$$\sigma > 0, \quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0, \quad d(\Lambda^*) = d(\Lambda_0).$$

By (c), the second case cannot hold unless

$$(d): \qquad \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = \delta = 0,$$

hence also

(e): $$\gamma_1 = \gamma_2 = \gamma_3 = 0,$$

because by (b),

$$\max(|\gamma_1|, |\gamma_2|, |\gamma_3|)$$

is of at most the same order as

$$\max(|a_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\delta|).$$

The equations (d) and (e) imply next that

$$A_1 = A_2 = A_3 = B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D = 0,$$

hence also

$$a_{11} = a_{22} = a_{33} = a_{12} = a_{23} = a_{32} = a_{13} = 0;$$

and so Λ^* coincides with Λ_0. This concludes the proof.

Although the proof just given is a pure *existence proof*, it can easily be altered so as to lead to the construction of a bounded star body K^* satisfying $K^* \prec K$.

The next theorems are all proved in a manner similar to that of Theorem M.

Physical Geography. — *Projective-geometric treatment of* O. LEHMANN's *theory of the transformation of steep mountain slopes.* By J. P. BAKKER and J. W. N. LE HEUX. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of April 27, 1946.)

Geomorphologic science, at its present stage of development, suffers from a lack of quantitative exactness. All too often discussions do not rise above qualitative aspects, with the result that communications, although in themselves most interesting and emanating from prominent authors, frequently remain quantitatively undefined, so that no decision in favour of one view or another can be taken without, in its turn, being to a great extent subjective.

WILLIAM MORRIS DAVIS has enriched geomorphology, in a large number of publications (Nos. 1 and 2) [1]), by his brilliant classification of co-existing types of landscapes. As he himself was already aware, the basis of his argument was always partly inductive and partly deductive. To the extent, however, that DAVIS applied the deductive method, it was certainly not pure as such. For DAVIS' purpose was to explain present landscape-types descriptively. He knew in his own mind, from the start, the type of landscape his deduction ought to produce. His argument accordingly directed itself to a definite aim, which endowed it in the first place with a final, not a deductive character. Such a method, whatever its merits, is hardly justified in natural science.

WALTHER PENCK, in his "Morphologische Analyse" (No. 3), attempted further to elaborate the deductive method. His premature death unfortunately prevented him from carrying out this work to its full extent. The idea before his mind's eye, namely the mathematical treatment of the theory of the development of mountain slopes, was never realized. His deductions are largely unacceptable, a fact which has been pointed out before, amongst others by J. P. BAKKER (*vide* Nos. 4, 5 and 6). We shall return to this later, in connection with LEHMANN's theory.

We are, therefore, still faced with the question: is it possible to trace a connection, in a way justified from a quantitatively exact standpoint, between unequal mountain-shapes existing side by side, such as valley- and mountain slopes, longitudinal profiles of rivers, etc.? Only in a few isolated cases, as in the study of the conditions of the formation of meander belts, longitudinal profiles, and deposits in flood plains does the science of physiologic morphology succeed in lifting a tip of the veil, and carrying the statement of the problem a little further (HJULLSTRÖM, LEIGHLY and others).

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[1]) These numbers refer to the list of literature at the close of this article.

When, however, it is a question of determining alterations in the shape of mountain- and valley-profiles in firm rocks, we are forced to admit that the processes of change take place at such an infinitely slow rate that it is only in very rare exceptions — as, for example, in the case of landslides — that the transition from one mountain-form to another, differing considerably from the first, can be observed directly. In the science of physiognomic morphology, therefore, in which direct evidence *ad oculos* is generally lacking, both the problem itself and the argumentation will necessarily be of a different character from that in physiologic morphology. The question, however, still remains: Is it possible, in physiognomic morphology (leaving aside morpho-tectonic problems), to arrive at a more exact argumentation on a mathematical basis?

In 1932, BAKKER published an article in the Dutch language in which he expressed the opinion that, if the signs were not deceptive, the development of a more theoretical morphology, based upon physico-mathematical treatment, was imminent (Cyclus-theorie en Morphologische Analyse, part 2, p. 17/18). Although doubt was expressed in some quarters as to the correctness of the author's view, a statement confirming it appeared the very next year in the form of OTTO LEHMANN's "Morphologische Theorie der Verwitterung von Steinschlagwänden" (No. 7).

As far as we are aware, OTTO LEHMANN's theory, which bases itself upon a law discovered as early as 1866 by the Rev. OSMOND FISHER (No. 8), and starts from rigorously defined premisses, is the only existing deduction in physiognomic morphology that is quantitatively accounted for. It is, however, merely a beginning; and many problems in which a rigorous deduction would seem possible, were left untouched by LEHMANN.

Apart from the above theory however, another and quite different way to arrive at greater exactness in physiognomic geomorphology might be followed. With the steady increase in the publication of photogrammetrically produced topographical detail-maps the possibility is created of determining with greater exactness the shapes of longitudinal and cross-profiles of valleys and mountain slopes than was hitherto possible. With the aid of nomographic methods it should be possible to open up an entirely new branch of this science, namely, geomorphologic curve-analysis, which may well prove to be able to throw new light upon many of the problems confronting us, such as that of the equilibrium-curves of rivers, about which discussion appears to have come to a deadlock. To this question, too, we shall return later. We shall now present LEHMANN's theory in the following new guise.

In LEHMANN's theory, the example is given of an evenly parallel receding steep mountain slope or cliff FS of known height h, with a great slope-angle β (fig. 1 and 2), bordered at the top by a horizontal plateau SR, and at the foot by an also approximately horizontal form FR', upon which falling angular debris may accumulate.

It is further assumed that the sheer wall FS is exposed exclusively to

the free play of weathering, any direct effect of lateral erosion, by rivers, glaciers, etc. being left out of consideration.

Numerous authors, both European and American, have described such steep rock-slopes, in respect of which a parallel recession as a result of weathering approximately may be assumed to have placed during some phase of their development. We know them from the upper mountain ranges of the Alpine type; from the cuesta- and mesa-landscapes of Western Europe and North America; from some "inselbergs" in the arid and semi-arid regions of Africa, South- and East-Asia, America, etc. Observations have also shown that, in those cases where the screes of angular debris are not deposited on the terrace *R'F* suddenly, by a landslide, but little by little, these screes protect a rocky nucleus with a convex cross-profile (*FABCR* in fig. 1); of this nucleus, of which we shall assume that its constitution remains unchanged, therefore, the exact shape has to be determined.

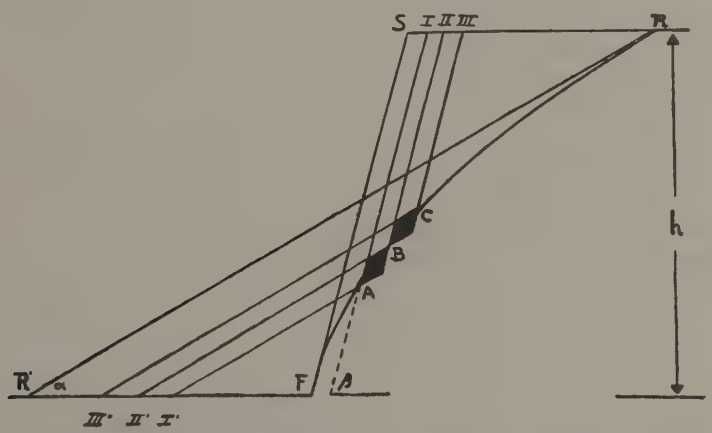

Fig. 1. Parallel recession of the steep slope *FS* and the simultaneous formation of screes [*I'A*, *II'B* etc.]. After the disappearance of the steep slope with angle β, the cross profile *R'R* of the screes also forms the tangent on the curve *FABCR* at the intersection point *R* with the plateau (after OTTO LEHMANN).

As early as 1866, OSMOND FISHER concluded that, if the volumes of ruptured rock per unit of time (i.e., in fig. 1, *SFAI*, *SFBII*, *SFCIII*, etc.) are equal to those of the fragments deposited on *R'F* (i.e., in fig. 1, *I'AF*, *II'FB*, *III'FC*, etc.), the curve *FABCR* will assume a parabolic shape. This is, in fact, quite correct; but it is a special case of a general theory put forward by LEHMANN, in which, e.g., the slope-angle of the cliff was introduced as a variable. Moreover, attention should be given before anything to the ratio between the volume of solid rock removed per unit of time from the steep mountain slope, and the screes volume accumulated at the base during the same time-unit. It is evident that FISHER's case is possible only when part of the fallen fragments is removed from the terrace *R'F* owing to outside influences e.g. by avalanches, since otherwise

the respective volumes of rock removed and debris deposited per unit of time, because of the greater pores-volume of the screes, can never be equal. However this may be, in order to arrive at an exact derivation, we shall begin by introducing the formula

$$\frac{\text{rock-volume}}{\text{screes-volume}} = \frac{1-c}{1},$$

in which c is constant.

We further take, following LEHMANN, the basic point of the steep slope F as zero-point in our co-ordination system, while imagining the stretches of rock and screes, respectively, I, II, III and I', II', III', to be infinitely thin. We may then neglect the black triangles at AB, BC, etc., in fig. 1 and regard the corresponding stretches of fallen rock and deposited debris as parallelograms. We further imagine the wall to be perpendicular to the plane of the drawing, so that two-dimensional figures may suffice, which enables us to replace, in our exposition, the volumes of the corresponding quantities of solid rock and debris by the areas (base $\times$ height) of the respective parallelograms. With the aid of fig. 2 we then get

$$\text{rock-volume} = (1-c) \times \text{screes-volume}$$

$$(dx - dy \cot \beta)(h-y) = (1-c)\left(dy - \frac{dx}{\cot \alpha}\right) y \cot \alpha$$

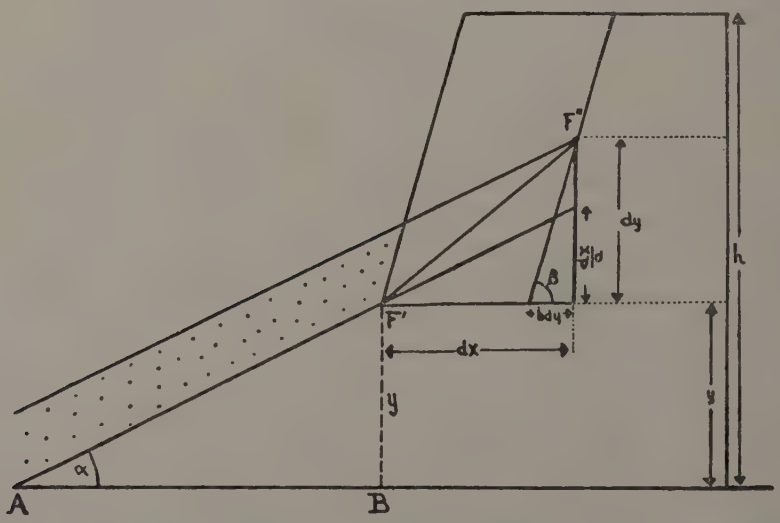

Fig. 2. Diagram from which the differential equation (1) may be obtained. $F'F''$ represents an infinitely small increase in the convex nucleus (after OTTO LEHMANN).

Now putting $\cot \beta = b$, and $\cot \alpha = a$ (co-tangent of the slope-angle of the screes) we get the following differential equation: —

$$(dx - bdy)(h-y) = (1-c)\,ay\left(dy - \frac{dx}{a}\right) \quad . \quad . \quad . \quad . \quad (1)$$

or: —

$$dx = \frac{bh + (a - ac - b)\,y}{h - cy}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By introducing the new constants

$$k = \frac{a - ac - b}{c}$$

$$l = h\,\frac{b}{a - ac - b}$$

$$m = \frac{h}{c},$$

we get

$$dx = k\,\frac{l + y}{m - y}\,dy = k\left(\frac{l + m}{m - y} - 1\right)dy. \quad . \quad . \quad . \quad . \quad (2a)$$

By integrating this formula we find the relation between x and y in the curve sought: —

$$x = k\,(l + m)\int \frac{dy}{m - y} - k\int dy + A, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which A is the integration constant.

Further, for $x = y = 0$, $A = k(l + m)\,{}^{e}\log m$,

$$x = k\,(l + m)\,{}^{e}\log\,\frac{m}{m - y} - ky \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is a logarithmic curve of a higher order: —

If $\beta = 90°$, or if $\cot\beta = 0$, then $k = a\,\frac{1 - c}{c}$; $l = 0$, and $m = \frac{h}{c}$

$$x = ah\,\frac{1 - c}{c^2}\,{}^{e}\log\,\frac{h}{h - cy} - a\,\frac{1 - c}{c}\,y \quad , \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It follows from this that it is *not* the angle of the wall β (90°) which has changed FISHER's parabola into a logarithmic curve, but the introduction of the constant c. In cases in which, owing to secondary factors, so many fragments have been removed that notwithstanding the greater pores-volume the screes volume is smaller than that of solid rock detached during the same period, it is better to take, for c, a negative number. Formula (5) for $\beta = 90°$ then becomes

$$x = ah\,\frac{1 + c}{c^2}\,{}^{e}\log\,\frac{h}{h + cy} + a\,\frac{1 + c}{c}\,y \quad . \quad . \quad . \quad . \quad (5a)$$

The derivation, for FISHER's parabola, for $c = 0$ and $\beta = 90°$ follows immediately from formula (2): —

$$dx = \frac{a\,y\,dy}{h}$$

$$y^2 = 2\,h\,x\tan\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Another special case arises when $c = -\infty$; from the differential equation (1) it then follows that

$$a\,y\,dy - a\,y\,\frac{dx}{a} = 0$$

$$y = \frac{1}{a}\,x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This equation refers to a straight line having the same slope-angle as the maximal slope of the screes in Nature in the rock-formation in question. This straight line exists already when $x = -100$, which means that in the complete or nearly complete absence of regular accumulation of debris at the base of the disintegrating steep mountain the latter must adopt a less steep slope with a straight-line profile and slope-angle α. We here meet with the phenomenon described as early as 1900 by E. RICHTER (No. 9) under the name of "schiefe Denudationsebene" (slanting denudation-surface), from the crests of the Alpine mountain ranges, and which also occurs frequently in the neighbourhood of the "Schliffgrenze" above the trough-shoulders of former glacier valleys. Such straight-line forms also occur, as is well known in the "island-mount"-landscapes of arid and semi-arid regions.

Since, according to the investigations by PIWOWAR and STINY (Nos. 10 and 11), the slope-angles of the screes, in Nature, may reach a maximum, according to the petrologic conditions of the rock, of between 26° and 43°, we may say that this determines more or less the slope-angle of RICHTER's straight-line denudation slope. We shall return to this point in a further publication.

It is finally possible to prove, that when the upstanding part of the mountain with slope-angle β has completely disappeared and that the screes have thereby reached the level of the top-plateau, their straight-line cross profile forms the tangent on the logarithmic curve at the intersecting point with the plateau.

In this case, $y = h$. According to the differential equation (2), we then get

$$\frac{dy}{dx} = \frac{h - cy}{bh + (a - ac - b)\,y} = \frac{h\,(1-c)}{ah\,(1-c)} = \frac{1}{a} \quad . \quad . \quad . \quad . \quad (8)$$

This is the tangent of the maximum slope-angle of the screes in Nature, as was already derived by FISHER for his own special case; but we are here dealing with a general law, valid for any slope-angle $\beta > \alpha$ of the steep wall.

Bearing in mind that the phenomena coming within the scope of LEHMANN's theory are examples taken from the upper region of the Alpine mountain range, where, besides the striking straightness of line in many slopes of from 26° to 43°, in crests and peaks, we may also find

finely curved lines underneath some of the screes, as in the case of "Drei Zinnen" (fig. 3*a* and 3*b*); and further, that kindred phenomena have been observed in Saxon Switzerland, in cuesta- and mesa-landscapes [2]), in the chalk cliff regions in the South of England and the "island-mounts" land-

Fig. 3*a*. "Drei Zinnen" in the Ampezza Dolomites near the former Austrian frontier. In the foreground the Paternsattel, where the convex slopes meet in a manner, resembling the theoretic interpretation shown in fig. 3*b*.

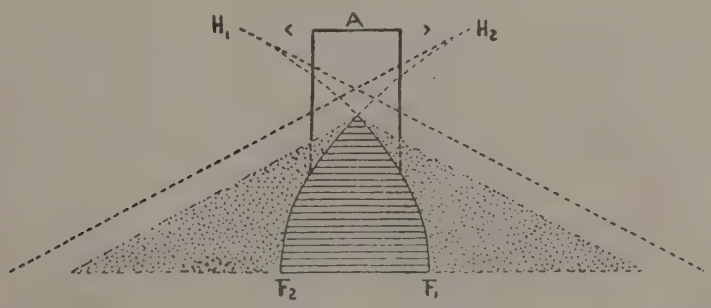

Fig. 3*b*. A narrow plateau A, which at first had the width F_1F_2, reduced on both sides by parallel recession. If the plateau had been wider, the screes would have reached the heights H_1 and H_2 (after OTTO LEHMANN).

[2]) Cf. also, with respect to this, the typical but more complicated forms of the Downs in South-England, shown in fig. 38 of ALBERT DE LAPPARENT's „Leçons de Géographie physique", p. 91, Paris (1896).

scapes of the arid and semi-arid territories, then it may not be too bold an assertion to say that all these phenomena comply to the same *general law*, which we propose henceforth to call *the steep mountain transformation law of* FISHER–LEHMANN, and which we shall formulate as follows:

In the case of slow and regular parallel recession of a steep mountain slope protruding above a horizontal form, a rocky nucleus is formed underneath the screes, of which nucleus the cross profile is a logarithmic curve of a higher order, whose form is dependent upon the slope-angle β *of the steep mountain, of the slope-angle* α *of the screes, and of the ratio between the volumes of rock removed and debris deposited* (fig. 4 and 5).

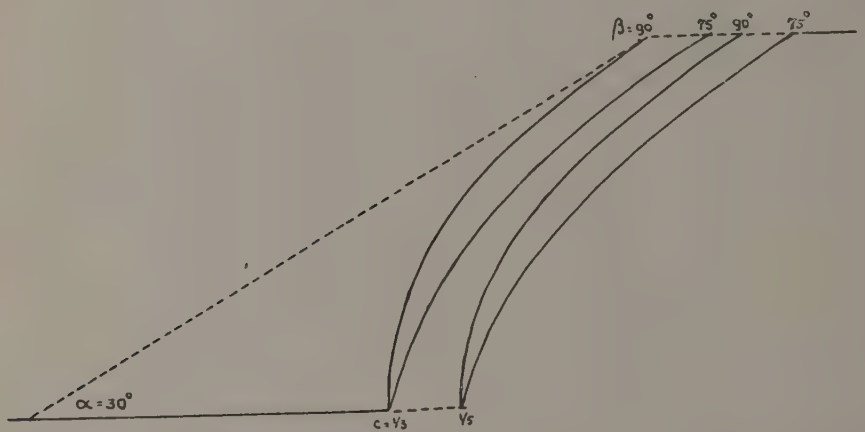

Fig. 4. The influence of the slope-angle β of the initial steep mountain on the curvature of the convex nucleus underneath the screes resp. for $c = \frac{1}{3}$ and $\frac{1}{5}$ and $\alpha = 30°$. As zeropoint of the system of coordinates we invariable took the footpoint of the convex curve (after LEHMANN).

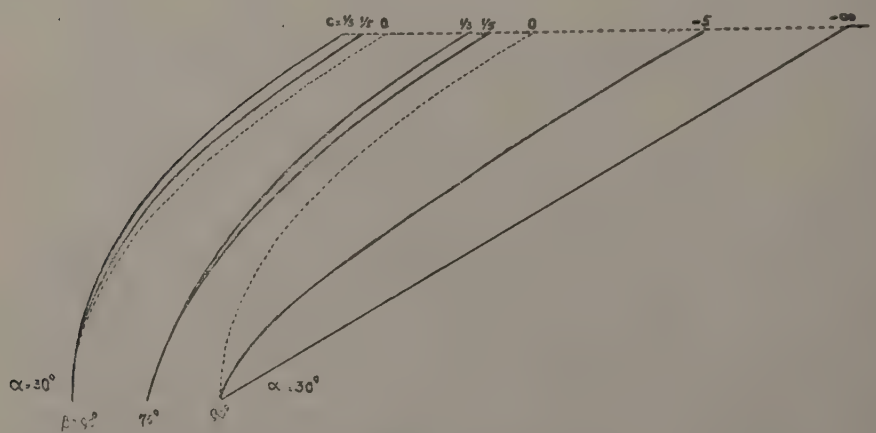

Fig. 5. The influence of the constant c on the form of the convex curves for $\alpha = 30°$ and $\beta = 75°$, $\beta = 90°$. For $c = -\infty$ we get RICHTER's "Denudationsböschung"; for $c = 0$ FISHER's parabola (after LEHMANN).

Of this law, the formulae (6), (7) and (8) are special cases.

Many geomorphologists, little accustomed as they are to an entirely exact treatment of their problems, will be loth to acknowledge the great significance of LEHMANN's theory. They should reflect, however, that the aim of all science is to reduce what is apparently incoherent to one common root-principle, which enables us to survey a vast field, with all its complications, from a central point of view. In the realization of this lay the great merit both of DAVIS and of LEHMANN's theory.

Those who are inclined to doubt the actual value of LEHMANN's theory will probably point out that, in many landscapes, there exist, side by side with forms apparently lending themselves to explanation with the aid of this theory, relief-types which do fit in with it at all. But how variable the premisses may be, even at very small distances! And the last thing we wish to argue is that, with LEHMANN's theory,the last word upon this question has been spoken. On the contrary, it will have to be sounded and elaborated at many points, for LEHMANN's theory is a special application of the more general theory of non-parallel recession of steep mountains; but it constitutes at least a first beginning of a truly exact treatment of physiognomic geomorphology.

When deviating forms are found in Nature, therefore, the theory should not be thrown overboard; in such cases it should be inquired to what extent the premisses have to be modified in order to find an explanation of the deviation found. This point will be made the subject of a number of future publications.

In attempting to explain such deviations in the way indicated, however, we shall find that the form in which LEHMANN has cast his theory does not readily lend itself to this purpose. LEHMANN himself mentions more than once the lengthy and painfully laborious task of constructing the curves [3]), in which he had to use logarithms with as may as 7 decimals so as to obtain smoothly running lines. The fact that he himself produced only very few such curves also points to the same thing.

The science of morphology, however, will in the future have need of a method allowing of a rapid construction of the curves required, thus rendering possible a ready comparison with the phenomena observable in Nature. This may be realized by making use of a diagrammatic method. When, for example, a formula has the general form

$$v = \mu f(x) = \mu \frac{ax + b}{cx + d}$$

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[3]) With the aid of formula (4) or the equations derived from it, LEHMANN determined 10 points of each curve. He further introduced y as the known factor in tenths of the wall height. Putting $h = 100$, and $y = 10$, 20, etc., we get x expressed in % for any given height of the cliff etc.

As long as $y < 50$, it appeared to be impossible to miss any of the points. For $y > 50$, computation of 2 or 3 points could in some cases be omitted.

in which μ is the modulus, and a, b, c and d, the constants, then it will lend itself to the use of a projective scale (*vide* No. 12).

It will be clear that the form of our differential equation (2),

$$\frac{dx}{dy} = \frac{bh + \{(1-c)\,a - b\}\,y}{h - cy} = w$$

satisfies this condition. This means that the co-tangents of the angles formed by the tangents on the profile sought, with the X-axis, are obtained by projective transformation of the regular Y-scale.

To this end three points of the projective scale sought are first fixed on the horizontal W-axis (fig. 6). After this the Y-axis, to be divided up regularly, is drawn through one of these points in an arbitrary direction, after which the projection-centre C is determined, from which the dividing points of the Y-axis are projected upon the W-axis.

We shall choose, as given points, B, $H(\infty)$ and $W(\infty)$. The points B and H are found by plotting, from an arbitrarily chosen starting point E, the distances $EB = b$ (for $y = 0$, hence numbered 0) and $EH = \frac{(1-c)\,a-b}{-c}$ (for $y = \infty$, hence numbered ∞).

We now draw the y-axis through the zero point B. In this case this is done perpendicular upon the W-axis. The point K is then determined, so that $BK = \frac{h}{c}$ (for $W = \infty$).

The projection centre C is now found by connecting the similarly numbered points of the W- and the Y-axis, which is easily done in this case by drawing, from H and K, lines parallel to the axes.

In this instance we have selected the following:

$$a = \cot 30° = 1{,}73205;$$
$$b = \cot 75° = 0{,}26795;$$
$$c = \tfrac{1}{3}.$$

We then find: $EH = -2{,}66$, and $\frac{h}{c} = 3h$.

The Y-scale is now projected from C upon the W-scale. This also implies that the co-tangents of the angles of the tangents on the profile sought with the W-axis, are found by measuring the distances from the points of the projective scale BA to the point E.

The angles themselves are found by drawing a circle with the radius 1, of which EA is the line of the co-tangents. This may be done, for example, by giving to $\frac{dx}{dy}$, in the formula (2), the value 1, so that LB obtains the value $0{,}60 \ldots h$. By projecting L on the W-axis we get the point N, numbered 1, while the centre point of the circle with radius 1 is found by plotting a perpendicular line in E, and giving to EM the value EN.

M may also be determined, of course, as the intersecting point of MB

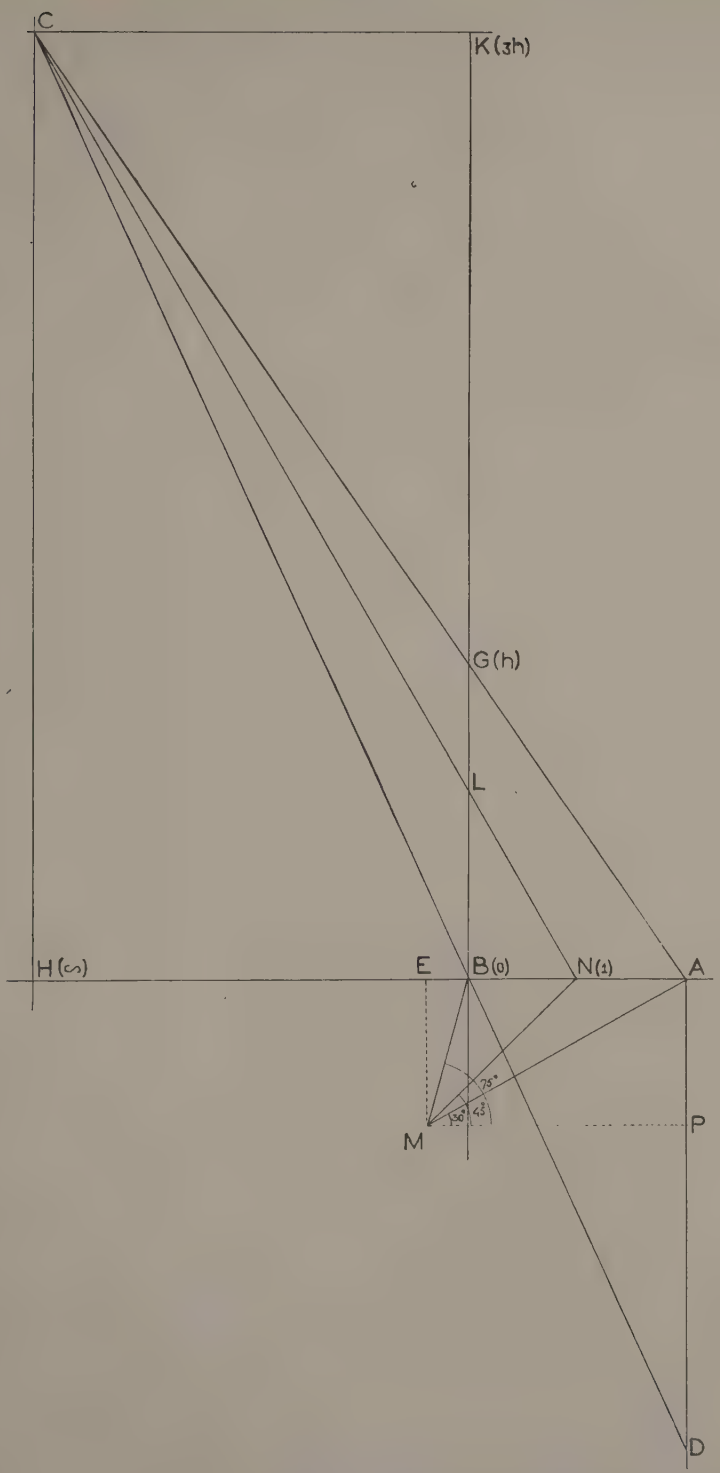

Fig. 6. See text.

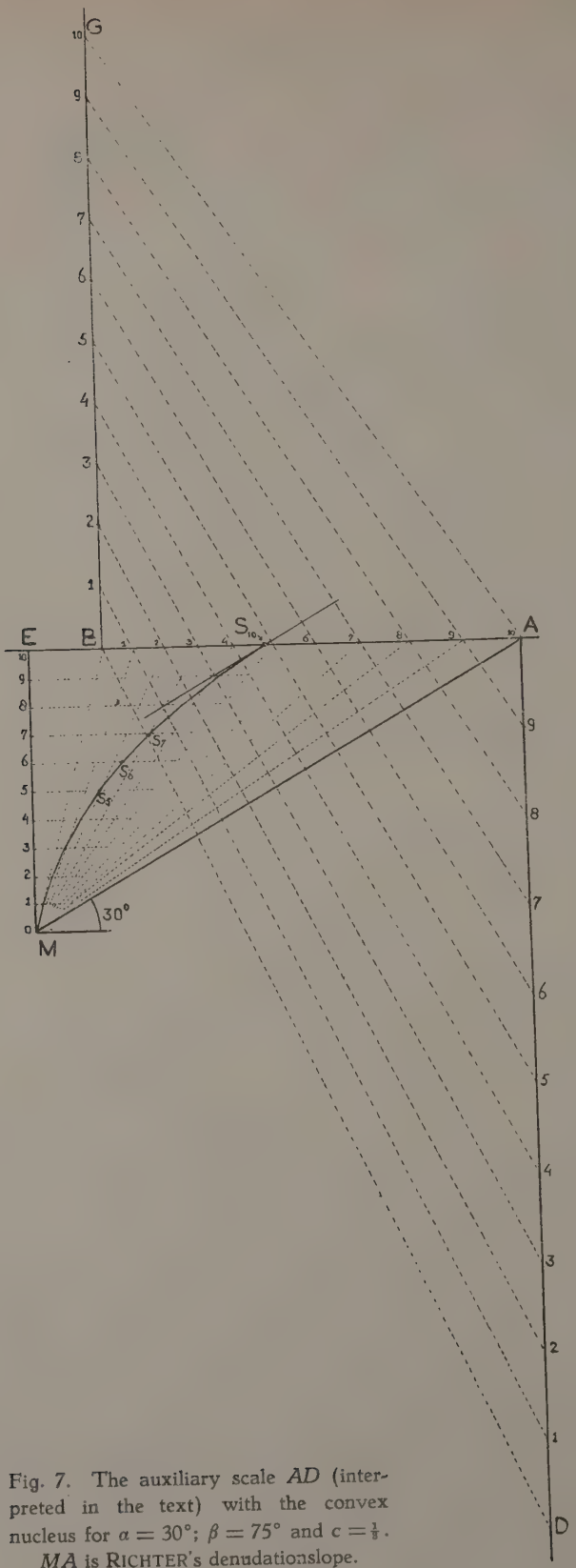

Fig. 7. The auxiliary scale *AD* (interpreted in the text) with the convex nucleus for $\alpha = 30°$; $\beta = 75°$ and $c = \frac{1}{8}$.
MA is RICHTER's denudationslope.

and MA, which must respectively form angles of 75° and 30° with the W-axis.

Between MB and MA lie the directions of all tangents on the profile sought. *This profile, therefore, must turn its convex side to the W-axis, intersecting the plateau above (in fig. 7 and 8, this plateau is, therefore, $S_{10}A$), at an angle α.*

The solution of the differential equations (1) and (2) is not required in order to arrive at this conclusion. Neither is this solution necessary to enable us to sketch the profile with sufficient exactness. For, the differential equation (2) gives the direction of the tangent at a point whose ordinate is known. As the starting point of our co-ordination system coincides with the basic point of the steep mountain slope with its straight-line profile, from which we started, we also know that to an ordinate zero belongs a direction angle $\beta = 75°$ of the new curve.

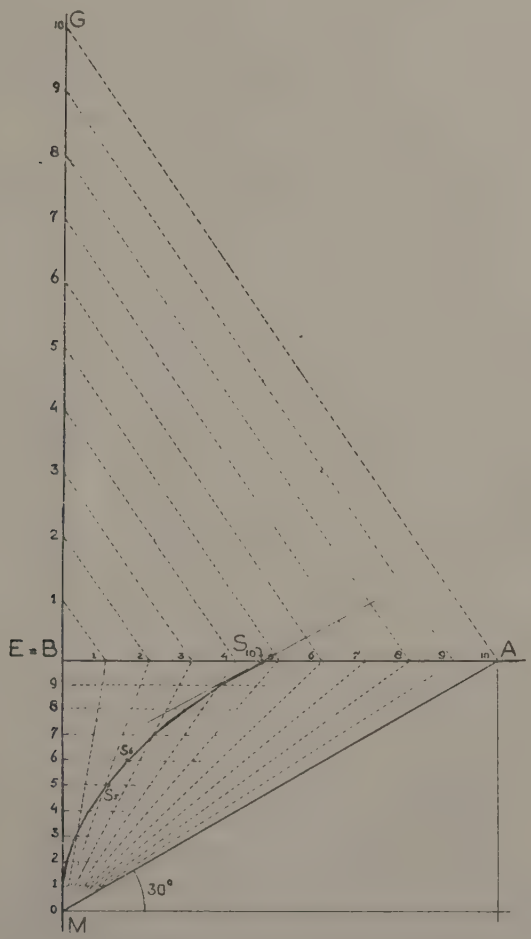

Fig. 8. The auxiliary scale used to obtain FISHER's parabola for $\alpha = 30°$, $\beta = 90°$, $c = 0$.

Now in order further to construct the curve, we must bear in mind that, in fig. 7, showing the (enlarged) projective scale BA, the intersecting point of the above-mentioned first tangent with the line $y = 1$ will not deviate much from the point with the ordinate 1 on the curve sought; the less so according as the parts of the y-axis are smaller. We may, therefore, draw a straight line through this intersecting point S_1, parallel to the direction $M - 1$; at the intersecting point S_2, thus obtained, a line parallel to the direction $M - 2$, and so on [1, 2 etc. are the points of the scale BA].

If this degree of exactitude is not considered sufficient, one may, instead of dividing the auxiliary scale AD into ten parts, divide it into twenty. Since, however, the irregularities occurring in Nature, will probably be of a greater order of magnitude, this will hardly be necessary in most cases.

From fig. 6 it follows at once, with the aid of similar triangles, that the auxiliary scale AD is equal to $1\frac{1}{2}$ GB, or put in a more general form,

$$AD = \frac{1}{1-c} GB.$$

The advantages of the method followed, apart even from the fact that it absolves one from the repeated solution of the differential equation, do not require much further comment.

With the aid of figures such as 6, 7 and 8 it becomes an easy matter to ascertain the influence which any change in α, β and c will have on the shape of the profile curve.

MA is, naturally, a case of RICHTER's "schiefe Denudationsebene", for $c = -\infty$ (i.e. $AD = 0$).

When $\beta = 90°$, and, therefore, $b = 0$, the points B and E will coincide, which, as we have seen, was one of the conditions of the appearance of FISHER's parabola sensu strictu; whilst $AD = GB$, so that the distance BA may be divided regularly in the ordinary way (fig. 8).

Further details concerning this subject will be discussed in the course of future publications.

We are indebted to Mr. A. J. WIGGERS, assistant for phys. Geography at the University of Amsterdam, for his kindness in preparing the figures for the press.

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Zoology. — *The mitotic activity during the regeneration of Polycelis nigra.* By ANNEKE M. E. VERHOEF. (From the Zoological Laboratory, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

Much has been written about the regeneration of Turbellarians; but few authors studied the way cells increase during this process.

N. M. STEVENS (1907), experimenting with three species of Planarians, found mitosis of parenchymous cells both in the regenerating and in the old part. She observed active migration of parenchymous cells from the old part to the new one. According to her, the rapid growth of the regeneration bud should be explained by the migration of cells towards the cut surface together with the mitosis in this part. The epidermis would be repaired by immigration of parenchyma cells and by amitosis.

After P. STEINMANN (1908) in the parenchyma near the wound mitosis can be found after some hours. In later stages, about 2 or 3 days after the operation, the typical mitoses appear through the whole body. In the regeneration tissue also mitoses were seen. The epidermis is repaired by amitosis.

P. LANG (1912) also observed mitosis in the regeneration bud. According to him, the parenchyma contains "Uebergangszellen" or changing cells. The transformation of these cells can be:

a. a dedifferentiation of differentiated cells into cells of an undifferentiated embryonic type (regeneration cells);

b. a metaplasia of differentiated cells into cells of an other differentiation. These cells are very important during the regeneration, but for the common processes of repair in normal life they also are necessary. During the dedifferentiation of an "Uebergangszelle" into a regeneration cell he often saw the beginning of mitosis. The epidermis would regain the necessary amount of cells by amitosis and partly by penetration of parenchyma cells into the ectoderm.

After W. C. CURTIS and L. M. SCHULZE (1924) formative cells are active elements in the production of new tissue; they divide by mitosis.

P. STEINMANN (1926) states in contradiction with the above mentioned work from 1908, that in the cell accumulation near the wound no mitoses are found in an amount worth mentioning. He says regeneration cells are capable of mitosis, but before they reach the regeneration bud. In his opinion organs in the old part of the body are dedifferentiated to obtain the necessary cells.

D. DRESDEN (1940) found no mitoses and declared: "Like other authors I could not detect any mitoses (or amitoses) in these cells — in the regeneration bud — and I consider it therefore quite probable that they arise by

dedifferentiation from old cells. In fact it is not at all likely that the number of cells in these pieces will increase during regeneration. The size of the regenerated animal is about half that of the original specimen and half the number of cells, therefore, will be sufficient for its constitution."

A. RIJVEN (unpublished) observed migration of cells towards the wound ("cellules libres" of PRENANT) and mitosis in the regeneration bud.

It seemed worthwhile to study the appearance of mitosis in normal and regenerating worms.

At the same time an investigation could be made into a possible daily periodicity of the mitotic activity. About this, plants and animals have been examined, and often a periodicity was found with a maximum during the night and a minimum at noon. The work of Mrs. DROOGLEEVER FORTUYN–VAN LEYDEN (1916, 1926) was repeated by miss CARLETON (1934) on young mice. The results of this work were: Under normal conditions a rhythm of 24 hours is found with a minimum at noon and a maximum at midnight. This periodicity is disturbed by continuous illumination or by an artificial changing of night and day.

Methods:

The experiments were carried out on *Polycelis nigra* out of ditches near Utrecht. The animals were divided into two groups:

a. the operation was performed at 8 a.m.

b. the operation was performed at 8 p.m.

Kept in the dark, the worms will regenerate quicker than in the light. So it was possible, that with the animals of group *b* the regeneration would start earlier than with those of group *a*.

The cutting was done with a razor blade on a glass plate covered with a thin layer of wax. By placing this plate on the stage of a binocular microscope it was possible to have the light transmitted from below. The worms were cut transversely through the pharynx into two halves of equal length. Three animals were operated at the same time, kept together in fresh water and afterwards were fixed at the same moment. The cutting of these three lasts one minute at the utmost, and so the moment of fixation could be figured out exactly for every group of three *Polycelis*. On account of a shortage of petri dishes the worms had to be kept three together, and afterwards it was impossible to say which posterior and anterior half belonged together. Therefore the halves described as one animal will not always belong to the same animal really, but are only the representatives of a group of three worms cut and fixed at the same moment.

During the first two days, a group of three was fixed every hour. The next two days it was done every three hours only.

The *Polycelis* of group *a* and *b* were treated in exactly the same way but for the moment of operation.

Normal worms were also fixed at intervals of three hours from 8 a.m. till 8 a.m. the next morning.

The fixation was carried out in the way prescribed by FEULGEN for the reaction on thymonucleic acid (fixation in sublimate-acetic acid).

The animals fixed at 1, 6, 12 hours etc. after the operation were sectioned in sagittal direction with a thickness of 5 μ. They were stained with FEULGEN's test on thymonucleic acid, which is an excellent way of staining the chromosomes and therefore very useful in the search for mitosis.

A map was drawn of every half and all mitoses perceived in the sections were marked on it with a cross.

Results:

Because of the rhabdites it is difficult to get a good idea of the proceedings in the epidermis. No mitosis was observed here. It seems to me, that except the first closing of the wound the epidermis takes no part in the process of regeneration.

In the inner parts of both normal and regenerating *Polycelis* many mitoses were found.

The animals had been exposed to the normal succession of night and day, which is, according to CARLETON, necessary to get a periodical rhythm of the mitotic activity. But in the pictures of the animals of one group no

Fig. 1. Distribution of mitoses in anterior half (left) and posterior half (right) of *Polycelis nigra*; 1 hour after operation.

periodicity can be seen. Also there is no fundamental difference between the worms of group *a*. and group *b*. There are some differences in the quantity of mitoses in the various animals, but these may be caused by diversities of size, age, etc.

But from the drawings something else is apparent. In the normal *Polycelis*, and also in the regenerating one during the first two days after the operation, the mitoses are distributed equally through the whole body. About 48 hours after the operation, however, one observes a change in the distribution of mitoses, viz., a concentration towards the base of the regeneration bud. This becomes more and more evident during the following days (cf. fig. 1 and 2).

Fig. 2. Distribution of mitoses in anterior half (left) and posterior half (right) of *Polycelis nigra*; 84 hours after operation.

As it was impossible to have all the drawings published, every map was divided transversely into four equal parts numbered I, II, III and IV; part I is remotest from the wound. For every part the percentage of the mitoses present was calculated. Now it was possible to make diagrams showing the change of the distribution during four days (cf. fig. 3 and 4). The ordinates represent the percentage of mitoses, the abscissae the time passed between operation and fixation. The average daily values are connected by lines. Every line represents one of the four parts.

On the first day after the operation all the values lie between 19 % and 29 %. During the following days I, II and III drop, and IV rises. There is no essential difference between the diagrams for the anterior and the

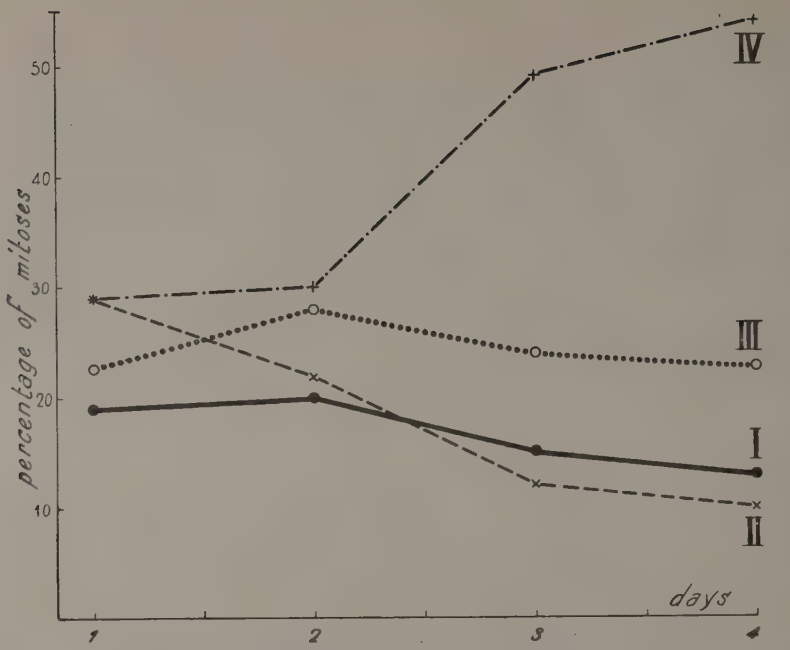

Fig. 3. Change in the distribution of mitoses in regenerating anterior halves of *Polycelis nigra* during the first 4 days after operation. I, II, III, IV = transverse quarters, cf. text.

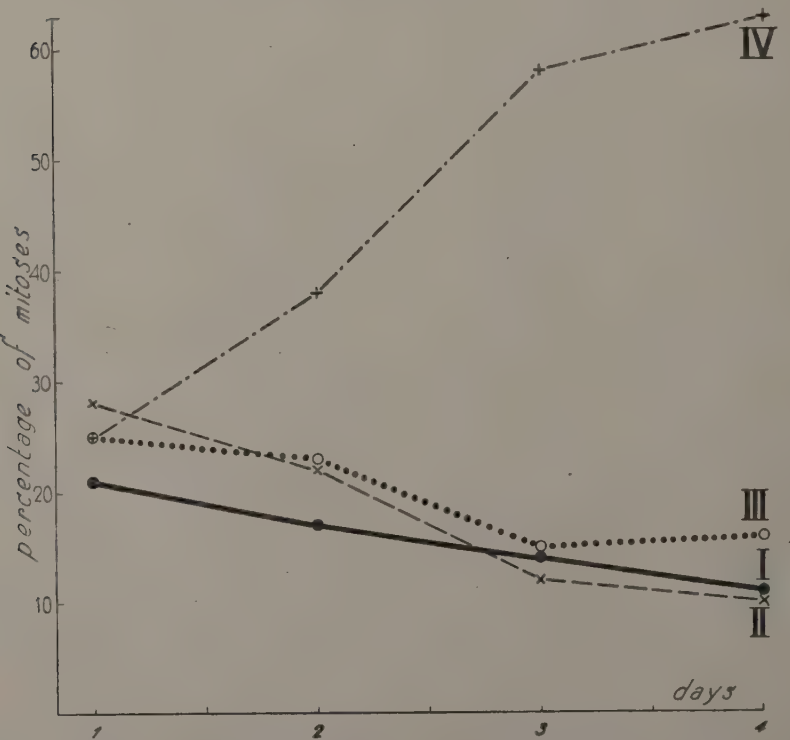

Fig. 4. Change in the distribution of mitoses in regenerating posterior halves of *Polycelis nigra* during the first 4 days after operation. I, II, III, IV = transverse quarters, cf. text.

posterior halves. In the latter, the concentration of the mitoses into part IV is somewhat more accentuated, and begins already on the second day after the operation, whereas it begins in the anterior halves not before the third day. In both, more than 50 % of the mitoses are concentrated in the quarter adjacent to the cut surface on the fourth day after the operation.

LITWILLER (1939) observed something like this with the regeneration of limbs of *Triturus*: "A definite peak of mitotic activity in both mesenchyme and epithelium is found near the base of the young regenerate and shifts distally in older regenerates. This is associated with differentiation in the main regions of the new limb."

If the same is going on during the regeneration of *Polycelis* this shift of mitotic activity ought to go on during the following weeks, always causing a maximum in the newly regenerated tissue.

This shift of the mitotic concentration could be caused by:

1. a shifting of cells — regeneration cells, formative cells — capable of a continuous mitotic activity. Then, only these cells are responsible for the growth and the regeneration involved.

2. a temporarily increased activity of the cells in the regenerating region.

As former authors often observed an active locomotion of cells during the regeneration, the explanation mentioned under 1. seems the most likely.

Perhaps more could be learned on closer investigation of the "Uebergangszellen" of LANG and the "cellules libres" of PRENANT.

Summary.

1. The distribution of mitoses was studied in regenerating *Polycelis nigra*.

2. No daily rhythm of mitotic activity could be observed.

3. In the first 2 days, the mitoses are distributed rather equally through the body.

4. About 48 hours after the operation, they begin to be concentrated in the neighbourhood of the regeneration bud. This concentration of the mitoses is in posterior halves somewhat more accentuated and begins a little earlier than in anterior halves.

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Zoology. — *The vitelline membrane of Limnaea stagnalis.* By OLGA HUDIG. (From the Zoological laboratory, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

The following investigation deals with the properties of the vitelline membrane of *Limnaea stagnalis*. The question of the nature of this membrane was raised in a paper of RAVEN (1946). On the one hand, it seemed to conform with the so-called vitelline membrane or chorion of other animal eggs, because:

1. the membrane is lifted locally from the surface of the egg by the second polar body;
2. it shows wrinkling in eggs shrinking in hypertonic solutions of urea and sucrose;
3. it can detach itself completely from the egg surface in sucrose solutions.

On the other hand, it seemed to take an active part in the cleavage of the egg. RAVEN and KLOMP (1946) described an abnormal cleavage in distilled water. The cleavage progresses normally until the blastomeres are perfectly rounded and only touch one another at one point. After that, the blastomeres flatten themselves a little against one another, then they remain in this condition till the next cleavage. No cleavage cavity is formed and in the cleavage furrow the membrane detaches itself from the egg surface, so that it forms a bridge from one blastomere to the other. Now RAVEN presumed this abnormal development to arise owing to the membrane not folding inward with the cleavage-furrow. In the normal development, this membrane was supposed to turn into a single partition-wall between the blastomeres, and the cleavage cavity would be formed by lenticular spaces originating in this wall and merging into one cavity. In this manner, all the cell walls of the embryo would be formed. Accordingly, the cell boundaries of the embryo could be traced back to the membrane originally enclosing the egg.

Earlier investigators (RABL, FOL, KOFOID) denied the presence of a vitelline membrane in Pulmonates. COMANDON and DE FONBRUNE (1935) observed that the embryo of *Limnaea limosa* leaves the vitelline membrane at the time of first ciliary movement. This was not observed by RAVEN in *Limnaea stagnalis*. The membrane was observed, however, in microscopic sections of uncleaved eggs.

The present investigation will chiefly concentrate on those moments in the development of the egg when the membrane can be observed distinctly under the microscope, viz.

1. during the formation of the polar bodies,
2. during the first cleavage,
3. in the young embryo, showing the first ciliary movement.

At the same time, experiments were made on the development after the first cleavage in various surroundings.

The writer wishes to acknowledge her indebtedness to Prof. CHR. P. RAVEN and to P. D. NIEUWKOOP for their direction and criticism.

Methods.

Eggs were obtained in the usual way by stimulation of the snails with *Hydrocharis* (cf. RAVEN and BRETSCHNEIDER 1942). In some cases the snails were taken in the act of laying and the egg-mass was pulled with forceps out of the vaginal orifice; in this way, the eggs could be treated at a very early stage. The first polar body is formed about an hour after oviposition (RAVEN and BRETSCHNEIDER, l.c.).

It is important to notice that in every egg-mass a regular sequence of developmental stages can be observed: those eggs that are laid last are the least developed. When the eggs had to be transferred to a special medium, the egg capsules were put around a drop of this fluid and pricked, so that the eggs immediately came into the desired medium. Then, they were twice more pipetted into a new drop of this medium in order to remove the egg capsule fluid.

As the animal pole, where the polar bodies are formed, in most cases is turned upwards, it is desirable to place the preparation vertically, when one wants to study the polar bodies. The following method proved most satisfactory: the eggs were put on a cover-glass in a small drop; the edges of the cover-glass were supplied with a layer of vaseline, and pressed on a slide. When one places the slide vertically, the drop remains suspended in the centre of the cover-glass, and, as it is shut off from the air by the vaseline, it cannot evaporate.

I. *Normal development.*

In an egg in the capsule, the vitelline membrane can be observed in the following stages of development:

a. The newly laid egg is surrounded by a hyaline layer which presumably is the vitelline membrane.

b. When both polar bodies are formed, the first, which is spherical, lies distinctly outside the membrane; the second lies under the membrane and is flattened (fig. 1).

c. When the first cleavage groove has indented the egg, the membrane is very distinct at the animal pole; it runs from one blastomere to the other. At the vegetative pole a membrane bridge can be observed too, during a short moment, when the blastomeres form a certain angle with each other (fig. 2).

d. When the embryo is ± two days old, it leaves the vitelline membrane. The embryo begins to rotate by ciliary action and in the end comes to lie free in the capsule fluid. The shrivelled membrane can always be

found, though it is very thin, because the polar bodies remain sticking to it (fig. 3).

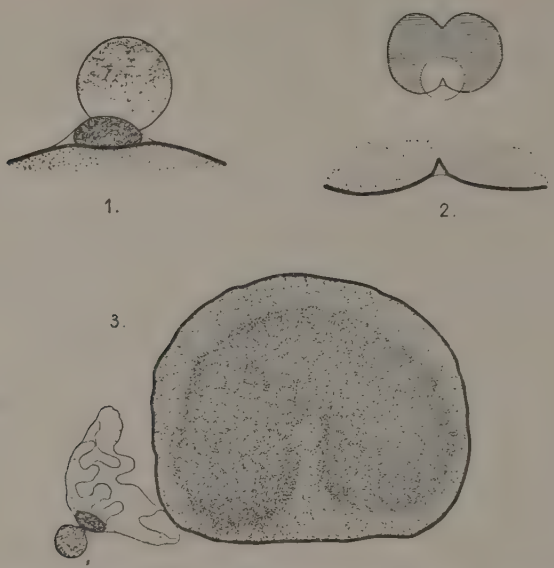

Fig. 1—3. Normal development. 1. Situation of polar bodies. 2. Momentary formation of membrane bridge at vegetative pole during first cleavage. 3. The rotating embryo has left the vitelline membrane.

II. *The development in various media.*

1. *Distilled water.*

Decapsulated eggs have been transferred to distilled water at various moments.

a. In a newly laid egg wrinkling of the membrane can be observed. The vitelline membrane detaches itself in many places from the surface and can be seen as a folded hyaline layer (fig. 4). The earlier the egg is placed into distilled water, the more the membrane seems to be folded.

b. The first polar body is now, like the second, formed inside the membrane (fig. 5).

More than a hundred eggs were observed. In about 60 of them, the polar bodies were in the field of vision. All but one of the first polar bodies were inside the membrane.

When the first polar body is being formed, the membrane is seldom visible. It can be clearly seen, however, that the first polar body is connected with the egg surface by a protoplasmic strand. The membrane is not distinctly visible till after the formation of the second polar body. The protoplasmic strand has by then disappeared, but sometimes the remains of it can be observed in the form of a small spherule.

The first polar body is formed inside the membrane, independent of the time at which the egg is transferred to distilled water, as long as its formation has not yet taken place at this moment.

With eggs of the same egg-mass the following experiments were made:

11 Eggs, which were still round, were transferred to distilled water. In 9 eggs, in which afterwards the polar bodies were visible, the first polar body was always inside the membrane.

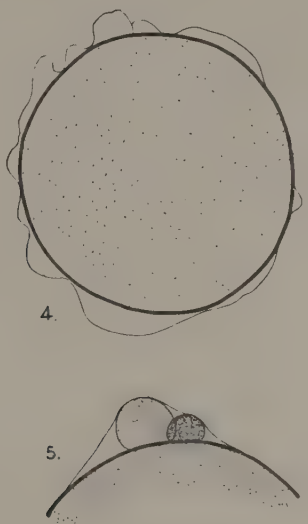

Fig. 4—5. Development in distilled water. 4. Swelling of vitelline membrane. 5. Both polar bodies are formed inside membrane.

20 Minutes later, 16 eggs, in which at the animal pole a hyaline cap had already been formed, were put into distilled water. In 10 eggs, in which the polar bodies could be brought into the field of vision, the first polar body was inside the membrane.

15 Minutes later again, 13 eggs, during the formation of the first polar body, were placed in distilled water. In 9 eggs the polar bodies were in the field of vision; in all of them the first polar body was inside the membrane.

c. After the cleavage in distilled water, membrane bridges are formed as described by RAVEN and KLOMP (l.c.). The membrane, although wrinkled, takes part in the movements of the egg surface, for, during the formation of the cleavage furrow it is taken inwards with the latter. When the cleavage furrow is complete, the blastomeres flatten a little against one another, but only for a short time; soon afterwards, they round themselves off again. At this moment, the membrane in the cleavage furrow loses contact with the egg surface and forms a kind of bridge between the two blastomeres. This process is independent of the time at which the eggs are placed in distilled water, from the moment of oviposition till after the first cleavage, as long as the blastomeres are still in touch at one point only.

When cleaved eggs, in which the 2 blastomeres are joined and a cleavage cavity has been formed, are transferred to distilled water, bridges can be seen after about half an hour. After the second cleavage, four bridges can be observed and the further development takes an abnormal course.

d. As has been described by RAVEN and KLOMP (1946), in distilled

water no cleavage cavity is formed; the blastomeres remain spherical till the next cleavage.

e. In distilled water, the development seldom proceeds beyond the 4-cell stage; as a rule, the embryos die in this stage, as described by RAVEN and KLOMP.

2. *Distilled water with egg capsule fluid.*

This mixture was chosen to imitate as nearly as possible the normal surroundings of the egg. In pure capsule fluid, the eggs soon die, probably because of the fluid losing water so quickly by evaporation, as soon as it is exposed to the air. A number of capsules were emptied into a small drop of water. The capsule fluid does not immediately mix with the water. The eggs with the adhering capsule fluid were then sucked up into a pipette and transferred to a cover-glass.

a. Of two egg-masses, 8 and 12 eggs, respectively, were studied. No wrinkling of the membrane took place, whereas control eggs in distilled water showed a distinct wrinkling.

b. 50 eggs were placed in the mixture before the formation of the first polar body. The first polar body nearly always forms outside the membrane.

c. In some eggs with adhering capsule fluid, it was investigated whether the membrane bridged over the cleavage furrow. As in normal eggs in the capsules, a bridge is observed at the vegetative pole only for a short moment, when the blastomeres form a certain angle.

When one puts eggs, that have been in distilled water till bridges have been formed, into this medium, the blastomeres join and flatten quickly and soon no more bridges can be observed.

d. In distilled water with capsule fluid, the cleavage takes a normal course (joining and flattening of the blastomeres, formation of cleavage cavity) independent of the moment at which the eggs are transferred from distilled water to this mixture. Even when bridges had already been formed, a normal cleavage takes place.

e. In this medium the eggs can live for several days. In a few cases, even the gastrula stage has been reached.

3. *Distilled water with crushed eggs.*

This medium was chosen because it has a particular influence on cleavage.

a. The eggs show some wrinkling of the membrane in this medium, but far less than the controls in distilled water.

b. In 8 out of 15 eggs, polar bodies could be observed. In all but one of these 8 eggs, the first polar body was inside the membrane. In 12 out of 16 eggs of another egg-mass, the polar bodies were visible. In 7 eggs the first polar body was inside, in 5 it was outside the membrane.

c. The eggs form bridges after the cleavage, but far less than in distilled

water. When one transfers eggs that have formed bridges in distilled water to distilled water with crushed eggs, the bridges remain visible.

d. A cleavage cavity is formed as in normal development.

In the first experiments in distilled water, the results were very irregular; sometimes, unexpectedly, a normal development took place: the blastomeres joined and flattened and a cleavage cavity was formed. It appeared that eggs lying near a crushed egg showed a normal development. In order to check this, the following experiments were made:
first egg-mass: 5 eggs ± 3 crushed eggs: development normal;
second egg-mass: 4 × (5 eggs with ± 5 crushed eggs): development normal.

e. These eggs can develop further into blastulae.

Out of 16 eggs + 8 crushed eggs, after one day 16 had developed into morulae; after three days 5 were left, which had grown into blastulae; on the fourth day, all were dead.

These embryos may be surrounded loosely by the vitelline membrane.

4. *$CaCl_2$-solutions.*

a. In eggs that have been transferred from distilled water to a 0,04 % solution of $CaCl_2$, the wrinkling of the membrane may be reduced a little, but never completely.

From two different egg-masses, 12 and 31 eggs resp. were placed in distilled water before the formation of the first polar body. All the eggs showed a distinct wrinkling of the membrane. From these eggs, 6 and 17 resp. were transferred to $CaCl_2$ 0.04 % and compared with the eggs remaining in distilled water. The wrinkling remained visible. 15 Eggs immediately put into $CaCl_2$ 0.04 % showed some wrinkling in a few cases only.

b. In low concentrations of $CaCl_2$, the first polar body is formed in most cases inside the membrane; in higher concentrations, it tends to lie outside.

A number of eggs were placed immediately in various solutions of $CaCl_2$. Table I gives the results.

TABLE I.

Concentration of $CaCl_2$	Eggs with visible polar bodies	First polar body	
		Inside membrane	Outside membrane
0.005%	10	9	1
0.00625%	30	25	5
0.0125%	32	11	21
0.025%	22	4	18
0.05%	15	9	6
0.1%	20	3	17
0.2%	13	4	9

c. In higher concentrations of $CaCl_2$, the bridges on the vegetative pole are visible for a moment only, as they are in normal eggs in the capsule. In 0.005 % and 0.00625 % the bridges can be observed distinctly.

d. On the development after the first cleavage, the following experiments were made:

39 Eggs in 0.04 % $CaCl_2$ all show a normal cleavage, the blastomeres flattening against each other and a cleavage cavity forming.

15 Eggs in 0.02 % $CaCl_2$ all show a normal cleavage.

22 Eggs in 0.01 % $CaCl_2$ all show a normal cleavage.

In 38 eggs in 0.005 % $CaCl_2$, the flattening of the blastomeres and the formation of the cleavage cavity are somewhat retarded; still, a more or less normal cleavage occurs. In still further dilutions, the eggs behave as in distilled water.

e. In 0.04 % $CaCl_2$ the eggs can develop into gastrulae.

11 Eggs; on the second day 9 had developed into morulae; on the fourth day 5 of them were still alive, and had developed into gastrulae; on the sixth day all were dead.

16 Eggs; on the second day 14 had developed into blastulae; on the fourth day 13 of them were still alive, and had developed into gastrulae; on the sixth day all were dead.

5. *Other Media.*

A few experiments were made with KCl and Na-citrate.

In 0.2 % KCl, the blastomeres do join and flatten and a cleavage-cavity is formed, but the eggs do not develop very far. The next day, they are always dead.

In Na-citrate, they show the same development as in KCl.

Eggs which, after a treatment of a few minutes with Na-citrate 0.02 %, were transferred to KCl, show the same development.

Discussion.

From the experiments we may conclude:

1. In *Limnaea stagnalis* a vitelline membrane is present. This membrane can be followed from the moment the egg is laid till it is left by the embryo.

a. In the uncleaved egg in the capsule, an outer hyaline layer can be seen. This might be due to optic effects, but it is more probable that there is a real membrane, as wrinkling of a similar transparent layer occurs in distilled water.

b. During the formation of the polar bodies, the membrane is visible, because the second polar body lifts it locally from the egg surface.

c. During the 1st cleavage, the membrane is visible for a moment at the vegetative pole, when the blastomeres form a certain angle at this pole. At the animal pole, the membrane is now clearly visible.

d. During further development the membrane remains visible in the neighbourhood of the second polar body. It can be observed with particular clearness in a medium of distilled water with crushed eggs, in which the embryo lies free in the membrane. Finally, the embryo leaves the membrane, when it begins to rotate by ciliary action.

2. The vitelline membrane must be considered to be a chorion. It does

not take an active part in the cleavage as RAVEN presumed. The cell walls cannot possibly originate from the vitelline membrane because:

a. The vitelline membrane continues to enclose the embryo during its development, and is finally cast off.

b. When eggs cleaved in distilled water, in which membrane-bridges have been formed, are placed in distilled water with capsule fluid, the blastomeres join and flatten and a cleavage cavity is formed. That the membrane should in this case fold inwards to form the partition-wall between the blastomeres is highly improbable. That this is not necessary for a normal development after the first cleavage is proved by the development in distilled water with crushed eggs, in which membrane bridges are clearly visible at both poles notwithstanding the joining of the blastomeres and the formation of a cleavage cavity.

c. When one puts normal eggs in the 2-cell stage, in which the blastomeres have joined and a cleavage cavity has been formed, into distilled water, bridges are, after some time, visible at both poles. After the second cleavage, four bridges are visible when the egg is looked at in polar view. Had the membrane followed the cleavage furrow to form the partition-wall between the blastomeres, no bridges could have come into existence.

Whether the vitelline membrane is strictly necessary for a normal cleavage of the egg is still an open question. In the beginning the membrane lies closely against the egg surface and may form an indispensable mechanical support. That the membrane surrounds the egg rather tightly is indicated by the fact that in hypotonic media the egg shows an under-normal swelling (RAVEN and KLOMP, l.c.).

However, the above-mentioned observations show that the vitelline membrane does not play such an important part in cleavage, as has been ascribed to it by RAVEN. The layer that has been observed by this author to be interposed between the joining blastomeres, thus forming the partition-wall between the cells, is, apparently, not the vitelline membrane proper, but a protoplasmic cortex layer lying immediately beneath this membrane.

3. About the properties of the membrane the following can be said:

It was supposed by RAVEN (1945) that the membrane undergoes a change between the formation of the first and the second polar body, because the first comes to lie outside, the second inside the membrane. The membrane indeed seems to get stronger in the course of development: it is invisible under the microscope during the formation of the first polar body, but during the formation of the second polar body it is clearly visible and is so strong that the second polar body is flattened. A similar thing is found in distilled water. In this medium the first polar body is formed inside the membrane too, but the latter is seldom visible during the formation of this body. Only after the formation of the second polar body the membrane is distinctly visible.

The first polar body is seen to remain connected with the egg surface by a protoplasmic strand this is visible in distilled water too.

The question now arose whether the first polar body really pierces the membrane. It might be supposed that it is always inside the membrane but that the latter, in the normal medium, is pulled so tightly round the polar body that this appears to lie outside. In this way the connecting strand between the first polar body and the egg surface could be explained. In distilled water, the vitelline membrane is so wide that an elastic enclosure of the first polar body does not occur. Both polar bodies then are lying loose inside the membrane, and a connecting strand between them or between the first polar body and the egg surface cannot be found. This view is supported further by the observation that the first polar body always comes to lie inside the membrane, when eggs in which the formation of this body is already in progress, are transferred to distilled water.

On the other hand, the connecting strand may be a spindle remnant; such remnants have been observed by RAVEN (1945) both in the maturation and the segmentation divisions in *Limnaea*. Moreover, in distilled water, one often finds besides the first and second polar body a small spherule inside the membrane, which may be a rest of an original connecting strand. These spherules are smaller than half the first polar body, so they cannot be a result of division of this body. Moreover, the latter has never been observed to divide; sometimes it prepares to do so, but the process is never brought to an end. Furthermore, the observation that the first polar body easily detaches itself from the egg surface, when the eggs are stirred by pressing the cover glass, speaks strongly against its enclosure by the vitelline membrane.

Anyhow, one must assume that the properties of the membrane alter between the formation of the first and the second polar body. In the first case, at the formation of the first polar body the membrane would be very plastic, so that this body remains completely surrounded by the membrane, whereas at the time of formation of the second polar body the membrane has lost its plasticity and offers a more elastic resistance to the formation of this polar body. In the second case, at the outset the membrane would be so thin that the first polar body can break through it, whereas at the formation of the second polar body the membrane is thicker and stronger, so that it cannot be pierced by this body.

4. In distilled water, the properties of the membrane alter immediately, because:

a. As soon as the egg is put into distilled water, it shows wrinkling of the surface. The membrane probably swells through absorption of water, which causes it to grow too wide for the egg. Then, it locally detaches itself from the egg surface and is thrown into folds.

Probably, this reaction is not reversible. The wrinkling can be reduced a little by $CaCl_2$ solutions, but never completely.

b. If one places the egg immediately before, or even during the

formation of the first polar body, in distilled water, this polar body is formed inside the membrane.

c. When cleaving eggs, in which the blastomeres are in touch in one point, are transferred to distilled water, membrane-bridges are formed.

5. In the experiments in other media, the polar bodies can be used as indicators of the reactions of the membrane: when the polar bodies show a normal position, one lying inside, the other outside, as in normal eggs in the capsule, then one can assume that the membrane reacts in the same way as it does in the capsule. When, however, the polar bodies both are situated inside the membrane, then the latter must have suffered an alteration.

It must be noted that a quite normal reaction of the membrane has never been attained in atypical media, because a few first polar bodies have always been formed inside the membrane.

The experiments in these media may be summarized as follows:

a. In diluted capsule fluid, the first polar body is usually formed outside the membrane.

b. In distilled water with crushed eggs, the first polar body is formed sometimes inside, sometimes outside the membrane.

c. In solutions of $CaCl_2$ varying from 0.2 % to 0.0125 %, the first polar body is formed in most cases outside the membrane; in 0.00625 % and further dilutions, it is usually formed inside the membrane.

6. In distilled water, the cleavage is abnormal; this abnormal cleavage occurs independently of the moment at which the eggs are placed in distilled water as long as the flattening of the blastomeres after the first cleavage has not yet begun. In eggs transferred to distilled water when the blastomeres have already joined, the next cleavage is abnormal. At the 4 cell-stage the embryo dies in distilled water.

In capsule fluid diluted with distilled water, the cleavage is normal. Eggs transferred from distilled water to capsule fluid develop normally independent of the moment when this takes place, even when membrane bridges have already been formed at this moment. So this process is completely reversible.

In this medium, the egg can develop as far as the gastrula stage.

In distilled water with crushed eggs, the cleavage is normal. Therefore, the egg itself must contain a substance which can cause a normal cleavage of other eggs after its liberation from a crushed egg. In view of the experiments with $CaCl_2$ solutions it is reasonable to suppose that the influence both of egg capsule fluid and of crushed eggs on cleavage is due to their contents of $Ca^{\cdot\cdot}$-ions.

In $CaCl_2$, the cleavage is normal in concentrations varying from 0.04 % to 0.005 %. In the latter concentration, the cleavage cavity is formed late and the blastomeres do not join quite so closely as they do in the higher concentrations.

In solutions of 0.04 %, the eggs can develop into gastrulae.

RAVEN and KLOMP (1946) supposed that the influence of $Ca^{\cdot\cdot}$-ions on cleavage might be attributed to a change in the properties of the vitelline membrane. As, however, my observations have shown that this membrane does not play such an important part in cleavage as has been presumed by RAVEN, it seems more probable that it is a deeper layer, belonging to the egg cortex proper, that is influenced by the presence or absence of $Ca^{\cdot\cdot}$-ions. As a matter of fact, these ions are known to have an important influence on the properties of biological membranes.

PASTEELS (1938) made experiments on the activation of eggs by $Ca^{\cdot\cdot}$- and $K^{\cdot}$-ions. It appeared that the $K^{\cdot}$-ions acted by liberating the $Ca^{\cdot\cdot}$-ions present in the egg cortex. When these $Ca^{\cdot\cdot}$-ions, by treatment with Na-citrate, had been precipitated, $K^{\cdot}$-ions had no more influence.

This possibility was also considered for the cleavage of *Limnaea stagnalis*, but it appeared from some preliminary experiments that not only KCl, but also Na-citrate in the absence of $Ca^{\cdot\cdot}$-ions were able to make the blastomeres join and to cause a cleavage cavity to be formed. These experiments are too fragmentary, however, for a definite conclusion to be drawn.

Summary.

1. The vitelline membrane is a pellicle which encloses the egg from the very beginning till it is left by the embryo.

2. Initially, it is very thin, but in the course of development it becomes stronger and more distinctly visible.

3. The membrane undergoes an alteration immediately after the egg is put into distilled water, probably through swelling, so that it is thrown into folds on the egg surface.

4. In consequence of this, the first polar body is not situated outside the membrane, as is normal, but inside.

5. This swelling does not or hardly occur in a mixture of capsule fluid and distilled water or a solution of $CaCl_2$.

6. The cleavage in distilled water is abnormal. Addition of capsule fluid, crushed eggs or $CaCl_2$ causes normal cleavage; the blastomeres join and flatten and a cleavage cavity is formed.

7. In a medium of distilled water with crushed eggs, the eggs can develop till the blastula stage.

In a mixture of capsule fluid and distilled water and in solutions of $CaCl_2$ 0.04 %, the eggs can develop till the gastrula stage.

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Zoology. — *Effects of oestrone and gonadotrophin administration in the male toad* (*Bufo bufo*). By G. J. VAN OORDT and H. KLOMP. (Zoological Institute, Dept. of Endocrinology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

It is a well-known fact that BIDDER's organ, which is to be found in all adult male and in some adult female Bufonids, represents a rudimentary ovary. This follows from (1) the microscopical structure and (2) the fact that in some cases BIDDER's organ may develop after castration into a functional ovary (HARMS, 1921; PONSE, 1924). Moreover, the adult male common toad (*Bufo bufo*) possesses distinct MÜLLERian ducts, which also develop into functional oviducts after ablation of the testes. It is a matter of course that several investigators have coordinated the presence of these two female organs in male toads.

In 1941 BRETSCHNEIDER and DUYVENÉ DE WIT gave a detailed description of the s.c. preovulation corpora lutea in the ovaries of the Bitterling (*Rhodeus amarus*), a hormone secreting function being ascribed to these structures. The latter occur also in considerable numbers in BIDDER's organ; therefore it is not impossible that the endocrine function of BIDDER's

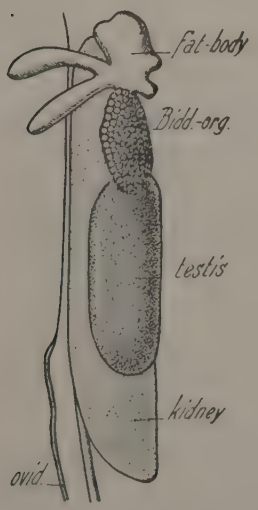

Fig. 1. Urinogenital organs of a normal male toad (*Bufo bufo*). × 4.

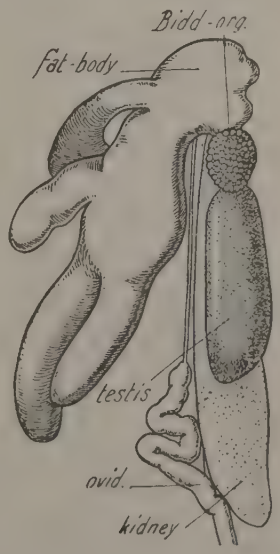

Fig. 2. Urinogenital organs of a male toad, after administration of 5 × 600 I.U. of oestrone. The wall of the oviduct is thickened and convoluted. × 4.

organ may also be attributed to them. The nature of this hypothetical hormone is not known. PONSE (1941) had no effects after injection of oestrone or of progesterone in female ovariectomized toads, whereas transplantation of *Bufo*-ovaries in ovariectomized females of the same species resulted in regeneration of the oviducts. She therefore concludes that the toad's ovary produces a hormone, different from the oestrogens or from progesterone of mammalian origin. On the other hand DE ALLENDE (1939) showed that the oviduct of the female toad *Bufo arenarum* develops under the direct influence of the hypophysis.

In our experiments only adult male common toads (*Bufo bufo*) were used. Oestrone, progesterone and gonadotrophins (pregnyl and gestyl) were administered in different doses by injection in the dorsal lymphsac, and their effects studied [1]).

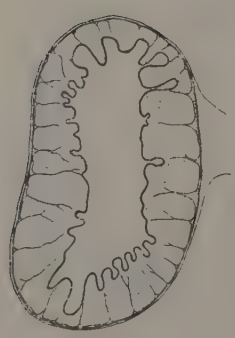

Fig. 3. Cross-section of the oviduct of a normal male toad. No mucous present. × 140.

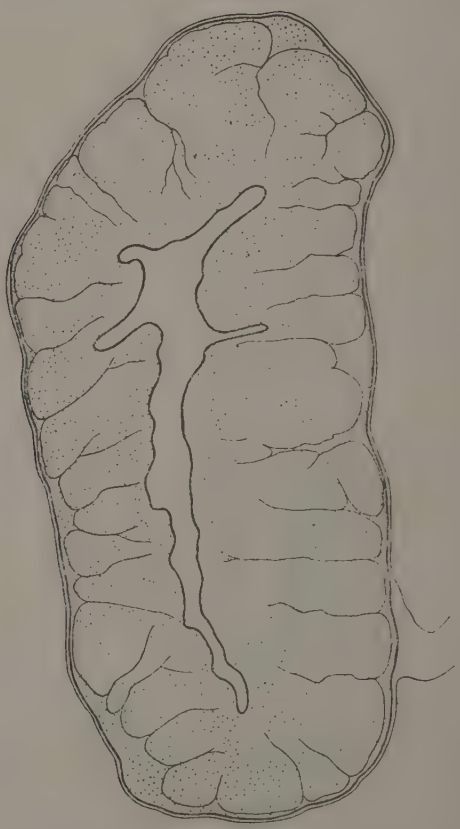

Fig. 4. Cross-section of the oviduct of a male toad after administration of 5 × 600 I.U. of oestrone. In the wall is much mucous present. × 140.

[1]) Our thanks are due to the Direction of "Organon" N.V., Oss, for providing us with these hormones.

The effect of oestrone. A total dosis of 600 and of 1100 I.U. of oestrone administered in February (1944) had no results, but ♂♂, injected in the end of July and in the beginning of August (1943) six times, every 2 days, with 500 I.U., showed 6 days after the last injection a distinct thickened and somewhat convoluted oviductwall (fig. 2). In the latter relatively large glands, secreting a mucous substance, were found (fig. 4). However, large quantities of oestrone (in total 5000—12000 I.U.) injected in February (1944) did not effect the wall of the oviduct. The number of experimental animals was not very great, but as in none of the control specimens this muccous substance was found, we are of opinion that at least during the summer months oestrone of mammalian origin is capable of inducing in the male toad a distinct thickening of the oviduct-wall and a secretion of mucous substance in its glands. Hence the oviduct is only sensitive for oestrone during a definite period of the year.

The effect of progesterone. Male toads which were injected in February (1944) with in total 60—500 γ progesterone did not show any effect, neither in BIDDER's organs nor in the MÜLLERian ducts.

The effect of gonadotrophins. According to HOUSSAY and GONZALEZ (1931) the hypophysis controls the development and function of BIDDER's organ. Whether the number of preovulation corpora lutea in this organ is regulated by the hypophysis, as is the case in the Bitterling (BRETSCHNEIDER and DUYVENÉ DE WIT, 1941), is not known. Therefore we paid special attention to their presence in BIDDER's organ of male toads, which were injected either with pregnyl, a pregnancy urine principle, or

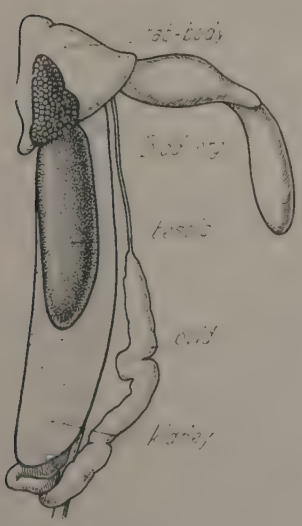

Fig. 5. Urinogenital organs of a male toad after administration of 5 × 500 I.U. of pregnyl. The oviduct is inflated and convoluted. × 4.

Fig. 6. Urinogenital organs of a male toad after administration of 4 × 200 + 3 × 400 I.U. of gestyl. The oviduct is inflated. × 4.

with gestyl, a pregnant mare serum preparation; however, an increase in the number of preovulation corpora lutea could not be established after the injection of these gonadotrophins.

On the other hand the testes were distinctly influenced by these gonadotrophins. In male toads, injected in the course of February (1944) with 200 I.U.—2500 I.U. of pregnyl or with 1500—2000 I.U. of gestyl, the intertubular tissue of the testes develops enormously and there is a great increase in the number of interstitial cells. Moreover, the sperm, which lie

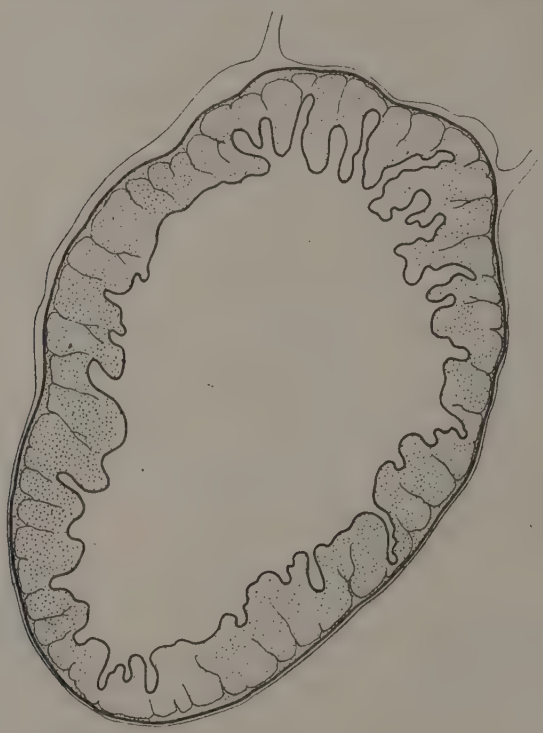

Fig. 7. Cross-section of the oviduct of a male toad after administration of 5 × 500 I.U. of pregnyl. The oviduct is inflated; its wall does not contain mucous. × 140.

in the controls in packages over SERTOLI's cells, their tails directed towards the lumen of the testis tubules, are loosened from each other, probably mobilized and practically all transported through the efferent ducts into the kidney tubules and even into the WOLFFian duct and its seminal vesicle. This was already the case 24 h. after one injection of 200 I.U. of pregnyl. Hence the effect of gonadotrophins on the testis of *Bufo* is comparable to that on the ovary of many Amphibians, in which ovulation is also easily induced by gonadotrophins.

Although in the female the ova pass into the body cavity by ovulation, whereas in the male the sperm are released and transported into the kidneys

and WOLFFian ducts, both these processes have so much in common that we will use the term s p e r m i a t i o n for this process in the male. As far as we are aware spermiation has hitherto only been induced in the frog (*Rana pipiens*) by RUGH (1937).

The oviducts of male toads are also influenced by gonadotrophins: they are inflated markedly (figs. 5 and 6), possessing a very wide lumen but

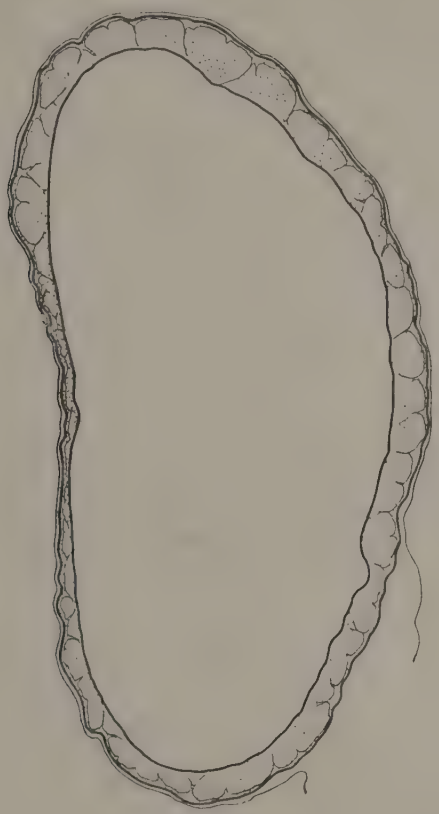

Fig. 8. Cross-section of the oviduct of a male toad after administration of 4 × 200 + 3 × 400 I.U. of gestyl. The oviduct is much inflated, its thin wall does not contain mucous. × 140.

their walls undergo no change: their thickness is about the same as that of the controls and mucous secreting glands are not present (figs. 7 and 8).

Finally it must be mentioned that the toads were intolerant of gestyl, becoming highly oedematous soon after its injection into the dorsal lymphsac.

S u m m a r y. Oestrone, injected in male toads, has a distinct influence on the MÜLLERian ducts when injected during the summer months; then mucous secreting glands develop in the wall of these ducts, which increase in diameter and are somewhat convoluted. Injections during the winter-

months (January and February) did not have any influence on the oviduct. The effects of gonadotrophins (pregnyl and gestyl) injected in male toads are (1) an enlargement of the tubular spaces and an increase in number of the interstitial cells, (2) an almost immediate releasing (s p e r m i a t i o n) of the sperm, which are transported through the efferent ducts into the kidney tubules and even into the WOLFFian duct, (3) an inflation of the MÜLLERian ducts, the walls of which did not increase in height nor develop mucous glands. BIDDER's organ is not effected by gonadotrophins.

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Mathematics. — *On the theory of linear integral equations.* V. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 27, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the papers I. II and IV, bearing the same title [1]). In this paper we shall consider linear integral equations

$$\int_{\triangle} K(x, y) f(y)\, dy - \lambda f(x) = g(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the space $L_2^{(m)}(\triangle)$ with kernel $K(x, y) = A(x)H(x, y)$, where $A(x)$ is a measurable, bounded and real function on the m-dimensional interval $\triangle$, and $H(x, y) \in L_2^{(2m)}$ is a positive (positive means here: of positive type) Hermitean kernel. Equations with a kernel $K(x, y)$ of this category are sometimes called of the *third kind*, and, if $A(x)$ takes on only the values $+1$ and -1, they are called of *polar* type (D. HILBERT). E. GARBE [2]) has discussed the equation of the third kind under the assumptions that $\triangle$ is a linear interval, $A(x)$ is continuous except for a finite number of jumps, $[A(x)]^{-1}$ is bounded [3]), and that the kernel $H(x, y)$ is continuous and *general* (that is, the functions $\int_{\triangle} H(x, y) f(y) dy$ are lying dense in the space $L_2^{(m)}(\triangle)$). Denoting the characteristic values $\neq 0$ of (1) by $\lambda_i (i = 1, 2, \ldots)$, and a corresponding H-orthonormal system of characteristic functions by $\psi_i(x)$ $(i = 1, 2, \ldots)$, he obtained the following results:

1°. If

$$a_i = \int_{\triangle \times \triangle} H(x, y)\, \overline{\psi_i(x)}\, f(y)\, dx\, dy$$

for an arbitrary $f(x) \in L_2$, then

$$\int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

uniformly in x,

2°.

$$H(x, y) = \Sigma \frac{\lambda_i^2 \psi_i(x) \overline{\psi_i(y)}}{A(x) A(y)},$$

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 194—204, 205—212 and 409—416 (1946).

[2]) E. GARBE, Zur Theorie der Integralgleichung dritter Art, Math. Annalen, **76**, 527—547 (1915).

[3]) Without stating it very clearly, GARBE uses the boundedness of $[A(x)]^{-1}$ in his formulae (10) and (34).

uniformly in x and y, or, writing

$$\int_{\triangle} H(x,y)\psi_i(y)\,dy = \chi_i(x), \text{ so that } \lambda_i\psi_i(x) = \int_{\triangle} K(x,y)\psi_i(y)\,dy = A(x)\chi_i(x),$$

$$H(x,y) = \Sigma\,\chi_i(x)\overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

uniformly in x and y.

We shall show now, as a consequence of other more general results, that GARBE's Theorems are valid under less stringent conditions upon the function $A(x)$. The formula (2) holds if $A(x) \neq 0$ almost everywhere on $\triangle$, and also, in the case that $\triangle$ is a linear interval, if $A(x)$ is, for every $x \in \triangle$, continuous to the left or to the right (in thise case it is therefore permitted that $A(x) = 0$ on a set of positive measure), and for the validity of (3) it is sufficient that $A(x) \neq 0$ almost everywhere in $\triangle$. We shall prove, moreover, that although the series $\Sigma\chi_i(x)\overline{\chi_i(y)}$ converges uniformly whenever $H(x,y)$ is continuous, its sum is not necessarily equal to $H(x,y)$ when $A(x) = 0$ in a subinterval of $\triangle$, even in the case that $H(x,y)$ is general.

§ 2. *The equation of the third kind.*

Let $K(x,y) = A(x)H(x,y)$, where $H(x,y) \in L_2^{(2m)}(\triangle)$ is a positive Hermitean kernel, and $A(x)$ is on $\triangle$ measurable, bounded and real. Then the linear transformation H in the space $L_2^{(m)}(\triangle)$, defined by

$$Hf = \int H(x,y)f(y)\,dy,$$

is completely continuous, self-adjoint and positive, while the linear transformation A, defined by

$$Af = A(x)f(x),$$

is bounded and self-adjoint. The completely continuous linear transformation $K = AH$ is then determined by

$$Kf = AHf = A(x)\int_{\triangle} H(x,y)f(y)\,dy = \int_{\triangle} K(x,y)f(y)\,dy.$$

As we know, the transformation K is symmetrisable relative to H, and we observe that every $f(x) \in L_2$, satisfying $Hf = 0$, satisfies also $Kf = 0$. The kernel $K(x,y)$ is therefore what we have called in IV a Marty-kernel. Supposing that

$$\|H(x,y)\|_{2m}^2 = \int_{\triangle\times\triangle} |H(x,y)|^2\,dx\,dy \neq 0,$$

so that H is not identical with the nulltransformation O, the theorems proved in IV may therefore be applied to the equation (1). We shall not

repeat them all here, and only pay attention to IV, Theorems 4, 6 and 11, since these may be replaced by stronger theorems. Instead of IV, Theorem 4 we have

Theorem 1. (*Expansion Theorem.*) *Writing*

$$a_i = (f, \chi_i) = \int_{\Delta} f(x)\overline{\chi_i(x)}dx$$

for an arbitrary $f(x) \in L_2$, *we have*

$$\int_{\Delta} K(x, y) f(y)\, dy \backsim \Sigma \lambda_i a_i \psi_i(x) + p(x),$$

$$\int_{\Delta} K_n(x, y) f(y)\, dy \backsim \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2),$$

where the function $p(x)$ *satisfies the relation*

$$Hp = \int_{\Delta} H(x, y)\, p(y)\, dy = 0$$

for almost every $x \in \Delta$.

Proof. Follows from I, Theorem 15.

Instead of IV, Theorem 6 we have

Theorem 2. *Let* $\lambda \neq 0$, *and let* $g(x) \in L_2$ *be H-orthogonal to all characteristic functions of* (1) *belonging to the characteristic value* λ (*If* λ *is no characteristic value,* $g(x)$ *is therefore arbitrary*). *Then every solution of* (1) *satisfies a relation of the form*

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x) + q(x),$$

where

$$a_i = \int_{\Delta} g(x)\overline{\chi_i(x)}\, dx \quad \text{for } \lambda_i \neq \lambda, \int_{\Delta} H(x, y)\, q(y)\, dy = 0$$

for almost every $x \in \Delta$, *and where* Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* $\psi_i(x)$ *has the value* $\int_{\Delta} f(x)\overline{\chi_i(x)}dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 17.

Instead of IV, Theorem 11, we have

Theorem 3. (*Expansion Theorem*). *We have*

$$K_2(x, y) - p_2(x, y) \backsim \Sigma \lambda_i^2 \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\int_{\Delta} H(x, z)\, p_2(z, y)\, dz = 0$$

almost everywhere in $\Delta \times \Delta$;

$$K_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqq 3); \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\int_{\Delta} K_n(x, x)\, dx = \sum_i \lambda_i^n \qquad (n \geqq 3). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Proof. The formulae (4) and (5), and also the formula (6) for $n \geq 4$ have already been proved in IV, Theorem 11. The only thing that remains to be proved is

$$\int_{\Delta} K_3(x, x)\, dx = \Sigma \lambda_i^3.$$

Now, in the proof of IV, Theorem 7 we have obtained the formula (4), stating that

$$\int_{\Delta} P(z, y)\, K(y, z)\, dy = \Sigma \lambda_i^2 \mid \chi_i(z) \mid^2$$

for almost every $z \in \Delta$, where

$$P(z, y) = \int_{\Delta} H(z, x)\, K(x, y)\, dx.$$

Hence

$$\int_{\Delta} H(z, x)\, K_2(x, z)\, dx = \int_{\Delta \times \Delta} H(z, x)\, K(x, y)\, K(y, z)\, dx\, dy =$$

$$\int_{\Delta} P(z, y)\, K(y, z)\, dy = \Sigma \lambda_i^2 \mid \chi_i(z) \mid^2,$$

or, observing that $A(z)\chi_i(z) = \lambda_i \psi_i(z)$,

$$K_3(z, z) = \int_{\Delta} K(z, x)\, K_2(x, z)\, dx = A(z) \int_{\Delta} H(z, x)\, K(x, z)\, dx =$$

$$\Sigma \lambda_i^2 A(z)\, \chi_i(z)\, \overline{\chi_i(z)} = \Sigma \lambda_i^3 \psi_i(z)\, \overline{\chi_i(z)}$$

for almost every $z \in \Delta$. This shows that

$$\int_{\Delta} K_3(z, z)\, dz = \Sigma \lambda_i^3.$$

Theorem 4. (*Expansion Theorem*). *If* $\Sigma\mu_i$ *converges, where*

$$\mu_i (i = 1, 2, \ldots)$$

is the sequence of characteristic values of the kernel $H(x, y)$, *we have*

$$K(x, y) - p(x, y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

almost everywhere in $\triangle \times \triangle$;

$$K_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \quad (n \geqq 2); \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\int_{\triangle} K_n(x, x)\, dx = \sum_i \lambda_i^n \quad (n \geqq 2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Proof. *If* $\Sigma\mu_i$ converges, the uniquely determined, positive self-adjoint transformation $H^{1/2}$ is of the form

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y)\, f(y)\, dy,$$

where $H_{1/2}(x, y) \epsilon L_2^{(2m)}$, so that the transformation $Q = AH^{1/2}$ is expressible as

$$Q f = \int_{\triangle} Q(x, y)\, f(y)\, dy,$$

where $Q(x,y) = A(x) H_{1/2}(x, y) \epsilon L_2^{(2m)}$. The theorem to be proved is therefore a consequence of IV, Theorem 12.

§ 3. *The case that* $H(x, y)$ *is continuous.*

Theorem 5. (*Expansion Theorem for the kernel*). *If* $H(x, y)$ *is continuous, then*

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

uniformly in $\triangle \times \triangle$, *where the function* $p(x, y)$ *satisfies the relation*

$$q(x, y) = \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

for every point $(x, y) \epsilon \triangle \times \triangle$.

Proof. Since $H(x, y)$ is continuous, the transformation $H^{1/2}$ is, on account of II, Theorem 10, given by

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y)\, f(y)\, dy,$$

where $\int_{\triangle} |H_{1/2}(x, y)|^2 \, dy$ is bounded. For almost every $x_1 \in \triangle$ and almost every $x_2 \in \triangle$ we have now

$$\int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy =$$

$$\int_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\} \overline{\{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\}} \, dy =$$

$$\int_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\} \{H_{1/2}(y, x_2) - H_{1/2}(y, x_1)\} \, dy =$$

$$H(x_2, x_2) - H(x_2, x_1) - H(x_1, x_2) + H(x_1, x_1);$$

consequently, since $H(x, y)$ is continuous in $\triangle \times \triangle$, there exists for any $\varepsilon > 0$ a number $\delta(\varepsilon) > 0$ such that

$$\int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy < \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

if only the distance

$$\varrho(x_1, x_2) = \left(\sum_{i=1}^{m} |x_1^{(i)} - x_2^{(i)}|^2\right)^{1/2}$$

of the points x_1 and x_2 satisfies the relation $\varrho(x_1, x_2) < \delta$, and $x_k (k = 1, 2)$ does not belong to a set $E_k \in \triangle (k = 1, 2)$ of measure 0. Further, on account of

$$\chi_i(x) = H\psi_i = H^{1/2} H^{1/2} \psi_i = H^{1/2} \Psi_i = \int_{\triangle} H_{1/2}(x, y) \, \Psi_i(y) \, dy,$$

holding for almost every $x \in \triangle$, we have

$$\chi_i(x_2) - \chi_i(x_1) = \int_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\} \, \Psi_i(y) \, dy$$

for almost every $x_1 \in \triangle$ and almost every $x_2 \in \triangle$; hence, in virtue of BESSEL's inequality (the system $\Psi_i(x)$ is orthonormal),

$$\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2 \leqslant \int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy. \quad . \quad (12)$$

for these values of x_1, x_2 and for arbitrary p.

From (11) and (12) we deduce that

$$\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2 < \varepsilon$$

for almost every $x_1 \epsilon \triangle$ and almost every $x_2 \epsilon \triangle$, if only $\varrho(x_1, x_2) < \delta$. Since however the functions $\chi_i(x) = \int_{\triangle} H(x, y)\psi_i(y)dy$ are continuous in $\triangle$, the function $\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2$ is continuous in x_1 and in x_2, so that the relation

$$\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2 \leqq \varepsilon$$

holds evidently for every pair of points $x_1, x_2 \epsilon \triangle$, if only $\varrho(x_1, x_2) < \delta$. Observing that p is arbitrary, we obtain finally

$$\sum_i |\chi_i(x_2) - \chi_i(x_1)|^2 \leqq \varepsilon \text{ for } \varrho(x_1, x_2) < \delta(\varepsilon);$$

in other words

$$\lim_{x_2 \to x_1} \sum_i |\chi_i(x_2) - \chi_i(x_1)|^2 = 0.$$

By MINKOWSKI's inequality we see now that

$$|(\sum_i |\chi_i(x_2)|^2)^{1/2} - (\sum_i |\chi_i(x_1)|^2)^{1/2}| \leqq (\sum_i |\chi_i(x_2) - \chi_i(x_1)|^2)^{1/2};$$

the sumfunction of the series $\Sigma |\chi_i(x)|^2$ is therefore a continuous function. Hence, on account of DINI's well-known Theorem, since the functions $|\chi_i(x)|^2$ are non-negative and continuous, the uniform convergence of $\Sigma |\chi_i(x)|^2$. The inequality

$$\Sigma |\chi_i(x)\,\overline{\chi_i(y)}| \leqq (\Sigma |\chi_i(x)|^2)^{1/2} \cdot (\Sigma |\chi_i(y)|^2)^{1/2}$$

shows then that $\Sigma \chi_i(x)\overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$. Writing

$$H(x, y) - p_1(x, y) = \Sigma \chi_i(x)\,\overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

we see, since both $H(x, y)$ and $\Sigma \chi_i(x)\overline{\chi_i(y)}$ are continuous in $\triangle \times \triangle$, that $p_1(x, y)$ is continuous in $\triangle \times \triangle$. Multiplying the relation (13) with $A(x)$ and writing $A(x)p_1(x, y) = p(x, y)$, we find

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x)\,\overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, and we know already (cf. Theorem 4) that

$$q(x, y) = \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

for almost every point $(x, y) \epsilon \triangle \times \triangle$. We have still to prove that $q(x, y) = 0$ everywhere in $\triangle \times \triangle$. For this purpose we observe that, since $H(x, z)$ is continuous, $q(x, y)$ is continuous in x for a fixed value of y, and, since

$$q(x, y) = \int_{\triangle} H(x, z)\, A(z)\, p_1(z, y)\, dz,$$

where $p_1(z, y)$ is continuous, $q(x, y)$ is continuous in y for a fixed value of x. Starting now with the fact that, for almost every $y \in \triangle$, $q(x, y) = 0$ for almost every $x \in \triangle$, we find on account of the continuity in x that, for almost every $y \in \triangle$, $q(x, y) = 0$ for every $x \in \triangle$; in other words, for every $x \in \triangle$ we have $q(x, y) = 0$ for almost every $y \in \triangle$. On account of the continuity in y we have therefore $q(x, y) = 0$ for every point $(x, y) \in \triangle \times \triangle$. This completes the proof.

Theorem 6. (*Expansion Theorem for the iterated kernels*). *If* $H(x, y)$ *is continuous, then*

$$K_n(x, y) = \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqq 2), \quad . \quad . \quad . \quad . \quad (14)$$

uniformly in $\triangle \times \triangle$.

Proof. The result for $n = 2$ follows immediately from the preceding theorem, since

$$\int_{\triangle} K(x, z)\, p(z, y)\, dz = A(x) \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

in $\triangle \times \triangle$. The relation

$$K_n(x, y) = \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n > 2)$$

follows easily by induction.

Theorem 7. (*Expansion Theorem*). *If* $H(x, y)$ *is continuous, and*

$$a_i = (f, \chi_i) = \int_{\triangle} f(x) \overline{\chi_i(x)}\, dx \text{ for an arbitrary } f(x) \in L_2,$$

we have

$$\int_{\triangle} K(x, y)\, f(y)\, dy - p(x) = \Sigma \lambda_i a_i \psi_i(x),$$

$$\int_{\triangle} K_n(x, y)\, f(y)\, dy = \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2),$$

uniformly in $\triangle$, *where* $p(x)$ *satisfies the relation*

$$\int_{\triangle} H(x, y)\, p(y)\, dy = 0$$

for every $x \in \triangle$.

Proof. The formula (10) implies

$$\int_{\triangle} K(x, y)\, f(y)\, dy - p(x) = \Sigma \lambda_i a_i \psi_i(x),$$

uniformly in $\triangle$, where

$$p(x) = \int_{\triangle} p(x, y) f(y)\, dy.$$

Hence

$$\int_{\triangle} H(x, y)\, p(y)\, dy = \int_{\triangle \times \triangle} H(x, y)\, p(y, z)\, f(z)\, dy\, dz = 0$$

for every $x \in \triangle$.

The formula (14) implies

$$\int_{\triangle} K_n(x, y) f(y)\, dy = \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2),$$

uniformly in $\triangle$.

A Hermitean kernel $A(x, y) \in L_2^{(2m)}$ is called a *general* kernel (D. HILBERT), when the set of all functions $Ag = \int_{\triangle} A(x, y) g(y) dy$ is lying dense in the space L_2; in other words, when, given $f(x) \in L_2$ and the number $\varepsilon > 0$, there exists a function $g(x) \in L_2$ such that

$$\int_{\triangle} | f(x) - \int_{\triangle} A(x, y)\, g(y)\, dy |^2\, dx < \varepsilon.$$

Theorem 8. *In order that the Hermitean kernel $A(x, y) \in L_2^{(2m)}$ be general, it is necessary and sufficient that $Af = 0$ should imply $f = 0$.*

Proof. Denoting by $\mu_i (i = 1, 2, \ldots)$ the sequence of all characteristic values $\neq 0$ of $A(x, y)$, and by $\varphi_i(x)$ a corresponding orthonormal system of characteristic functions, it is not difficult to prove that the condition that $Af = 0$ should imply $f = 0$ is equivalent with the condition that the system $\varphi_i(x)$ is complete (that is, the finite linear combinations $\Sigma a_i \varphi_i(x)$ are lying dense in L_2).

Let now $Af = 0$ imply $f = 0$, and let $f(x) \in L_2$ and $\varepsilon > 0$ be given. Since the system $\varphi_i(x)$ is orthonormal and complete, we have $f = \Sigma a_i \varphi_i$ with $a_i = (f, \varphi_i)$. Taking the index N such that $\| f - \sum_{i=1}^{N} a_i \varphi_i \|^2 < \varepsilon$, and writing $a_i = \mu_i b_i$, we have for $g = \sum_{i=1}^{N} b_i \varphi_i$ the relation

$$Ag = \sum_{i=1}^{N} b_i A \varphi_i = \sum_{i=1}^{N} \mu_i b_i \varphi_i = \sum_{i=1}^{N} a_i \varphi_i,$$

hence

$$\int_{\triangle} | f(x) - \int_{\triangle} A(x, y)\, g(y)\, dy |^2\, dx = \| f - Ag \|^2 = \| f - \sum_{i=1}^{N} a_i \varphi_i \|^2 < \varepsilon.$$

The kernel $A(x, y)$ is therefore general.

Conversely, if $A(x, y)$ is general, the elements $Ag = \Sigma \mu_i (g, \varphi_i)\varphi_i$ are lying dense in L_2, so that the system $\varphi_i(x)$ is complete. Then however, as we have seen, $Af = 0$ implies $f = 0$.

Let $f(x)$ be a measurable function on the interval $\triangle$, E_1 the set where $f(x) = 0$, $E_2 = \triangle - E_1$ the set where $f(x) \neq 0$.

Definition. *We shall say that $f(x)$ possesses the property (G) when every measurable set $E_3 \subset E_2$, for which meas. $(E_2 - E_3) = 0$, is lying dense in E_2, in other words, when E_2 is contained in the closure $\bar{E}_3$ of E_3.*

Theorem 9. *If $f(x) \neq 0$ almost everywhere in $\triangle$, $f(x)$ possesses the property (G).*

Proof. Since now meas. E_2 = meas. $\triangle$, we have for every E_3 for which meas. $(E_2 - E_3) = 0$, also meas. E_3 = meas. $\triangle$. This implies $E_2 \subset \bar{E}_3 = \triangle$, so that $f(x)$ possesses the property (G).

Theorem 10. *If $f(x)$ is continuous in $\triangle$, $f(x)$ possesses the property (G). In the case that $\triangle$ is a linear interval, it is even sufficient to suppose that $f(x)$ is, for every $x \in \triangle$, continuous to the left or to the right.*

Proof. Let $f(x)$ be continuous in $\triangle$. Then the set E_1 is closed, so that E_2 is open (relative to $\triangle$); in other words, E_2 contains only internal points. Given now the set $E_3 \subset E_2$ such that meas. $(E_2 - E_3) = 0$, every neighbourhood of a point $x \in E_2$ must contain points of E_3; hence $E_2 \subset \bar{E}_3$. Every continuous function possesses therefore the property (G). In the case that $\triangle$ is a linear interval, the same proof holds if only $f(x)$ is, for every $x \in \triangle$, continuous to the left or to the right.

Theorem 11. *(Expansion Theorem). If $H(x, y)$ is continuous and general, and if $A(x)$ possesses the property (G), then*

$$K(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad \ldots \ldots \quad (15)$$

uniformly in $\triangle \times \triangle$, and, writing $a_i = (f, \chi_i) = \int_{\triangle} f(x)\overline{\chi_i(x)}dx$ for an arbitrary $f(x) \in L_2$,

$$\int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x), \quad \ldots \ldots \quad (16)$$

uniformly in $\triangle$. Furthermore

$$\int_{\triangle} K(x, x)\, dx = \Sigma \lambda_i. \quad \ldots \ldots \quad (17)$$

Proof. By Theorem 5 we have

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, where $p(x, y) = A(x)p_1(x, y)$, the function $p_1(x, y)$ is continuous in $\triangle \times \triangle$, and $\int_{\triangle} H(x, z)p(z, y)dz = 0$ in $\triangle \times \triangle$. Since $H(x, y)$ is a general kernel, $Hp = 0$ implies $p = 0$, so that, for every $y \in \triangle$, $p(x, y) = 0$ for almost every $x \in \triangle$. Denoting by $E_1 \subset \triangle$ the set where $A(x) = 0$, and by E_2 the complementary set where $A(x) \neq 0$, we see that $p(x, y) = A(x)p_1(x, y) = 0$ for $x \in E_1$. Furthermore

$$p_1(x, y) = p(x, y)/A(x) = 0$$

almost everywhere in E_2; hence, in a set $E_3 \subset E_2$ for which meas. $(E_2 - E_3) = 0$. In virtue of the continuity of $p_1(x, y)$ the relation $p_1(x, y) = 0$ holds also for $x \in \bar{E}_3$. But, $A(x)$ possessing the property (G), we have $E_2 \subset \bar{E}_3$, so that $p_1(x, y) = 0$, and therefore also $p(x, y) = 0$, for $x \in E_2$ Having established thus that, for every $y \in \triangle$, $p(x, y) = 0$ for $x \in E_1$ and $x \in E_2$, we see that $p(x,y) = 0$ in $\triangle \times \triangle$, hence

$$K(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

uniformly in $\triangle \times \triangle$.

The formulae (16) and (17) follow immediately from (15).

Theorem 12. *Let $H(x, y)$ be continuous and general, and $A(x)$ possess the property (G). Let furthermore $\lambda \neq 0$, and $g(x) \in L_2$ be H-orthogonal to all characteristic functions of $K(x, y)$, belonging to the characteristic value. (If λ is no characteristic value, $g(x)$ may be any function belonging to L_2). Then the solution of the equation (1) is given by*

$$f(x) = -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

where $a_i = \int_{\triangle} g(x)\overline{\chi_i}(x)dx$ for $\lambda_i \neq \lambda$, Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of $\psi_i(x)$ is arbitrary, and the series

$$\Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x)$$

converges uniformly in $\triangle$.

Proof. By the preceding theorem we have

$$g(x) + \lambda f(x) = \int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i b_i \psi_i(x),$$

uniformly in $\triangle$, where $b_i = (f, \chi_i)$. Since

$$\lambda_i b_i = \lambda_i (f, \chi_i) = \lambda_i (f, H\psi_i) = (f, HK\psi_i) = (HKf, \psi_i) =$$
$$(Kf, H\psi_i) = (g + \lambda f, \chi_i) = a_i + \lambda b_i,$$

we find $b_i = -a_i/(\lambda - \lambda_i)$ for $\lambda_i \neq \lambda$. Furthermore, the solution $f(x)$ being determined except for a characteristic function of $K(x, y)$, belonging to the characteristic value λ, the coefficients b_i may be taken arbitrary for those values of i for whih $\lambda_i = \lambda$. Hence

$$f(x) = -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

uniformly in $\triangle$.

Theorem 13. (GARBE's *Theorem*). *If $H(x, y)$ is continuous and general, and if moreover $A(x) \neq 0$ for almost every $x \in \triangle$, then*

$$H(x, y) = \Sigma \chi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

uniformly in $\triangle \times \triangle$.

Proof. In the proof of Theorem 5 we have seen that $\Sigma \chi_i(x) \overline{\chi_i(y)}$ converges uniformly. Furthermore, since $A(x)$ possesses the property (G), we have by Theorem 11

$$K(x, y) = A(x) H(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)} = A(x) \Sigma \chi_i(x) \overline{\chi_i(y)}$$

for every point $(x, y) \in \triangle \times \triangle$. Hence, for every $y \in \triangle$,

$$H(x, y) = \Sigma \chi_i(x) \overline{\chi_i(y)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

for those values of x for which $A(x) \neq 0$, that is, for almost every $x \in \triangle$. Since however both $H(x, y)$ and $\Sigma \chi_i(x) \overline{\chi_i(y)}$ are continuous in $\triangle \times \triangle$, the relation (18) holds for every point $(x, y) \in \triangle \times \triangle$.

Remark. It is not difficult to show that $A(x) \neq 0$ almost everywhere in $\triangle$ is the necessary and sufficient condition that $Af = A(x) f(x) = 0$ should imply $f = 0$ in the space L_2. This condition therefore is, for a measurable, bounded and real function $A(x)$, the analogue of the condition to be general for a Hermitean kernel $A(x, y) \in L_2^{(2m)}$.

It may be asked whether, in the case that $A(x) = 0$ in a set of positive measure, the relation

$$H(x, y) = \Sigma \chi_i(x) \overline{\chi_i(y)}$$

remains valid. We shall show that this is not necessarily true. Let, for this purpose, $\triangle$ be the linear interval $0 \leq x \leq 2\pi$, and $t_i(x)$ the orthonormal trigonometrical system, hence

$t_1(x) = (2\pi)^{-1/2}$, $t_{2n}(x) = \pi^{-1/2} \cos nx \; (n \geq 1)$, $t_{2n+1}(x) = \pi^{-1/2} \sin nx \; (n \geq 1)$.

It is well-known that the system $t_i(x)$ is complete in the space $L_2(0, 2\pi)$ of all functions $f(x)$ for which $|f(x)|^2$ is summable over $\triangle$.

Let now the continuous, general, positive Hermitean kernel $H(x, y)$ be defined by

$$H(x, y) = \sum_i i^{-4} t_i(x) t_i(y);$$

it has the characteristic values $i^{-4}(i = 1, 2, \ldots)$ with the corresponding characteristic functions $t_i(x)$. The transformation $H^{1/2}$ corresponds then with the continuous, general, positive Hermitean kernel

$$H_{1/2}(x, y) = \sum_i i^{-2}\, t_i(x)\, t_i(y),$$

having the characteristic values i^{-2} with the characteristic functions $t_i(x)$. Furthermore we define the bounded self-adjoint transformation A by $Af = A(x)f(x)$, where

$$A(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq \pi, \\ 1 & \text{for } \pi < x \leqq 2\pi. \end{cases}$$

Let us consider now the periodic function $f(x)$, defined by

$$f(x) = \begin{cases} x^5(\pi - x)^5 & \text{for } 0 \leqq x \leqq \pi, \\ 0 & \text{for } \pi \leqq x \leqq 2\pi. \end{cases}$$

The fourth derivative of $f(x)$ is periodic and continuous; if therefore the Fourier series of $f(x)$ is $\Sigma(f, t_i)t_i(x) = \Sigma b_i t_i(x)$, we find without any difficulty by partial integration that there exists a constant M such that $|b_i| \leq Mi^{-4}$ $(i = 1, 2, \ldots)$, so that $f(x) = \Sigma b_i t_i(x)$, uniformly in $\triangle$. Defining now $a_i = i^2 b_i$ $(i = 1, 2, \ldots)$, we have $|a_i| \leq Mi^{-2}$, which shows that $\Sigma a_i t_i(x)$ also converges uniformly in $\triangle$. Writing $\Psi(x) = \Sigma a_i t_i(x)$, we find

$$H^{1/2}\Psi = \Sigma a_i H^{1/2} t_i = \Sigma a_i i^{-2} t_i = \Sigma b_i t_i = f,$$

while from the definitions of $A(x)$ and $f(x)$ follows immediately $A(x)f(x) = 0$; hence

$$H^{1/2} A H^{1/2} \Psi = H^{1/2} A f = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Denoting the H-orthonormal characteristic functions with characteristic values $\neq 0$ of the kernel $K(x, y) = A(x)H(x, y)$ by $\psi_i(x)$, we know by I, Theorem 18 that the functions $\Psi_i = H^{1/2}\psi_i$ are the orthonormal characteristic functions with characteristic values $\neq 0$ of the self-adjoint transformation $H^{1/2}AH^{1/2}$. From (19) follows therefore that $\Psi(x)$ is orthogonal to all functions $\Psi_i(x)$. Writing $\widetilde{\Psi}(x) = \Psi(x)/\|\Psi\|$, the system $\{\widetilde{\Psi}(x), \Psi_i(x)\}$ is orthonormal, and, since for every $x \in \triangle$

$$\chi_i(x) = H\psi_i = H^{1/2}\Psi_i = \int_{\triangle} H_{1/2}(x, y)\, \psi_i(y)\, dy,$$

$$f(x)/\|\Psi\| = H^{1/2}\widetilde{\Psi} = \int_{\triangle} H_{1/2}(x, y)\, \widetilde{\Psi}(y)\, dy,$$

we find in virtue of BESSEL's inequality

$$\Sigma |\chi_i(x)|^2 + |f(x)|^2/\|\Psi\|^2 \leqq \int_{\triangle} |H_{1/2}(x, y)|^2\, dy = H(x, x),$$

so that, on account of $f(x) \neq 0$ for $0 < x < \pi$,

$$\Sigma \chi_i(x)\,\overline{\chi_i(x)} < H(x, x)$$

for these values of x. The functions $H(x,y)$ and $\Sigma\chi_i(x)\overline{\chi_i(y)}$ being continuous in $\triangle \times \triangle$, there exists consequently for every point $(x_0,x_0) \in \triangle \times \triangle$, subject to $0 < x_0 < \pi$, a two-dimensional neighbourhood $E(x_0) \subset \triangle \times \triangle$ such that

$$H(x, y) \neq \Sigma \chi_i(x)\,\overline{\chi_i(y)}$$

for $(x, y) \in E(x_0)$.

§ 4. *An example.*

We shall illustrate the theorems, proved in the preceding paragraph, by an example showing that the functions $p(x)$ and $p(x, y)$, occurring in the Theorems 7 and 5, need not vanish identically.

Let $\triangle$ be the linear interval $[0, 2\pi]$, and $\triangle_1, \triangle_2, \triangle_3$ the subintervals $[0, \pi/2]$, $(\pi/2, \pi)$, $[\pi, 2\pi]$. The orthonormal system of functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ is defined by

$$\varphi_1(x) = \begin{cases} (2/\pi)^{1/2}\,|\sin 2x| & \text{for } x \in \triangle_1 + \triangle_2, \\ 0 & \text{for } x \in \triangle_3, \end{cases}$$

$$\varphi_2(x) = \begin{cases} -(2/\pi)^{1/2}\,\sin 2x & \text{for } x \in \triangle_1 + \triangle_2, \\ 0 & \text{for } x \in \triangle_3, \end{cases}$$

$$\varphi_3(x) = \begin{cases} 0 & \text{for } x \in \triangle_1 + \triangle_2, \\ -(2/\pi)^{1/2}\,\sin x & \text{for } x \in \triangle_3. \end{cases}$$

The positive, self-adjoint transformation H is then defined by

$$Hf = \int_{\triangle} H(x, y)\, f(y)\, dy,$$

where

$$H(x, y) = \varphi_1(x)\,\varphi_1(y) + \varphi_3(x)\,\varphi_3(y).$$

The kernel $H(x, y)$ is evidently continuous, and

$$H\varphi_1 = \varphi_1,\ H\varphi_2 = 0,\ H\varphi_3 = \varphi_3.$$

Let now the bounded, self-adjoint transformation A be given by $Af = A(x)f(x)$, where

$$A(x) = \begin{cases} -1 & \text{for } x \in \triangle_1, \\ 1 & \text{for } x \in \triangle_2 + \triangle_3. \end{cases}$$

Then

$$A\varphi_1 = \varphi_2,\ A\varphi_2 = \varphi_1,\ A\varphi_3 = \varphi_3;$$

hence

$$K(x, y) = A(x) H(x, y) = \varphi_2(x) \varphi_1(y) + \varphi_3(x) \varphi_3(y).$$

Since evidently $H^{1/2} = H$, the self-adjoint transformation $\widetilde{K} = H^{1/2} A H^{1/2}$ is corresponding with the kernel

$$\widetilde{K}(x, y) = \int H(x, z) K(z, y) dz = \varphi_3(x) \varphi_3(y).$$

As we know, $K(x, y)$ and $\widetilde{K}(x, y)$ have the same characteristic values $\neq 0$; hence, since $\widetilde{K}(x, y)$ has evidently only the characteristic value $\lambda = 1$, different from 0, with the characteristic function $\varphi_3(x)$, the kernel $K(x, y)$ has also $\lambda_1 = 1$ as the only characteristic value $\neq 0$ with the H-normal characteristic function $\psi_1(x) = \varphi_3(x)$. Observing that $\chi_1(x) = H\psi_1 = H\varphi_3 = \varphi_3(x)$, so that $\lambda_1 \psi_1(x) \overline{\chi_1(y)} = \varphi_3(x) \varphi_3(y)$, we have therefore

$$K(x, y) - p(x, y) = \lambda_1 \psi_1(x) \overline{\chi_1(y)},$$

where $p(x, y) = \varphi_2(x) \varphi_1(y) \neq 0$ for x and y in the interior of $\triangle_1$ or $\triangle_2$, and $\int_{\triangle} H(x, z) p(z, y) dz = 0$, as required by Theorem 5.

Furthermore, by Theorem 7,

$$\int K(x, y) f(y) dy = \lambda_1 a_1 \psi_1(x) + p(x),$$

where

$$a_1 = \int_{\triangle} f(x) \chi_1(x) dx \quad \text{and} \quad Hp = \int_{\triangle} H(x, y) p(y) dy = 0.$$

Taking $f(x) = \varphi_1(x)$, we have

$$\int_{\triangle} K(x, y) f(y) dy = K\varphi_1 = A H \varphi_1 = \varphi_2(x) \quad \text{and} \quad a_1 = (\varphi_1, \varphi_3) = 0;$$

hence

$$\varphi_2(x) = p(x),$$

which shows that $p(x) \neq 0$ for x in the interior of $\triangle_1$ and $\triangle_2$. Evidently $Hp = H\varphi_2 = 0$, as required.

ERRATA

in: "*Equations for elastic solids in spherical coordinates, etc.*" by F. A. VENING MEINESZ, Proceedings Vol. XLVIII, 1945, p. 469.

p. 470, equation 3*c*: σ_r should be σ_z

p. 471, line 9: $\triangle^2$ should be $\nabla^2 =$

equation 7*a*, 7*b* and 7*c*:

$$\left(\nabla P - a\varrho' g \frac{\partial\theta}{\partial z}\right) \text{ should be } \left(\nabla P + a\varrho' g \frac{\partial\theta}{\partial z}\right)$$

equation 7*e*: $\frac{m}{m+1} \frac{\partial^2 \sigma}{\partial x\, \partial z}$ should be $\frac{m}{m+1} \frac{\partial^2 \sigma}{\partial x\, \partial z}$

p. 473, equation 13: $\partial\varphi$ should be ∂y (for P_y as well as for v)

p. 475, equation 22*a* and 22*b*: θ_1 should be θ_0 and v_0 should be v_1

p. 476, equation 28*a*: μ should be η

equation 28*b*: m^2 should be s^2

line 6 from the bottom: ∂v_0 should be ∂v_1

p. 477, equation 32*b*: m^2 should be s^2

equation 33: φ_0 should be p_0

equation 34*a*: o_λ should be σ_λ

p. 480, equation 44*b*, second line: insert the sign + between

$$\frac{\partial P_0}{\partial \varrho} \text{ and } \frac{m-2}{2(m-1)}$$

p. 481, equations 47*b* and 47*c*: e should be e_0

p. 482, line 9: $m = 0$ should be $s = 0$

p. 483, equation 54*c*: $\frac{\partial^2 K_n}{\partial 8^2}$ should be $\frac{\partial^2 K_n}{\partial \delta^2}$

equation 55: μ should be η

line 2 of § 5: "in these case of" should be "in the case of"

p. 485, equation 59: The first term should be provided with negative sign

p. 486, line 8: read "to e_0" instead of "e_0".

Aerodynamics. — *Some problems of the motion of interstellar gas clouds.* I. By J. M. BURGERS. (Mededeling No. 48 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of May 25, 1946.)

1. *Introduction.* — The following considerations have arisen from discussions with prof. dr. J. H. OORT at Leiden, who suggested most of the problems and to whose interest and advice I am greatly indebted in preparing these notes [1]). They must be taken, nevertheless, as a very preliminary attempt towards the application of aerodynamics in analysing some of the intricate riddles that are presented by the results of astronomical observations on the interstellar material within our Galaxy. Apart from the difficulties inherent in aerodynamics itself or arising when it is applied to matter in a state of such extreme dilution, astronomical observation, notwithstanding the powerful methods of photography and spectroscopy and the beauty of all what has already been revealed, still gives us only very scanty information about the structure and the motion of the cosmic clouds. In attempting to make a few calculations many guesses must be introduced, as otherwise it is impossible to construct simple pictures which can serve as starting points for a provisional theory. The reader therefore must take for granted that the cases treated in the following pages are no more than examples, which it is hoped may elucidate a few features, and perhaps point to a way for further work.

In the calculations the interstellar gas is considered as an ideal gas, to which the ordinary equations of aerodynamics can be applied. Although various elements and even certain chemical combinations are present (Ca, Na, K, Ca^+, Ti^+, Fe^+, CH, CH^+, CN), the bulk (about 99,9 % of the number of atoms) is formed by hydrogen. We will assume hydrogen atoms to be the only constituent, and take the gas law in the form: $p/\varrho = R\,T/m$, with p = pressure, ϱ = density, T = absolute temperature, R = molecular gas constant = $8{,}315.10^7$ (cm-gram-second units will be applied throughout), and m = molecular weight = 1 for ordinary monatomic hydrogen. The ratio of the specific heats is taken as $k = 5/3$.

Dust particles ("smoke") are present in the interstellar gas, and it is mainly due to their presence that clouds become visible, either as dark

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[1]) OORT also kindly put at my disposal the manuscript of the George Darwin lecture "*Some phenomena connected with interstellar matter*", delivered by him before the Royal Astronomical Society in London on May 10, 1946, and the results of his discussions with prof. dr. H. A. KRAMERS at Leiden. Some data used in the present communication have been taken from this manuscript, which is to be published in the Monthly Notices Roy. Astron. Society 1946.

masses, screening off the light of the stars, or as luminous patches when stars of sufficient luminosity are near. (Light emitted by the gas itself is visible in some cases, probably in consequence of an important rise in temperature produced by collisional effects as will be considered below.) No account, however, has been taken of the dust particles in the calculations. They will give a small contribution to the density and perhaps might have some influence upon the viscosity of the gas.

The average density of the interstellar material is of the order of 3.10^{-24} gr/cm^3. It seems probable that a considerable fraction of the material is concentrated in more or less separate clouds, with an average density of the order of 10^{-22} gr/cm^3, an average radius of 20 parsecs $\cong 6 . 10^{19}$ cm (1 parsec $= 3{,}083 . 10^{18}$ cm), and consequently a mass of approximately $M = 9 . 10^{37}$ gr (46000 times the mass of the sun); and distances apart of the order of 65 parsecs $\cong 2 . 10^{20}$ cm (or more). In the intermediate regions the density then will be much below $3 . 10^{-24}$. These clouds must take part in the general rotation of the Galaxy so as approximately to balance the gravitational attraction towards the galactic centre. At the sun's distance from the centre (approx. 30.000 lightyears $= 3{,}2 . 10^{22}$ cm) this rotational velocity is 300 to 320 km/sec $= 3{,}0$ à $3{,}2 . 10^7$ cm/sec. Superposed upon this general motion the clouds have peculiar motions of the order of 15 to 20 km/sec ($1{,}5$ à $2{,}0 . 10^6$ cm/sec).

A density of 10^{-22} gr/cm^3 corresponds to $n = 60$ hydrogen atoms per cm^3 (mass of a H-atom : $1{,}66 . 10^{-24}$ gram). Taking as the collisional diameter in the sense of the classical kinetic theory of gases the diameter of the first BOHR orbit $\sigma = 1{,}1 . 10^{-8}$ cm, the old formula for the free path gives: $l = 1/(\pi n \sigma^2) = 4{,}38 . 10^{-9}/\varrho$, which for $\varrho = 10^{-22}$ leads to $l \cong 4{,}4 . 10^{13}$ cm (the distance earth-sun is nearly $1{,}5 . 10^{13}$ cm).

It will be evident that the possibility of obtaining a Maxwellian velocity distribution and the normal relation between pressure, density and temperature can be found only in elements of volume with dimensions large compared with l. In the case of the cloud around the star Merope of the Pleiades the diameter of the illuminated portion is about 500″, which at a distance of 100 parsecs amounts to about $7{,}5 . 10^{17}$ cm. This cloud shows stratifications or waves with a thickness of approximately 8″ $= 1{,}2 . 10^{16}$ cm. In the Cygnus nebulae NGC 6960 and 6992 there are extremely thin strips or sheets, with an apparent thickness from 1″ to 5″. Taking the distance to be 350 parsec $= 1{,}08 . 10^{21}$ cm, these thicknesses amount from 0,5 to $2.5 . 10^{16}$ cm. The density is supposed to be somewhat smaller than 10^{-22} and to correspond to 30 atoms per cm^3, giving $\varrho = 5 . 10^{-23}$ and $l = 8{,}7 . 10^{13}$ cm. In both cases the applicability of the ideal gas laws to the thinnest observable sheets seems possible.

The temperature of the interstellar gas is assumed to be of the order of 10.000°. The mean molecular velocity then is given by: $c_m = \sqrt{3 R T/m} = 1{,}58 . 10^6$ cm/sec. The velocity of sound and velocities of expansion are of the same order of magnitude. In certain cases (clouds produced by the

explosion of a nova) velocities of the order of 1000 km/sec $= 10^8$ cm/sec have been observed. In all these cases there is no need for applying relativity corrections.

As the general gravitational field in the Galaxy regulates the motion of a cloud as a whole, it will not have much effect upon expansion or compression phenomena. The gravitational field of a cloud itself perhaps may be of some importance. Taking by way of example a spherical cloud of mass $M = 9 \cdot 10^{37}$ gr, having a mean density $\bar{\varrho} = 10^{-22}$, application of EMDEN's results for the case of static "polytropic" equilibrium [2]) gives for the density at the centre $\varrho_0 = 6{,}0 \cdot 10^{-22}$; for the temperature at the centre $T_0 = 650°$; and for the pressure at the centre $p_0 = 3{,}24 \cdot 10^{-11}$ (the radius comes out as $6 \cdot 10^{19}$ cm as should be). Hence the pressure due to the gravitational attraction inside the spherical mass in the neighbourhood of the centre becomes comparable with the pressure $8{,}3 \cdot 10^{-11}$ calculated for a temperature of 10.000° and a density of 10^{-22}. — At the surface of this spherical cloud the value of g is equal to $1{,}67 \cdot 10^{-9}$. In the case of the expansion problem considered in the next section we will make an estimate of its influence.

Viscosity, heat conduction and radiation in the first instance will be left out of account. A few remarks concerning their possible effects will be made in connection with some results.

Finally the calculations will refer to motions in one dimension only, so as to obviate the mathematical difficulties connected with problems of spherical expansion.

2. *Equations of motion. — Application to a case of simple expansion in a vacuum.* — The equations for the one-dimensional motion of an ideal gas, in the absence of gravity, have the form:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial x} = -\varrho\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = -(k-1)\,T\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad (3)$$

$$p = RT\varrho/m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Viscosity, heat conduction and radiation have been neglected; u is the velocity of the gas, while D/Dt in the usual way indicates the total derivative with respect to t for an observer moving with a particular element of volume. For such an observer combination of (2) and (3) gives: $T/\varrho^{k-1} =$ constant,

[2]) R. EMDEN, Gaskugeln (Leipzig 1907), Tab. 4, p. 79, for the case $k = 5/3$, and the formulae of p. 69, pp. 96/97.

expressing POISSON's isentropic relation for an individual element of volume. This relation holds so long as the differential equations can be applied; it breaks down, however, in so-called shock waves where $\partial u/\partial x$ and $\partial T/\partial x$ assume such large values that viscosity and heat conduction become important and lead to an increase of entropy.

As the density in a stellar cloud is much larger than that in the space around it, we must expect that it will expand, with an accompanying compression of the surrounding matter. The simplest case is that of an originally homogeneous mass, expanding into a vacuum, assuming that for $t<0$ we had: $p=p_0$, $\varrho=\varrho_0$ for $x<0$, and $p=0$, $\varrho=0$ for $x>0$. Introducing the velocity of sound

$$c=\sqrt{k\,p/\varrho}=\sqrt{k\,RT/m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(counted positive in the same direction as u) and making use of the relation $p/\varrho^k=$ constant, the solution can be put into the form [3]):

$$\left.\begin{aligned} u &= \frac{2}{k+1}\,\frac{x}{t}+\frac{2}{k+1}\,c_0 \\ c &= \frac{k-1}{k+1}\,\frac{x}{t}-\frac{2}{k+1}\,c_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This solution is limited to a region defined by two values of x/t. The first one characterizes the front of the expansion wave; here:

$$(x/t)'=2\,c_0/(k-1)=u;\ c=0 \text{ (expansion to zero density);}$$

the second one determines the back of the wave, where

$$(x/t)''=-c_0;\ u=0.$$

In the case $k=5/3$ the front moves outward with 3 times the velocity of sound in the original state of the gas, while the back of the wave moves inward with just this velocity. For $T=10.000°$, $m=1$, $k=5/3$ we obtain: $c_0=1{,}18\,.\,10^6$ cm/sec; $u_{\text{front}}=3{,}53\,.\,10^6$ cm/sec (35,3 km/sec). In a year $=3{,}16\,.\,10^7$ sec the front moves outward over $1{,}12\,.\,10^{14}$ cm, which at a distance of 100 parsecs would correspond to 0,075″ and thus perhaps might be just appreciable. Nevertheless it will be evident that a cloud with a diameter of $7{,}5\,.\,10^{17}$ cm for many centuries will retain its appearance.

Estimation of the influence of gravity. — In order to take gravity into account, a term $-g$ must be added to the right hand side of eq. (1). In connection with the restriction to one-dimensional motion we take g to be a

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[3]) Compare: J. M. BURGERS, Over de eendimensionale voortplanting van drukstoringen in een ideaal gas, Versl. Nederl. Akademie v. Wetenschappen, afd. Natuurkunde. **52**, 478 (1943).

constant. Introducing u, c as dependent variables, eqs. (1) and (2) can be transformed into the system:

$$\left.\begin{aligned}\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}\left(u+\frac{2}{k-1}c\right)&=-g\\ \left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{dx}\right\}\left(u-\frac{2}{k-1}c\right)&=-g\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

This system can be solved with the aid of its characteristics, which are determined by $(dx/dt)_I = u + c$; $(dx/dt)_{II} = u - c$. Passing over the details and taking for simplicity the same initial state as considered above (although this cannot be an equilibrium state), the solution is found to be:

$$\left.\begin{aligned}u&=\frac{2}{k+1}\frac{x}{t}+\frac{2}{k+1}c_0-\frac{k}{k+1}gt\\ c&=\frac{k-1}{k+1}\frac{x}{t}-\frac{2}{k+1}c_0+\frac{k-1}{2(k+1)}gt\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

The front of the wave, determined again by $c = 0$, moves according to the equation:

$$x' = 2\,c_0\,t/(k-1) - \tfrac{1}{2}gt^2 = 3\,c_0\,t - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad (9)$$

while the back, where $c = -c_0$, is to be found at:

$$x'' = -c_0\,t - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

At the back of the wave $u = -gt$; likewise for all values of x to the left of those determined by (10) $u = -gt$, which means that here the whole mass of the gas is moving with the acceleration $-g$ just as a solid body would do.

This latter result is of no particular importance, as it is a consequence of the condition assumed for $t = 0$, no account having been taken of the possibility of a compression (which would require the introduction of a boundary condition for some negative value of x). It will be evident, however, that the influence of gravity will be of importance only when gt becomes equal to, say c_0. With $g = 1{,}67 \cdot 10^{-9}$ as found in the example at the end of section 1, this will be the case after a lapse of time of $7 \cdot 10^{14}$ sec $= 2{,}2 \cdot 10^{7}$ year. Hence no great error is made by neglecting the influence of gravity in expansion and compression phenomena of clouds during comparatively short periods.

3. *Expansion with compression of the surrounding gas of low density.* — When for $t < 0$ we have a gas (I) with a density 10^{-22} to the left of $x = 0$, and to the right a gas (II) with a density of, say, $3 \cdot 10^{-24}$, both gases originally being at the same temperature T_0, the expansion of the denser gas (I) will produce a compression wave in gas (II), which com-

pression wave, owing to the high velocities involved, will take the form of a shock wave. The expansion of gas (I) will extend to a pressure equal to that produced by the passage of the shock wave through gas (II). As the front velocity $(x/t)'$ of the expansion region is smaller than the gas velocity u when the expansion does not proceed to zero density, a region of constant velocity, density and pressure appears between the front of the expansion region and the boundary separating gas I from gas II.

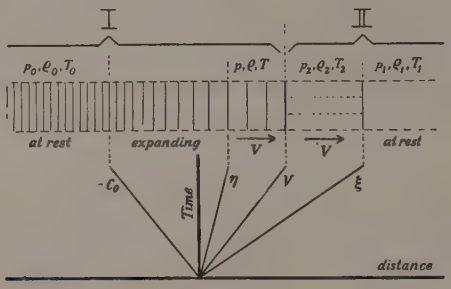

Fig. 1.

The propagation of the waves has been represented schematically in fig. 1. Starting from the right hand side, the shock wave in gas II (original state: p_1, ϱ_1, T_1; zero velocity) moves with the velocity ξ. Behind this wave gas II moves with the constant velocity V, its state being defined by p_2, ϱ_2, T_2. We then have [4]):

$$\xi = \frac{k+1}{4} V + \sqrt{c_1^2 + \frac{(k+1)^2 V^2}{16}}\,; \quad p_2 = p_1 + \varrho_1 V \xi\,. \quad . \quad (11)$$

The boundary between the two gases moves with the velocity V and so does gas I in the region between this boundary and the front of the expansion wave, the latter moving with the velocity $(x/t)' = \eta$. At this front eqs. (6) give:

$$u = \frac{2}{k+1}(\eta + c_0);\; c = \frac{k-1}{k+1}\eta - \frac{2}{k+1} c_0. \quad . \quad . \quad . \quad (12)$$

At the same time:

$$p = \left(\frac{\varrho_0^k}{p_0}\right)^{1/(k-1)} \left(\frac{c^2}{k}\right)^{k/(k-1)}; \quad \varrho = \left(\frac{\varrho_0^k}{p_0}\right)^{1/(k-1)} \left(\frac{c^2}{k}\right)^{1/(k-1)} \quad . \quad . \quad (13)$$

At this front we must have $u = V$; moreover with regard to the boundary between gas I and gas II the condition $p = p_2$ must be fulfilled.

The equations can be solved numerically. With the data:

(gas I) $\varrho_0 = 100.10^{-24}$; $T_0 = 10^4$; $p_0 = 83{,}2.10^{-12}$; $c_0 = 1{,}18.10^6$;
(gas II) $\varrho_1 = 3.10^{-24}$; $T_1 = 10^4$; $p_1 = 2{,}5.10^{-12}$; $c_1 = 1{,}18.10^6$,

[4]) See the paper mentioned in footnote [3]), p. 477, eqs. (9) and (A).

we obtain:

$$\eta = 0{,}42.10^6;\ u = V = 1{,}20.10^6;\ \xi = 2{,}22.10^6,$$

and for the state of gas I in the region of constant velocity $u = V$:

$$p = 10{,}5.10^{-12};\ \varrho = 29.10^{-24};\ T = 4370°;\ c = -0{,}78.10^6,$$

while for gas II in the region behind the shock wave:

$$p_2 = 10{,}5.10^{-12};\ \varrho_2 = 6{,}5.10^{-24};\ T_2 = 19360°.$$

Owing to the compression the rarer gas takes a much higher temperature than the expanding denser gas. The region of this high temperature increases in width at the rate $\xi - V = 1{,}02 \cdot 10^6$ cm/sec (10,2 km/sec).

With regard to the shock wave front it must be remarked that when account is taken of viscosity and heat conduction, it is found that the sudden change of state in reality is a continuous one, although with very steep gradients. The thickness of the transition layer is of the order $\mu/\varrho V$, where μ is the viscosity [5]). Assuming the viscosity to be given by $\mu = \frac{1}{3}\ \varrho\, l\, c_m$, with $l = 4{,}3 \cdot 10^{-9}/\varrho$, we find, for $T = 10000°$, $c_m = 1{,}58 \cdot 10^6$ cm/sec: $\mu \cong 0{,}0023$ [6]). The thickness of the layer becomes of the order of magnitude of l. This generally will be inappreciable from the observational point of view. In the transition region the Maxwellian velocity distribution will not hold; a kinetic treatment of the phenomena in this region would be necessary, which, however, will be extremely difficult.

4. *Collision of two clouds.* — The examples given in the preceding sections will appear artificial, and the reader may ask how the state assumed for $t = 0$ can have arisen. This, however, is a reflection of our general lack of knowledge about the fields of motion in the interstellar gas, which makes it necessary to start from imagined cases. The examples were meant to illustrate that when regions of highly different densities occur side by side, expansion and compression phenomena certainly will be present; their order of magnitude will be apparent from the results obtained.

Clouds of a much higher density than that of the average surrounding gas can be produced in star explosions. Such clouds in the first instants (the first few years or perhaps decades of their existence) will move with enormous velocities, compared to which the velocity of expansion often

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[5]) Compare *e.g.*: R. BECKER, Stosswelle und Detonation, Zeitschr. f. Physik **8**, p. 339, 1922; G. I. TAYLOR and J. W. MACCOLL, The mechanics of compressible fluids, section **6** (W. F. DURAND, Aerodynamic Theory, Berlin 1935, vol. III, Div. H, p. 218).

[6]) The value of μ also can be calculated with the aid of SUTHERLAND's formula: $\mu = \mu_0 \sqrt{T/273} \cdot (1 + C/273)/(1 + C/T)$. Here μ_0 (at $T = 273°$) is proportional to $\sqrt{m}/\sigma^2$. For H_2 ($m = 2$; $\sigma = 2{,}730 \cdot 10^{-8}$ cm) $\mu_0 = 0{,}000085$, $C = 72$; for He ($m = 4$; $\sigma = 2{,}174 \cdot 10^{-8}$ cm) $\mu_0 = 0{,}000189$, $C = 80$. Estimating for H_1 ($m = 1$; $\sigma = 1{,}1 \cdot 10^{-8}$ cm) $\mu_0 = 0{,}00037$, $C = 76$, the formula gives $\mu = 0{,}0029$ at $T = 10000°$, which is slightly larger than the value found above.

may appear negligible. In certain cases such clouds perhaps may impinge upon each other; or they may impinge upon a portion of the interstellar gas with more than minimum density; also the gradual loss of velocity experienced in moving through the interstellar gas deserves attention. Certain problems illustrating these possibilities will be treated now.

The simplest case is that of a "head on" collision of two clouds, having both the same density and temperature, with parallel boundary planes. In each cloud a shock wave will appear. When we introduce a system of coordinates with respect to which the two clouds have equal and opposite velocities of absolute magnitude V, the situation is a symmetrical one (comp. fig. 2). The two shock waves move outwards with the absolute velocity ξ; between them the gases are at rest. Indicating the original state of each gas by p_0, ϱ_0, T_0; and the compressed state between the shock waves by p', ϱ', T', we have [7]):

$$\left.\begin{aligned} V+\xi &= \frac{k+1}{4}V+\sqrt{c_0^2+\frac{(k+1)^2V^2}{16}} \\ p' &= p_0+\varrho_0 V(V+\xi);\ \varrho' = \varrho_0(V+\xi)/\xi \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

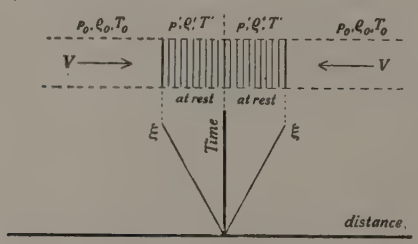

Fig. 2.

When V is large compared with c_0, we may use the approximations:

$$\xi \cong \tfrac{1}{2}(k-1)V;\ p' \cong \tfrac{1}{2}(k+1)\varrho_0 V^2;\ \varrho' \cong (k+1)\varrho_0/(k-1) \quad (14a)$$

from which:

$$RT' \cong \tfrac{1}{2}(k-1)V^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

By way of example we take $\varrho_0 = 10^{-22}$; $T_0 = 10000°$; $V = 10^7$ cm/sec (100 km/sec). With $k = 5/3$ as before we find: $\xi \cong V/3$; $p' \cong 1{,}33 \cdot 10^{-8}$; $\varrho' \cong 4\,\varrho_0$; $T' \cong 400.000°$.

In the more general case indicated in fig. 3, where gas I, moving with the velocity V_1, overtakes gas II (velocity V_2), the solution still can be obtained by means of straightforward algebra. In the case of large

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[7]) These formulae follow from eqs. (9), (*A*), (*B*) of the paper mentioned in footnote [3]), if it is observed that relatively to the uncompressed gas the shock wave moves with the velocity $V+\xi$ and the compressed gas with the velocity V.

velocities the following approximations are obtained, where $\varepsilon = \sqrt{\varrho_2/\varrho_1}$ (supposed to be < 1):

$$\left.\begin{aligned} &p' = p'' \cong \tfrac{1}{2}(k+1)\,\varrho_2\,(V_1 - V_2)^2/(1+\varepsilon)^2 \\ &\qquad \varrho' \cong (k+1)\,\varrho_1/(k-1) \quad ; \quad \varrho'' \cong (k+1)\,\varrho_2/(k-1) \\ &RT' \cong \tfrac{1}{2}(k-1)\,\varepsilon^2 (V_1 - V_2)^2/(1+\varepsilon)^2 ; \; RT'' \cong \tfrac{1}{2}(k-1)(V_1 - V_2)^2/(1+\varepsilon)^2 \end{aligned}\right\} \quad (16)$$

so that $T''/T' \cong \varrho_1/\varrho_2$. — Particular cases, *e.g.* $V_2 = 0$, can be deduced from these formulae. It will be seen that the temperature of the rarer gas increases to a much higher degree than that of the denser gas. The thickness of the zone of high temperature in gas II increases at the rate: $\xi_2 - V^* \cong \tfrac{1}{2}(k-1)\,(V_1 - V_2)/(1+\varepsilon)$.

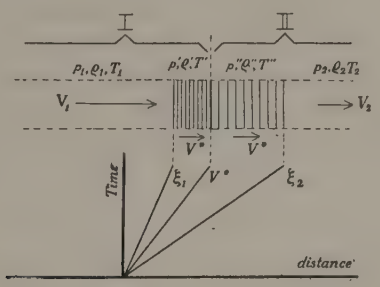

Fig. 3.

[It must be observed that with the very high temperatures and consequent high molecular velocities obtained in collisions between clouds with velocities of 50 km/sec and more, dissociation (ionization) of the hydrogen atoms must be taken into account. In all collisions with sufficiently high relative velocity of the colliding atoms, there will be a loss of energy in consequence of excitation or of the production of ions, which energy afterwards is radiated out in the form of light quanta. There is thus a continuous loss of energy in the compressed gas, at a rate which depends upon the number of favourable collisions. In those cases where this loss of energy is considerable, the temperature of the gas behind the shock wave will decrease, and a temperature gradient will be set up, which will influence the density and the pressure.] [8])

[8]) In the case of the Nebula south of ζ Orionis (IC 434) the impression is gained of a dark cloud of greater density penetrating into a luminous cloud of smaller density, while bright fringes appear just along the boundary. These fringes are rather sharply limited on the side of the dark cloud (in particular around the "Dark Bay"), while they shade off gradually towards the side of the smaller density. Similar features are observed in other cases. With the "Crab Nebula" in Taurus a thin luminous layer seems to precede at some distance the expanding central luminous mass. The spectrum of this thin layer shows many "forbidden" lines, which makes it probable that excitation here is due to collisions in a gas of not too small density.

When an explanation of these features is asked for, the picture developed in the text is too simple and presents the difficulty that it leads to clear cut frontiers limiting the

5. *Cloud with high velocity impinging upon gas at rest.* — We will consider what will happen when a cloud moving with a very high velocity, after having passed through a region of extremely low and negligible density, suddenly impinges upon a gas at rest of sufficient density to make itself felt. Such a case according to OORT seems to be present with a part of the cloud produced in the explosion of Nova Persei in 1901. It is estimated that this cloud moves with a velocity $V_1 = 1200$ km/sec $=$ $= 1{,}2 \cdot 10^8$ cm/sec.

During the first instants after the collision of the cloud with interstellar gas at rest of appreciable density, the solution appropriate to the case is the same as the one considered in the second part of section 4, with $V_2 = 0$. In the interstellar gas a shock wave appears, moving with the velocity: $\xi_2 = \frac{1}{2}(k+1) V_1/(1+\varepsilon) = \frac{4}{3} V_1/(1+\varepsilon)$, while the boundary separating the cloud from the interstellar gas moves with the velocity: $V^* = V_1/(1+\varepsilon)$. Hence the region of compressed interstellar gas broadens at the rate: $\xi_2 - V^* = \frac{1}{2}(k-1) V_1/(1+\varepsilon) = \frac{1}{3} V_1/(1+\varepsilon)$. The temperature in this region, calculated without taking account of radiation, would rise to the value: $T'' = \frac{1}{2}(k-1)(V_1^2/R)/(1+\varepsilon)^2 = 5{,}77 \cdot 10^7/(1+\varepsilon)^2$. — The shock wave in the cloud has the forward velocity

$$\xi_1 = V_1 - \tfrac{1}{2}(k+1)\,\varepsilon\, V_1/(1+\varepsilon),$$

and thus relatively to the material of the cloud moves backward with the velocity:

$$V_1 - \xi_1 = \tfrac{1}{2}(k+1)\,\varepsilon\, V_1/(1+\varepsilon) = \tfrac{4}{3}\,\varepsilon\, V_1/(1+\varepsilon) \quad . \quad . \quad . \quad (22)$$

When the cloud has a finite thickness L and is assumed to have constant

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regions of compression and of increased temperature. However, it has been mentioned already that heat conduction and viscosity turn shock waves into continuous transition regions; at the boundary where the denser and the rarer gas meet with equal pressures and mass velocities (but with different temperatures), also diffusion will play a part. Now in those cases where on one side we have a high density combined with low temperature and on the other side a low density with higher temperature, it is possible that somewhere in the transition region optimal conditions for the emission of radiation will be found. This perhaps could give an explanation of the appearance of narrow bright fringes. In those cases where a shock wave would bring a theoretical increase of temperature of 100.000° or more, optimal conditions for radiation might be found possibly not in the region of high temperature, but in a sheet of the transition layer with some intermediate temperature.

It must be observed nevertheless that as soon as kinetic energy in collisions of atoms is lost in consequence of excitation and ionization (leading to emission of radiant energy) the phenomena become much more complicated. It is possible that owing to this loss of energy no constant condition is obtained behind the shock wave; in certain cases perhaps even the shock wave itself may be absent. Before a satisfactory description can be given of the relations to be expected in such cases and the problem of sharp and diffuse boundaries can be attacked, a more penetrating investigation therefore will be necessary, based upon equations in which due account is taken of radiation and diffusion phenomena, etc.

values of p, ϱ, T throughout, this shock wave will reach the back of the cloud after the interval $\tau = L/(V_1 - \xi_1)$. When tentatively we estimate the thickness of the Nova Persei cloud at $L = 0{,}5'' = 4{,}1 \cdot 10^{15}$ cm (assuming a distance of 550 parsecs $= 1{,}7 \cdot 10^{21}$ cm), then with $V_1 = 1{,}2 \cdot 10^8$, $\varrho_2/\varrho_1 = 0{,}01$, $\varepsilon = 0{,}1$ we find $V_1 - \xi_1 = 1{,}45 \cdot 10^7$ and $\tau = 2{,}83 \cdot 10^8$ sec $= 9$ years.

At the end of this interval the whole cloud has undergone a compression; it moves as a whole with the velocity $V^* = 1{,}09 \cdot 10^8$ and has obtained the temperature T'. Its thickness has been reduced to: $L - \tau(V_1 - V^*) = (k-1)\,L/(k+1) = 1{,}03 \cdot 10^{15}$, that is in the inverse ratio of the increase in density (ϱ'/ϱ_1).

At the back of the cloud the shock wave is reflected as an expansion wave. The front of this wave has the velocity $c'' = \sqrt{k R T''} = 8{,}1 \cdot 10^7$ relatively to the gas, and thus after a further interval of time of approximately 0,4 year reaches the front boundary of the cloud. As soon as the expansion wave has reached the front of the cloud, the velocity and the pressure will decrease here, so that also the compression produced in the interstellar gas will become less. From now onward the whole phenomenon assumes a rather complicated form [9]). To work it out in details does not seem promising in view of the uncertainties of the available data. In particular the assumption of a homogeneous state in the original cloud is an oversimplification of the case.

In view of this situation OORT put forward the question whether a stationary solution could be found, describing a case where owing to the general slowing down of the cloud a "barometric" gradient of the pressure and the density is set up. OORT supposes that such cases may be found in the very thin cloud strips, observed in NGC 6960 in Cygnus. An approximate solution of this type will be constructed in the next section.

(To be continued.)

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[9]) Certain phenomena of this nature have been investigated in the paper mentioned in footnote [3]).

Aerodynamics. — *Some problems of the motion of interstellar gas clouds.* II *). By J. M. BURGERS. (Mededeling No. 48 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of May 25, 1946.)

6. *Quasi-stationary solution with "barometric" pressure gradient.* — Our starting point again will be eqs. (1) — (4). In order to be able to take a formal account of radiation loss, of ψ erg per cm^3 and per sec, we multiply eq. (3) by $R/(k-1)$ and to the right hand side add the term $-\psi/\varrho$, so that it takes the form:

$$\frac{R}{k-1}\frac{DT}{Dt}=\frac{R}{k-1}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=-RT\frac{\partial u}{\partial x}-\frac{\psi}{\varrho} \quad . \quad . \quad . \quad (3a)$$

The most convenient way for constructing an appropriate solution is to start from the *streamlines,* for which tentatively the following equation is assumed:

$$x=\varphi_0(t)-s\,\varphi_1(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

A streamline is defined by $s =$ constant. We take $s = 0$ as the front of the moving gas layer and restrict s to positive values; further at $t = 0$ (present epoch) we take: $\varphi_1(0) = 1$. Hence the motion of the front is defined by: $x = \varphi_0(t)$, and s measures the distance behind the front at $t = 0$. With constant t we have for any function of x and t:

$$\partial/\partial x=-(1/\varphi_1)\,.\,\partial/\partial s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From (17) we obtain for the velocity:

$$u=(dx/dt)_{\text{for constant } s}=\dot{\varphi}_0-s\dot{\varphi}_1 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(using dots to denote derivatives with respect to t, while an accent will be used for derivatives of functions of s). This gives: $\partial u/\partial x = \dot{\varphi}_1/\varphi_1$.

For the density the following function is taken:

$$\varrho=\varrho^*(s)/\varphi_1(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which satisfies the equation of continuity. Making use of (4), eq. (1) then gives:

$$\ddot{\varphi}_0-s\ddot{\varphi}_1=\frac{\varrho^{*\prime}}{\varrho^*}\frac{RT}{\varphi_1}+\frac{R}{\varphi_1}\frac{\partial T}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Tentatively for ϱ^* we assume the formula:

$$\varrho^*=\varrho_m\,e^{-\beta s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

*) Part I has appeared in these Proceedings **49** (1946), p. 589.

which gives the desired exponential decrease of the density with s. The coefficient ϱ_m determines the maximum value (ϱ_m/φ_1) of the density at the front of the advancing layer. Equation (21) now can be considered as a differential equation for T as a function of s; its integral is [10]):

$$RT = -\ddot{\varphi}_0\,\varphi_1/\beta + (1+\beta s)\,\ddot{\varphi}_1\,\varphi_1/\beta^2 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The pressure at the front of the advancing layer: $p_{fr} = R\,T_{fr}\,\varrho_{fr} = = -\varrho_m(\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2)$, must be equal to the pressure generated in the interstellar gas into which our layer is penetrating. We will not enter into a detailed investigation of the shock waves and other phenomena set up in this gas, but assume that the "impact pressure" is given by $\frac{1}{2}(k+1)\,\varrho_0\,u_{fr}{}^2$, where ϱ_0 is the original density of the interstellar gas, while u_{fr}, the front velocity of the advancing layer, is equal to $\dot{\varphi}_0$. Hence we obtain:

$$\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2 = -\tfrac{1}{2}(k+1)\,(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Introduction of (23) into (3a) finally gives, after multiplication by $-(k-1)$:

$$\frac{\ddot{\varphi}_0\,\varphi_1}{\beta}\left(\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + k\,\frac{\dot{\varphi}_1}{\varphi_1}\right) - \frac{(1+\beta s)\,\ddot{\varphi}_1\,\varphi_1}{\beta^2}\left(\frac{\dddot{\varphi}_1}{\ddot{\varphi}_1} + k\,\frac{\dot{\varphi}_1}{\varphi_1}\right) = (k-1)\,\frac{\psi}{\varrho} \quad . \quad (25)$$

In general ψ/ϱ will be a complicated function of s, and it will not be possible to satisfy (25) in an exact way, which shows that the assumption (22) apparently was too restricted. We therefore start by investigating the case $\psi = 0$, and provisionally assume $\ddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$. Then to a first approximation (24) gives:

$$\ddot{\varphi}_0/\beta \cong -\tfrac{1}{2}(k+1)\,(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

from which:

$$u_{fr} = \dot{\varphi}_0 = \frac{2\,\varrho_m}{(k+1)\,\varrho_0}\,\frac{1}{\beta\,(t+t_0)} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

t_0 being an integration constant. At the same time eq. (25) reduces to:

$$k\,\dot{\varphi}_1/\varphi_1 \cong -\ddot{\varphi}_0/\ddot{\varphi}_0 = 2/(t+t_0) \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

from which, taking account of the condition $\varphi_1(0) = 1$:

$$\varphi_1 = \{(t+t_0)/t_0\}^{2/k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

With the same degree of approximation (23) becomes:

$$RT \cong -\frac{\ddot{\varphi}_0\,\varphi_1}{\beta} = \frac{2\,\varrho_m}{(k+1)\,\varrho_0}\,\frac{1}{\beta^2\,t_0^2}\left(\frac{t_0}{t+t_0}\right)^{2-2/k} \quad . \quad . \quad . \quad . \quad (28)$$

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[10]) A term proportional to $e^{\beta s}$ might be added to this solution, which term could be written $\varphi_1\varphi_2\,e^{\beta s}$, where φ_2 is an arbitrary function of t. This term, however, has been discarded. The discrepancy observed in eq. (25) cannot be removed by introducing such a term. — As will be seen below we suppose that the second term of (23) is of minor importance.

As the function φ_1 determines a gradual expansion of the moving gas layer, (28) expresses the adiabatic decrease of T consequent upon this expansion, which is obtained as radiation is neglected.

7. Continuation. — Numerical data for a thin strip or sheet of cloud in NGC 6960. — Influence of radiation loss. — From data supplied by OORT (in view of their vagueness slightly adjusted so as to obtain a satisfactory fit in calculating the temperature) we take: $u_{fr} = 7{,}0 \,.\, 10^6$ (70 km/sec) at the present epoch ($t = 0$); $\varrho_m = 5{,}2 \,.\, 10^{-23}$ (31,5 H-atoms per cm^3); $\varrho_0 = 10^{-24}$. Then (26) gives: $u_{fr} = 39/\beta(t + t_0)$, so that for $t = 0$: $\beta t_0 = 5{,}57 \,.\, 10^{-6}$. The timescale evidently is proportional to the linear scale, which according to (22) is fixed by $1/\beta$. Taking $1/\beta$ equal to $2{,}5 \,.\, 10^{16}$ cm (the estimated visual thickness of a sheet is about 5″), we find $\beta = 4{,}0 \,.\, 10^{-17}$, giving: $t_0 = 1{,}39 \,.\, 10^{11}$ sec $= 4410$ year.

From (26) and (27) we now calculate for $t = 0$: $\ddot{\varphi}_0 = -5{,}03 \,.\, 10^{-5}$; $\dddot{\varphi}_1 = +1{,}24 \,.\, 10^{-23}$; $\dddot{\varphi}_1/\beta = 3{,}1 \,.\, 10^{-7}$, so that the assumption $\dddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$ appears to be satisfied. The expansion determined by φ_1 is very gradual; with $t = 4410$ year we have $\varphi_1 = 2{,}3$.

Further from (28) we deduce, again for $t = 0 : T = 15100°$, which is in accordance with the astronomical estimate.

We now turn back to the radiation loss. From data supplied by OORT it follows that in the present problem the radiation loss cannot be neglected. We might try to obtain a better approximation than that given by (25*a*) by taking for ψ/ϱ a function of the time alone, calculated with an average value of the density, replacing (25) by:

$$k \frac{\dot{\varphi}_1}{\varphi_1} = -\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + \frac{(k-1)\,\beta}{\ddot{\varphi}_0 \varphi_1} \frac{\psi}{\varrho} . \quad . \quad . \quad . \quad . \quad . \quad (25b)$$

Without integrating this equation it will be evident that the radiation loss tends to decrease the rate of expansion and even may turn it into a contraction. The assumption $\dddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$ does not seem to be impaired by the correction. As the temperature is given by $R\,T \cong -\ddot{\varphi}_0\varphi_1/\beta$, the decrease of T with time becomes faster than that given previously by (28).

Although the solution is incomplete, the following conclusions seem possible. The fact that the astronomical data for u_{fr}, ϱ_m, ϱ_0 lead to the correct value for T, can be taken as an indication that the assumption concerning the "impact pressure" experienced in moving through the interstellar gas is not far from the truth. The result concerning φ_1 gives an acceptable explanation of the mechanism by which the very thin sheets of cloud for long times can retain their elegant appearance. The estimate for the timescale depends directly upon the assumption concerning the impact pressure. It appears rather short when considered in relation to the distance between the two nebulae NGC 6960 and 6992 ($155' = 48{,}7 \,.\, 10^{18}$ cm), which are believed to have originated from a single explosion and thus should have travelled each about $24{,}4 \,.\, 10^{18}$ cm. As formula (26) cannot

be applied to velocities approaching that of light, we may ask what would be the distance covered in the interval of time evolved since u_{fr} decreased from 10.000 km/sec (10^9 cm/sec; $t + t_0 = 30{,}9$ year) until its present value of 70 km/sec ($7 \cdot 10^6$ cm/sec; $t + t_0 = 4410$ year). This distance amounts to $4{,}83 \cdot 10^{18}$ cm, which is less than $^1/_5$ of the total amount travelled since the explosion (to cover the remaining $19{,}6 \cdot 10^{18}$ cm at 10^9 cm/sec would require ca. 620 year). Evidently eq. (26) must be considered as an approximation valid for the present epoch only, and the decrease of velocity with time originally must have taken place at a slower rate [11]).

8. *Possibility of surface waves on interstellar clouds.* — Thus far we have restricted to the consideration of motion in one dimension only. It is evident that the real motions and currents in interstellar space will be of a much more complicated character. A comparison with atmospheric movements in many cases will have impressed itself on the mind of the observer. In particular the question has been brought forward whether certain types of structure, resembling cirrus clouds in the earth's atmosphere, may indicate the presence of waves or perhaps of vortices, such as can arise owing to the instability of the motion of fluid layers sliding over each other with different velocities. As an instance of such structures the clouds around the Pleiades must be mentioned; in particular the cloud around the star Merope shows a marked periodic structure. In certain parts of these clouds, *e.g.* in those around Alcyone and between Maia and Merope, several systems of periodic structure seem to be present, crossing each other at rather large angles. It is probable that the illuminated patches all belong to one single cloud, extending at least over about $1^\circ = 5{,}4 \cdot 10^{18}$ cm, and that the periodic structures in the various patches are connected with each other.

When it is attempted to explain such structures by assuming a wave

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11) The relations from which the timescale has been deduced, can be put in the form:

$$p_{fr} = R\,T_{fr}\,\varrho_{fr} = \tfrac{1}{2}(k+1)\,\varrho_0\,u_{fr}^2 = -M_1\,(d\,u_{fr}/dt),$$

where $M_1 = \varrho_{fr} \cdot L$ (M_1 being the mass of the whole layer per unit area and L its effective thickness). This equation gives: $d\,u_{fr}/dt = -R\,T_{fr}/L$, and thus fixes the present timescale from very simple data.

With the formulae of the text: $\varrho_{fr} = \varrho_m/\varphi_1$; $L = \varphi_1/\beta$. The assumption of a smaller value of β (leading to thinner sheets of cloud) reduces both the timescale and the distance covered. Larger values of β are in contradiction with observation. The assumption of a substantially smaller value of ϱ_0, which would lead to a greater value of $\beta\,t_0$, would change the value of the temperature in the same ratio, and thus seems unlikely. It may be thought that perhaps originally the clouds have moved through a portion of the interstellar gas of smaller density, but the problem then presents itself whether both nebulae have suffered the same adventure.

No account has been taken of the possibility of lateral motions (a divergent motion in the plane of the sheet might produce a continuous reduction of thickness), but in view of the elementary character of the relation between temperature, density and impact pressure, such motions are not likely to lead to an appreciably different timescale.

formation of similar nature as observed with certain atmospheric clouds. several difficulties present themselves. The wave motion in the atmosphere, arising between layers of air sliding over each other, is controlled by gravity. Furthermore in atmospheric clouds the wave pattern becomes visible in consequence of condensation of water vapour in those elements of volume which have suffered a reduction in temperature through their upward displacement. In the case of the interstellar gas an increased condensation of dust particles perhaps may be connected with a local increase in density and thus could make visible periodic changes of the density.

As to gravity, local gravitational fields due to separate clouds, as have been considered in section **1**, may bring with them values of g of the order of $1{,}67 \cdot 10^{-9}$. An estimate of the general gravitational field of the Galaxy can be obtained by taking the mass inside a sphere described with the distance from the centre of the Galaxy to the sun as radius (ca. $3 \cdot 10^{22}$ cm) equal to $2{,}4 \cdot 10^{11}$ times the mass of the sun, giving $M_{\text{Gal}} = 4{,}7 \cdot 10^{44}$ gr, which makes the value of g in our neighbourhood $3{,}5 \cdot 10^{-8}$. The general gravitational field, however, will be compensated by the centripetal acceleration of the clouds, so long as these move freely. Residual effects might be found when clouds collide with each other and produce abnormal accelerations in their surface layers. These accelerations even may exceed considerably the value of g calculated from the gravitational attraction: in the example treated in the preceding section the acceleration in the thin sheet is $du/dt \doteq -u/(t + t_0) = -5 \cdot 10^{-5}$ at the present epoch. It is not easy to decide whether fields produced in this way can regulate the appearance of wave motions, supposing that the colliding masses at the same time should have a tangential velocity with respect to each other. As one of the two colliding masses will suffer a deceleration and the other an acceleration, the equivalent g-vectors will have opposite directions, pointing towards the surface of separation. This produces a situation different from that found in ordinary wave motion, which appears in stratified material where gravity everywhere acts in the same direction, pointing from the region of lower density to that of higher density.

A comparison between interstellar and atmospheric motions might be made by applying the theory of similarity, according to which similar motions can exist when in the first place there is geometrical similarity in pattern of motion and in density distribution, while in the second place REYNOLDS' number $Re = \varrho U\lambda/\mu$, FROUDE's number $Fr = U^2/g\lambda$ and MACH's number $Ma = U/c$ must have the same values in the cases to be compared (U: a characteristic value of the velocities to be considered; λ: wavelength; ϱ: density, for the interstellar gas approx. 10^{-22}, for air at the height of say 4 km 0,0008; μ: viscosity, for the interstellar gas approx. 0,0023, for air 0,00017; c: velocity of sound). All ordinary cases of wave motion are such in which Ma is far below unity and then is of no importance. We therefore take FROUDE's number first, and consider a case of stationary wave motion along the surface of separation between two layers

of material, having the densities ϱ, ϱ', the surface of separation being perpendicular to the local gravity vector with density increasing in the direction to which the vector points, the layers sliding over each other with velocities U, U' (measured with respect to the wave pattern). Then the wavelength must satisfy the relation [12]):

$$(\varrho - \varrho')\, g\, \lambda / 2\pi = \varrho\, U^2 + \varrho'\, U'^2 . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

provided λ is small compared with the thickness of the layers. In this case $Fr = (\varrho - \varrho')/2\pi\varrho - \varrho' U'^2/\varrho g\lambda$, from which appears $Fr < 1/2\pi$. Hence in order to obtain a wavelength of $3 \cdot 10^{16}$ cm as found in the Merope cloud, we must have $U^2/g < 5 \cdot 10^{15}$, which would require either rather small values of the velocity (*e.g.*: $U = 2 \cdot 10^3$ with g of the order 10^{-9}) or very high values of g, of the order 10^{-7} and more, provided of course we have to do with stationary (stable) wave motion.

With larger values of U^2/g the wave motion is unstable and in course of time increases in amplitude. This in itself is not impossible, but when U^2/g considerably exceeds the value mentioned the observed wavelength must be determined by some other cause; one might think of the thickness of some intermediate layer. Problems referring to wave motion in stratified systems have been treated by RAYLEIGH, TAYLOR, and others [13]). But as there is no direct clue to the case which should be chosen as a basis for comparison and as we miss any trustworthy datum about the value of g, no promising way for the construction of an appropriate solution as yet is to be seen.

It might be that we should discard any reference to the influence of gravity, and should look exclusively to features of the velocity distribution in order to find an explanation for the observed value of λ. The question arises if viscosity can play a part. If we assume a velocity U of $5 \cdot 10^5$ cm/sec, the value of REYNOLDS' number $\varrho U\lambda/\mu$ comes out as 650. This appears rather small when compared with the results of some theoretical investigations on the stability of laminar motion when viscosity is effective, which would make us to expect a value above 15000 [14]). The supposition of a much higher value of U, however, would bring us to velocities above

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[12]) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 377 (Art. 234, form. 5).

[13]) RAYLEIGH, Theory of Sound (London 1945), Vol. II, Ch. XXI (p. 376 seqq.). — G. I. TAYLOR, Effect of variation in density on the stability of superposed streams of fluid, Proc. Roy. Soc. (London) A **132**, p. 499, 1931. — See also: V. BJERKNES, J. BJERKNES, H. SOLBERG und T. BERGERON, Physikalische Hydrodynamik (Berlin 1933). Kap. VIII, IX u. X (pp. 305—421).

[14]) See H. SCHLICHTING, Zur Entstehung der Turbulenz bei der Plattenströmung. Göttinger Nachrichten Math.-physik. Kl., 1933 (II, no. 38), p. 181. From fig. 3 (p. 197) the minimum value of $U_m \delta^*/\nu$ is read off as 575 (comp. p. 202; $\nu = \mu/\varrho$) with $\alpha\delta^* = 0{,}23$, where $\alpha = 2\pi/\lambda$; this gives $U_m \lambda/\nu = 15700$. The maximum value of $\alpha\delta^*$ is about 0,28 with $U_m \delta^*/\nu =$ ca. 860; $U_m \lambda/\nu = 19300$. In both cases the waves are just on the limit between stability and instability; experimental observations point to the appearance of waves well inside the domain of instability, with still greater values of $U_m \lambda/\nu$.

the speed of sound, in which case the problem takes a wholly different character. Not much is known concerning the formation of waves in this case [15]). Further investigations are necessary before the question raised in this section can be settled.

9. *Considerations on the motion of the interstellar gas in the Galaxy as a whole.* — OORT has raised the problem whether the erratic motions and forms of the interstellar clouds may be due to effects depending upon the variation of galactic rotation with the distance from the centre of the Galaxy. It may be supposed that rotational motion with a regular velocity distribution ("laminar" rotation) would prove to be unstable, and that a form of turbulence should set in. In this connection one may first make an estimate of the magnitude of REYNOLDS' number for the rotation of the gas in the Galaxy. Assuming a radius of 50000 light years $= 5 . 10^{22}$ cm, a circumferential velocity of 260 km/sec $= 2{,}6 . 10^{7}$ cm/sec, $\varrho = 3 . 10^{-24}$ and $\mu = 0{,}0023$, we obtain: $Re = 1{,}7 . 10^{9}$. This value is high, though not excessive. In experimental situations (rotating basin filled with water) values of $8 . 10^{6}$ probably can be reached; in atmospheric cyclones with a radius of (often far) over 100 km values of the order 10^{9} certainly occur. The boundary conditions in the latter cases, however, are wholly different from those existing in the Galaxy, and this prevents a direct comparison.

It is extremely difficult to investigate the stability of laminar rotation, in particular as account must be taken of changes in density and temperature, while even the changes in the gravitational field may be of some importance. According to JEANS gravitational instability in a homogeneous gas may arise for disturbances of a period λ_g exceeding a certain limit of the order $c\sqrt{\pi/\gamma\varrho}$ (c: velocity of sound; γ: gravitation constant $= 6{,}67 . 10^{-8}$; ϱ_0: original density) [16]). With $T = 10.000°$, $c = 1{,}18 . 10^{6}$, $\varrho_0 = 3 . 10^{-24}$ this limit becomes $4{,}7 . 10^{21}$ cm. In view of the dimensions of the Galaxy such a form of instability consequently may play its part. Its timescale is of the order $\lambda_g/c = 4{,}0 . 10^{15}$ sec $= 1{,}3 . 10^{8}$ year. As the distances between the interstellar clouds and their linear dimensions are much smaller than the value of λ_g just mentioned, other causes must be operative as well, and these evidently should be found in the rotation.

We cannot attack this problem at present, but a few remarks may be added. A laminar rotation of the interstellar gas as a whole would require a nearly complete balance between gravitational attraction and centrifugal force, as otherwise extremely large pressure gradients would be necessary. Estimating the value of g in our neighbourhood as $\gamma M_{Gal}/r^2 = 3{,}13 . 10^{37}/r^2$, we find $U = 5{,}6 . 10^{18}/\sqrt{r}$. Viscous friction in

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15) See for some provisional results (which rather would point to a greater stability of surfaces of separation with velocities exceeding that of sound): J. ACKERET, Ueber Luftkräfte bei sehr grossen Geschwindigkeiten, Helv. Phys. Acta **1**, p. 301, 1928.

16) J. H. JEANS, Astronomy and Cosmogony (Cambridge, 1928), p. 340.

this case will produce a resultant force per unit volume of magnitude: $-0{,}75\,\mu U/r^2$, which at the sun's distance from the centre of the Galaxy ($r = 3 \cdot 10^{22}$) amounts to $1{,}92 \cdot 10^{-48}\,U$. With the density $3 \cdot 10^{-24}$ this would lead to an acceleration of $-6{,}4 \cdot 10^{-25}\,U$. The timescale thus arrived at is so enormous that we must conclude that viscous friction is inefficient to regularize the rotation.

A force will also be produced by the resistance of the stars. Taking as example the sun, and writing U_{rel} for its motion relatively to the gas, we find, curiously enough, that REYNOLDS' number for this relative motion $\varrho\, U_{rel}\, R_{sun}/\mu$ (with $R_{sun} = 7 \cdot 10^{10}$ cm) even for $U_{rel} = 300$ km/sec $= 3 \cdot 10^{7}$ cm/sec comes out as $2{,}74 \cdot 10^{-3}$, i.e. far below unity. Hence we must apply STOKES' law of resistance. With the star density in the neighbourhood of the sun $N = 1/(3 \cdot 10^{56})$ the force per unit volume becomes: $6\pi\mu R_s\, U_{rel}\, N = 10^{-47}\, U_{rel}$. Although comparable with the internal viscous friction, this force just as little can play an important part.

Mathematics. — *On the theory of linear integral equations.* VI. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of May 25, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the papers I, II and IV, bearing the same title [1]). Furthermore we shall make use of V, Theorem 8, stating that a Hermitean kernel $A(x, y) \in L_2^{(2m)}$ is general if and only if $Af = \int_{\triangle} A(x, y) f(y)\, dy = 0$ implies $f(x) = 0$. In this paper we shall consider linear integral equations

$$\int_{\triangle} K(x, y) f(y)\, dy - \lambda f(x) = g(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the space $L_2^{(m)}(\triangle)$ with kernel

$$K(x, y) = \int_{\triangle} A(x, z) H(z, y)\, dz,$$

where both kernels $A(x, y)$ and $H(x, y)$ belong to the space $L_2^{(2m)}(\triangle)$ and are Hermitean, while moreover $H(x, y)$ is supposed to be positive (positive means here: of positive type). A. J. PELL [2]) has discussed the equation of this kind, and, denoting the characteristic values of (1) by $\lambda_i (i = 1, 2, \ldots)$, and a corresponding H-orthonormal system of characteristic functions by $\psi_i(x) (i = 1, 2, \ldots)$, she found under the assumptions that $A(x, y)$ is continuous, while $H(x, y)$ is bounded, the following main result:

If $a_i = \int_{\triangle \times \triangle} H(x, y) \overline{\psi_i(x)} f(y)\, dx\, dy$ for an arbitrary $f(x) \in L_2$, then

$$\int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x) + p(x),$$

uniformly in x, where $Hp = \int_{\triangle} H(x, y) p(y)\, dy = 0$.

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 194—204, 205—212 and 409—423 (1946).

[2]) A. J. PELL, Applications of biorthogonal systems of functions to the theory of integral equations, Transactions of the Am. Math. Soc. **12**, 165—180 (1911).

We shall prove, under the sole assumption that the Hermitean kernel $A(x, y)$ and the positive Hermitean kernel $H(x, y)$ belong to $L_2^{(2m)}$, that

$$K(x, y) - p(x, y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\chi_i(x) = \int_{\triangle} H(x, y) \psi_i(y) \, dy \qquad (i = 1, 2, \ldots)$$

and

$$\int_{\triangle} H(x, z) p(z, y) \, dz = 0 .$$

almost everywhere in $\triangle \times \triangle$. Furthermore

$$K_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \quad (n \geqq 2); \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\int_{\triangle} K_n(x, x) \, dx = \sum_i \lambda_i^n \qquad (n \geqq 2). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Moreover, writing $a_i = \int_{\triangle} f(x) \overline{\chi_i(x)} \, dx$ for an arbitrary $f(x) \in L_2$, we have

$$\int_{\triangle} K(x, y) f(y) \, dy \backsim \Sigma \lambda_i a_i \psi_i(x) + p(x), \quad . \quad . \quad . \quad . \quad (5)$$

$$\int_{\triangle} K_n(x, y) f(y) \, dy \backsim \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2), \quad . \quad . \quad . \quad (6)$$

where

$$Hp = \int_{\triangle} H(x, y) p(y) \, dy = 0.$$

Thereafter we shall find sufficient conditions under which the convergence in mean in (2), (3), (5) and (6) may be replaced by uniform convergence (it will be proved that the result of A. J. PELL holds under very general conditions (cf. Theorem 4)), and the functions $p(x, y)$ add $p(x)$ in (2) and (5) vanish.

§ 2. *The equation with kernel* $K(x, y) = \int_{\triangle} A(x, z) H(z, y) \, dz$.

Let

$$K(x, y) = \int_{\triangle} A(x, z) H(z, y) \, dz,$$

where $A(x, y)$ and $H(x, y)$ are Hermitean kernels, belonging to the space $L_2^{(2m)}(\triangle)$, while, moreover, $H(x, y)$ is positive (of positive type). Then the linear transformations A and H in the space $L_2^{(m)}(\triangle)$, defined by

$$Af = \int_{\triangle} A(x, y) f(y)\, dy,$$

$$Hf = \int_{\triangle} H(x, y) f(y)\, dy,$$

are completely continuous and self-adjoint, while H is, moreover, positive. The completely continuous transformation $K = AH$ is then determined by

$$Kf = AHf = \int_{\triangle\times\triangle} A(x, z) H(z, y) f(y)\, dz\, dy = \int_{\triangle} K(x, y) f(y)\, dy.$$

As we know, the transformation K is symmetrisable relative to H (since $HK = HAH$ is self-adjoint), and we observe that every $f(x) \in L_2$, satisfying $Hf = 0$, satisfies also $Kf = AHf = 0$. The kernel $K(x, y)$ is therefore what we have called in IV a Marty-kernel. Supposing that

$$\| H(x, y) \|_{2m}^2 = \int_{\triangle\times\triangle} |H(x, y)|^2\, dx\, dy \neq 0,$$

so that H is not identical with the nulltransformation O, the theorems proved in IV may therefore be applied to the equation (1). We shall not repeat them all here, and only pay attention to IV, Theorems 4, 6 and 11, since these may be replaced by stronger theorems. Instead of IV, Theorem 4, we have

Theorem 1. (*Expansion Theorem*). *Writing* $a_i = (f, \chi_i) = \int_{\triangle} f(x)\overline{\chi_i(x)}dx$ *for an arbitrary* $f(x) \in L_2$, *we have*

$$\int_{\triangle} K(x, y) f(y)\, dy \backsim \Sigma \lambda_i a_i \psi_i(x) + p(x), \quad . \quad . \quad . \quad . \quad (5)$$

$$\int_{\triangle} K_n(x, y) f(y)\, dy \backsim \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2), \quad . \quad . \quad . \quad (6)$$

where the function $p(x)$ *satisfies the relation*

$$Hp = \int_{\triangle} H(x, y) p(y)\, dy = 0$$

for almost every $x \in \triangle$.

Proof. Follows from I, Theorem 15.

Instead of IV, Theorem 6, we have

Theorem 2. *Let $\lambda \neq 0$, and let $g(x) \in L_2$ be H-orthogonal to all characteristic functions of* (1), *belonging to the characteristic value λ. (If λ is no characteristic value, $g(x)$ is therefore arbitrary). Then every solution of* (1) *satisfies a relation of the form*

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x) + q(x),$$

where $a_i = \int_{\triangle} g(x) \overline{\chi_i(x)}\, dx$ *for* $\lambda_i \neq \lambda$, $\int_{\triangle} H(x, y)\, q(y)\, dy = 0$ *for almost every* $x \in \triangle$, *and where* Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* $\psi_i(x)$ *has the value* $\int_{\triangle} f(x) \overline{\chi_i(x)}\, dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 17.

Instead of IV, Theorem 11 we have

Theorem 3. (*Expansion Theorem for the kernels*). *We have*

$$K(x, y) - p(x, y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad \ldots \quad (2)$$

where

$$\int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

almost everywhere in $\triangle \times \triangle$;

$$K_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqq 2); \ldots \quad (3)$$

$$\int_{\triangle} K_n(x, x)\, dx = \sum_i \lambda_i^n \qquad (n \geqq 2) \ldots \quad (4)$$

Proof. The formulae (2) and (3) will follow from IV, Theorem 9, if only we prove that $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ converges in mean. For this purpose we observe that the system $\Psi_i = H^{1/2} \psi_i$ is orthonormal, so that by BESSEL's inequality

$$\left.\begin{aligned} &\Sigma |(H\psi_i, f)|^2 = \Sigma |(\psi_i, Hf)|^2 = \Sigma |(Hf, \psi_i)|^2 = \\ &\Sigma |(H^{1/2} f, \Psi_i)|^2 \leqq \| H^{1/2} f \|^2 = (Hf, f) \leqq \| Hf \| \cdot \| f \| \leqq \| H \| \cdot \| f \|^2 \end{aligned}\right\}. \quad (7)$$

for any $f(z) \in L_2$. Taking now $f(z) = \overline{A(x, z)}$, this function belongs to L_2 for almost every $x \in \triangle$, so that

$$\lambda_i \psi_i(x) = \int_{\triangle} K(x, y) \psi_i(y)\, dy = \int_{\triangle} A(x, z) \left[\int_{\triangle} H(z, y) \psi_i(y)\, dy \right] dz = (H\psi_i, f)$$

for these values of x. Hence, by (7),

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqslant \|H\| \cdot \|f\|^2 = \|H\| \cdot \int_{\Delta} |A(x,z)|^2 \, dz, \quad . \quad , \quad (8)$$

so that

$$\Sigma \lambda_i^2 \int_{\Delta} |\psi_i(x)|^2 \, dx \leqslant \|H\| \cdot \int_{\Delta \times \Delta} |A(x,z)|^2 \, dx \, dz. \quad . \quad . \quad . \quad (9)$$

The system $\Psi_i(x) = H^{1/2} \psi_i$ being orthonormal, we have for arbitrary complex numbers a_i,

$$\|\sum_{i=p}^{q} a_i \chi_i(y)\|^2 = \|H^{1/2} \sum_{i=p}^{q} a_i \Psi_i(y)\|^2 \leqslant \|H^{1/2}\|^2 \cdot \|\sum_{i=p}^{q} a_i \Psi_i(y)\|^2 =$$

$$= \|H^{1/2}\|^2 \sum_{i=p}^{q} |a_i|^2,$$

or, taking $a_i = a_i(x) = \lambda_i \overline{\psi_i(x)}$, and integrating then over x,

$$\int_{\Delta \times \Delta} |\sum_{i=p}^{q} \lambda_i \psi_i(x) \overline{\chi_i(y)}|^2 \, dx \, dy \leqslant \|H^{1/2}\|^2 \sum_{i=p}^{q} \lambda_i^2 \int_{\Delta} |\psi_i(x)|^2 \, dx.$$

By (9), the expression on the right hand side tends to 0 as $p, q \to \infty$, the same is therefore true of the expression on the left hand side, which shows the convergence in mean of $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$.

From this convergence in mean of $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ follows now, by IV, Theorem 10, that

$$\int_{\Delta} K_2(x,x) \, dx - \int_{\Delta} r(x,x) \, dx = \Sigma \lambda_i^2,$$

$$\int_{\Delta} K_n(x,x) \, dx = \sum_i \lambda_i^n \quad (n \geqslant 3),$$

where

$$r(x,y) = \int_{\Delta} K(x,z) \, p(z,y) \, dz.$$

The formula (4) will be proved, therefore, if we can show that $r(x,x) = 0$ for almost every $x \in \Delta$. This follows from the fact that

$$\int_{\Delta} H(y,z) \, p(z,x) \, dz = 0$$

for almost every $x \in \triangle$, which implies

$$r(x,x) = \int_{\triangle} K(x,z)\, p(z,x)\, dz = \int_{\triangle \times \triangle} A(x,y)\, H(y,z)\, p(z,x)\, dy\, dz =$$

$$= \int_{\triangle} A(x,y) \left[\int_{\triangle} H(y,z)\, p(z,x)\, dz \right] dy = 0$$

for almost every $x \in \triangle$.

§ 3. *The case that $A(x,y)$ is continuous in mean.*

Definition. We shall say that the Hermitean kernel $A(x,y) \in L_2^{(2m)}$ is *continuous in mean* if

$$\int_{\triangle} |A(x,y)|^2\, dy = \int_{\triangle} |A(y,x)|^2\, dy$$

is finite for every $x \in \triangle$, and

$$\lim_{x_2 \to x_1} \int_{\triangle} |A(x_2,y) - A(x_1,y)|^2\, dy = 0$$

for $x_1, x_2 \in \triangle$.

It is immediately to be seen that whenever $A(x,y)$ is continuous in mean, the function $h(x) = \int_{\triangle} A(x,y)\, f(y)\, dy$ is continuous in $\triangle$ for every $f(x) \in L_2$, so that the same is true of

$$g(x) = \int_{\triangle} K(x,y)\, f(y)\, dy = \int_{\triangle} A(x,y) \left[\int_{\triangle} H(y,z)\, f(z)\, dz \right] dy$$

for every $f(x) \in L_2$. In particular the characteristic functions

$$\psi_i(x) = \lambda_i^{-1} \int_{\triangle} K(x,y)\, \psi_i(y)\, dy$$

are continuous in $\triangle$.

Theorem 4. (*Expansion Theorem*). *If $A(x,y)$ is continuous in mean, and $a_i = (f, \chi_i) = \int_{\triangle} f(x)\, \overline{\chi_i(x)}\, dx$ for an arbitrary $f(x) \in L_2$, we have*

$$\int_{\triangle} K(x,y)\, f(y)\, dy = \sum \lambda_i\, a_i\, \psi_i(x) + \tilde{p}(x),$$

$$\int_{\triangle} K_n(x,y)\, f(y)\, dy = \sum_i \lambda_i^n\, a_i\, \psi_i(x) \qquad (n \geqq 2),$$

uniformly in $\triangle$, *where* $\tilde{p}(x)$ *is continuous in* $\triangle$ *and satisfies the relation*

$$\int_{\triangle} H(x, y)\, \tilde{p}(y)\, dy = 0$$

for almost every $x \in \triangle$. *In the case that* $H(x, y)$ *is a general kernel,* $\tilde{p}(x)$ *vanishes identically.*

Proof. In the proof of the preceding theorem we have seen that

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqq \|H\| \cdot \int_{\triangle} |A(x, z)|^2\, dz$$

for those values of x, for which $g(z) = \overline{A(x, z)}$ belongs to L_2. It is not difficult to prove that now, since $A(x, y)$ is continuous in mean, there exists a constant M such that

$$\int_{\triangle} |A(x, z)|^2\, dz \leqq M^2,$$

so that $g(z) = \overline{A(x, z)}$ belongs to L_2 for every $x \in \triangle$. This shows that

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqq \|H\| \cdot M^2$$

for all $x \in \triangle$. Furthermore, since $a_i = (f, \chi_i) = (f, H^{1/2} \Psi_i) = (H^{1/2} f, \Psi_i)$ and the system $\Psi_i(x)$ is orthonormal, we have

$$\Sigma |a_i|^2 \leqq \|H^{1/2} f\|^2,$$

so that, if $\varepsilon > 0$ is given,

$$\sum_{i=p} |a_i|^2 \leqq \varepsilon^2 / \|H\| \cdot M^2$$

for sufficiently large p. The inequality

$$\sum_{i=p} |\lambda_i a_i \psi_i(x)| \leqq \left(\sum_{i=p} |a_i|^2\right)^{1/2} \cdot \left(\sum_{i=p} \lambda_i^2 |\psi_i(x)|^2\right)^{1/2} \leqq \varepsilon$$

shows then that the series $\Sigma \lambda_i a_i \psi_i(x)$ converges uniformly in $\triangle$. Since both its sum and $\int_{\triangle} K(x, y) f(y)\, dy$ are continuous in $\triangle$, we may write

$$\int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x) + \tilde{p}(x),$$

where $\tilde{p}(x)$ is continuous in $\triangle$. From Theorem 1 it follows that $p(x) = p(x)$ for almost every $x \in \triangle$, where $p(x)$ is the function occurring in that theorem, satisfying therefore $Hp = 0$. Then we have also $H\tilde{p} = 0$. This completes the proof of the first part.

That $\sum_i \lambda_i^n a_i \psi_i(x)$ $(n \geqq 2)$ converges also uniformly in $\triangle$ is now trivial. On account of Theorem 1 its sum is equal to $\int_{\triangle} K_n(x, y) f(y)\, dy$ for almost every $x \in \triangle$. But, since both this sum and this function are continuous in $\triangle$, equality holds for every $x \in \triangle$, hence

$$\int_{\triangle} K_n(x, y) f(y)\, dy = \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqq 2),$$

uniformly in $\triangle$.

Assuming now that $H(x, y)$ is a general kernel, it follows from $H\tilde{p} = 0$ that $\tilde{p}(x) = 0$ for almost every $x \in \triangle$. The continuity of $\tilde{p}(x)$ implies then that $\tilde{p}(x)$ vanishes identically.

Theorem 5. *Let $A(x, y)$ be continuous in mean, and $H(x, y)$ be general. Let furthermore $\lambda \neq 0$, and $g(x) \in L_2$ be H-orthogonal to all characteristic functions of $K(x, y)$, belonging to the characteristic value λ (If λ is no characteristic value, $g(x)$ may be any function belonging to L_2). Then the solution of* (1) *is given by*

$$f(x) = -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

where $a_i = \int_{\triangle} g(x) \overline{\chi_i(x)}\, dx$ *for* $\lambda_i \neq \lambda$, Σ' *denotes that for those values of* i *for which* $\lambda_i = \lambda$ *the coefficient of* $\psi_i(x)$ *is arbitrary, and the series*

$$\Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x)$$

converges uniformly in $\triangle$.

Proof. By the preceding theorem we have

$$g(x) + \lambda f(x) = \int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i b_i \psi_i(x),$$

uniformly in $\triangle$, where $b_i = (f, \chi_i)$. Since

$$\lambda_i b_i = \lambda_i (f, \chi_i) = \lambda_i (f, H\psi_i) = (f, HK\psi_i) = (HKf, \psi_i) = \\ = (Kf, H\psi_i) = (g + \lambda f, \chi_i) = a_i + \lambda b_i,$$

we find $b_i = -a_i/(\lambda - \lambda_i)$ for $\lambda_i \neq \lambda$. Furthermore, the solution $f(x)$ being determined except for a characteristic function of $K(x, y)$, belonging to the characteristic value λ, the coefficients b_i may be taken arbitrary for those values of i for which $\lambda_i = \lambda$. Hence

$$f(x) = -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

uniformly in $\triangle$.

For later purposes we prove

Theorem 6. *If $A(x,y)$ is continuous in mean, the series $\Sigma\lambda_i^2|\psi_i(x)|^2$ converges uniformly in $\triangle$.*

Proof. In the same way as we have proved (8), we may find

$$\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2 \leqq \|H\|\int_{\triangle}|A(x_2,z)-A(x_1,z)|^2\,dz,$$

hence

$$\lim_{x_2\to x_1}\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2=0.$$

By MINKOWSKI's inequality we see that

$$|(\Sigma\lambda_i^2|\psi_i(x_2)|^2)^{1/2}-(\Sigma\lambda_i^2|\psi_i(x_1)|^2)^{1/2}| \leqq (\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2)^{1/2};$$

the sumfunction of the series $\Sigma\lambda_i^2|\psi_i(x)|^2$ is therefore a continuous function. Hence, on account of DINI's well-known Theorem, since the functions $\lambda_i^2|\psi_i(x)|^2$ are non-negative and continuous, the uniform convergence of $\Sigma\lambda_i^2|\psi_i(x)|^2$.

§ 4. *The case that $A(x,y)$ and $H(x,y)$ are both continuous in mean.*

Theorem 7. *If $H(x,y)$ (but not necessarily $A(x,y)$) is continuous in mean, the series $\Sigma\lambda_i^2|\chi_i(x)|^2$ converges uniformly in $\triangle$.*

Proof. We observe first that, if $H(x,y)$ is continuous in mean, the functions $\chi_i(x)=\int_{\triangle}H(x,y)\,\psi_i(y)\,dy$ are continuous in $\triangle$. For every $f(y)\in L_2$ we have $(f,A\chi_i)=(Af,H^{1/2}\Psi_i)=(H^{1/2}Af,\Psi_i)$, hence, the system $\Psi_i(x)$ being orthonormal,

$$\Sigma|(A\chi_i,f)|^2=\Sigma|(f,A\chi_i)|^2 \leqq \|H^{1/2}Af\|^2 \leqq \|H\|\cdot\|A\|^2\cdot\|f\|^2. \quad (10)$$

on account of BESSEL's inequality. Taking now $f(y)=\overline{H(x,y)}$, there exists, since $H(x,y)$ is continuous in mean, a constant M such that

$$\|f\|^2=\int_{\triangle}|H(x,y)|^2\,dy \leqq M$$

for every $x\in\triangle$. Furthermore

$$(A\chi_i,f)=(AH\psi_i,f)=(K\psi_i,f)=\lambda_i(\psi_i,f)=\lambda_i\int_{\triangle}H(x,y)\,\psi_i(y)\,dy=\lambda_i\chi_i(x),$$

so that by (10)

$$\Sigma\lambda_i^2|\chi_i(x)|^2 \leqq \|H\|\cdot\|A\|^2\cdot M$$

for every $x\in\triangle$.

Substituting now $f(y) = \overline{H(x_2, y) - H(x_1, y)}$ in the relation (10), we find in the same way

$$\sum \lambda_i^2 |\chi_i(x_2) - \chi_i(x_1)|^2 \leqslant \|H\| \cdot \|A\|^2 \int_{\triangle} |H(x_2, y) - H(x_1, y)|^2 dy,$$

hence

$$\lim_{x_2 \to x_1} \sum \lambda_i^2 |\chi_i(x_2) - \chi_i(x_1)|^2 = 0.$$

By MINKOWSKI's inequality follows now again that the sumfunction of $\sum \lambda_i^2 |\chi_i(x)|^2$ is a continuous function, which implies, on account of DINI's Theorem, the uniform convergence of $\sum \lambda_i^2 |\chi_i(x)|^2$.

Theorem 8. (*Expansion Theorem for the iterated kernels*). *If both* $A(x, y)$ *and* $H(x, y)$ *are continuous in mean, then*

$$K_n(x, y) = \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqslant 2),$$

uniformly in $\triangle \times \triangle$.

Proof. It is not difficult to prove that, if both $A(x, y)$ and $H(x, y)$ are continuous in mean, all kernels $K_n(x, y)$ $(n \geqslant 1)$ are continuous in $\triangle \times \triangle$. Since, by the Theorems 6 and 7, the series $\sum \lambda_i^2 |\psi_i(x)|^2$ and $\sum \lambda_i^2 |\chi_i(x)|^2$ are converging uniformly in $\triangle$, the inequality

$$\sum_i |\lambda_i^n \psi_i(x) \overline{\chi_i(y)}| \leqslant (\sum_i \lambda_i^2 |\psi_i(x)|^2)^{1/2} \cdot (\sum_i \lambda_i^{2n-2} |\chi_i(y)|^2)^{1/2}$$

shows that, for $n \geqslant 2$, the series $\sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$. By Theorem 3 (3) the sumfunction is equal to $K_n(x, y)$ almost everywhere in $\triangle \times \triangle$. But, both $K_n(x, y)$ and this sumfunction being continuous in $\triangle \times \triangle$, equality is seen to hold for every point $(x, y) \in \triangle \times \triangle$. This completes the proof.

§ 5. *The case that* $A(x, y)$ *is continuous in mean and* $H(x, y)$ *is continuous.*

Theorem 9. (*Expansion Theorem for the kernel*). *If* $A(x, y)$ *is continuous in mean and* $H(x, y)$ *is continuous, then*

$$K(x, y) - p(x, y) = \sum \lambda_i \psi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, *where*

$$q(x, y) = \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

for every point $(x, y) \in \triangle \times \triangle$.

Proof. [3]) Since $H(x, y)$ is continuous, the transformation $H^{1/2}$ is, on account of II, Theorem 10, given by

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy,$$

[3]) The beginning of this proof is identical with the beginning of the proof of V, Theorem 5. For the sake of completeness, however, we shall give the proof in full.

where $\int_{\triangle} |H_{1/2}(x, y)|^2\, dy$ is bounded. For almost every $x_1 \in \triangle$ and almost every $x_2 \in \triangle$ we have now

$$\int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2\, dy =$$

$$\int_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\}\, \{H_{1/2}(y, x_2) - H_{1/2}(y, x_1)\}\, dy =$$

$$= H(x_2, x_2) - H(x_2, x_1) - H(x_1, x_2) + H(x_1, x_1);$$

consequently, there exists for every $\varepsilon > 0$ a number $\delta(\varepsilon) > 0$ such that

$$\int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2\, dy \leqq \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for almost every $x_1 \in \triangle$ and $x_2 \in \triangle$, if only the distance $\varrho(x_1, x_2)$ of the points x_1 and x_2 satisfies $\varrho(x_1, x_2) < \delta$. Furthermore we have

$$\chi_i(x_2) - \chi_i(x_1) = \int_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\}\, \Psi_i(y)\, dy;$$

hence by BESSEL's inequality (the system $\Psi_i(x)$ is orthonormal) and (11)

$$\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2 \leqq \int_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2\, dy \leqq \varepsilon \quad . \quad (12)$$

for almost every $x_1 \in \triangle$ and $x_2 \in \triangle$, if only $\varrho(x_1, x_2) < \delta$. Since however the functions $\chi_i(x)$ are continuous in $\triangle$, the left hand side of (12) is continuous in x_1 and x_2, so that (12) holds for all $x_1, x_2 \in \triangle$, satisfying $\varrho(x_1, x_2) < \delta$. Observing now that p is arbitrary, we obtain

$$\lim_{x_2 \to x_1} \Sigma |\chi_i(x_2) - \chi_i(x_1)|^2 = 0,$$

so that, by the device used already several times, we see that $\Sigma |\chi_i(x)|^2$ converges uniformly in $\triangle$. By Theorem 6 the same is true of $\Sigma \lambda_i^2 |\psi_i(x)|^2$; the inequalities

$$\Sigma |\lambda_i \psi_i(x) \overline{\chi_i(y)}| \leqq (\Sigma \lambda_i^2 |\psi_i(x)|^2)^{1/2} \cdot (\Sigma |\chi_i(y)|^2)^{1/2},$$

$$\Sigma |\chi_i(x) \overline{\chi_i(y)}| \leqq (\Sigma |\chi_i(x)|^2)^{1/2} \cdot (\Sigma |\chi_i(y)|^2)^{1/2}$$

show then that the series $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ and $\Sigma \chi_i(x) \overline{\chi_i(y)}$ converge uniformly in $\triangle \times \triangle$. Writing

$$H(z, y) - p_1(z, y) = \Sigma \chi_i(z) \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

we see that $p_1(z, y)$ is continuous in $\triangle \times \triangle$. Multiplying (13) with

$A(x, z)$, and writing $p(x, y) = \int_{\triangle} A(x, z)\, p_1(z, y)\, dz$, we find, since we may use term by term integration on account of the uniform convergence,

$$K(x, y) - p(x, y) = \Sigma \int_{\triangle} A(x, z)\, \chi_i(z)\, \overline{\chi_i(y)}\, dz = \Sigma \lambda_i \psi_i(x)\, \overline{\chi_i(y)}$$

for every point $(x, y) \in \triangle \times \triangle$. It follows from Theorem 3 that

$$q(x, y) = \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

for almost every point $(x, y) \in \triangle \times \triangle$. But, as may be seen easily, $q(x, y)$ is continuous in $\triangle \times \triangle$, so that $q(x, y) = 0$ everywhere in $\triangle \times \triangle$. This completes the proof.

Theorem 10. *If $A(x, y)$ is continuous in mean, and $H(x, y)$ is continuous and general, then*

$$K(x, y) = \Sigma \lambda_i \psi_i(x)\, \overline{\chi_i(y)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

uniformly in $\triangle \times \triangle$.

Proof. By the preceding theorem we have

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x)\, \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, where

$$p(x, y) = \int_{\triangle} A(x, z)\, p_1(z, y)\, dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$p_1(x, y)$ is continuous in $\triangle \times \triangle$, and

$$\int_{\triangle} H(x, z)\, p(z, y)\, dz = 0.$$

Since $H(x, y)$ is now general, $Hp = 0$ implies $p = 0$, so that, for every $y \in \triangle$, $p(x, y) = 0$ for almost every $x \in \triangle$. It follows however from (15) that $p(x, y)$ is continuous in x for every $y \in \triangle$; hence $p(x, y) = 0$ in $\triangle \times \triangle$, which proves (14).

Theorem 11. *If $A(x, y)$ is continuous in mean and general, while $H(x, y)$ is continuous and general, then*

$$H(x, y) = \Sigma \chi_i(x)\, \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$.

Proof. We know already (cf. the proof of Theorem 9) that $\Sigma \chi_i(x)\, \overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$, and by the preceding theorem we have

$$K(z, y) - \Sigma \lambda_i \psi_i(z)\, \overline{\chi_i(y)} = \int_{\triangle} A(z, x) \{ H(x, y) - \Sigma \chi_i(x)\, \overline{\chi_i}\, \overline{(y)} \}\, dx = 0.$$

Hence, since $A(z, x)$ is general, for every $y \in \triangle$,

$$H(x, y) - \Sigma \chi_i(x) \overline{\chi_i(y)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for almost every $x \in \triangle$. Both $H(x, y)$ and $\Sigma \chi_i(x) \overline{\chi_i(y)}$ being however continuous in $\triangle \times \triangle$, the relation (16) holds for every point $(x, y) \in \triangle \times \triangle$, so that

$$H(x, y) = \Sigma \chi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$.

Remark. It is not difficult to prove that, if $A(x, y)$ is not general, (16) is not necessarily true.

§ 6. *An example.*

We shall illustrate the theorems, proved in the preceding paragraphs. by an example showing that the functions $\tilde{p}(x)$ and $p(x, y)$, occurring in the Theorems 4 and 9, need not vanish identically.

Let, for this purpose, $\triangle$ be the linear interval $0 \leqq x \leqq 2\pi$, and $t_i(x)$ the orthonormal trigonometrical system, hence

$$t_1(x) = (2\pi)^{-1/2}, \; t_{2n}(x) = \pi^{-1/2} \cos n x \; (n \geqq 1), \; t_{2n+1}(x) = \pi^{-1/2} \sin n x \; (n \geqq 1).$$

It is well-known that the system $t_i(x)$ is complete in the space $L_2(0, 2\pi)$ of all functions $f(x)$ for which $|f(x)|^2$ is summable over $\triangle$. We determine now the sequences $\mu_3, \mu_4, \ldots$ and $\nu_3, \nu_4, \ldots$ of positive numbers such that

$$\lim \mu_i = \lim \nu_i = 0, \; \mu_3 > \mu_4 > \ldots, \; \nu_3 > \nu_4 > \ldots, \; \sum_3 \mu_i < \infty$$

and $\sum_3 \nu_i < \infty$. The Hermitean kernel $A(x, y)$ and the positive Hermitean kernel $H(x, y)$ are then defined by

$$A(x, y) = \tfrac{1}{2}\{t_1(x) + t_2(x)\}\{t_1(y) + t_2(y)\} - \tfrac{1}{2}\{t_1(x) - t_2(x)\}\{t_1(y) - t_2(y)\} + \sum_3^\infty \nu_i t_i(x) t_i(y),$$

$$H(x, y) = t_1(x) t_1(y) + \sum_3^\infty \mu_i t_i(x) t_i(y).$$

It is clear that both series are converging uniformly in $\triangle \times \triangle$, both $A(x, y)$ and $H(x, y)$ are therefore continuous in $\triangle \times \triangle$. We find now

$$K(x, y) = \int_\triangle A(x, z) H(z, y) \, dz = t_2(x) t_1(y) + \sum_3^\infty \nu_i \mu_i t_i(x) t_i(y),$$

and we observe that

$$Ht_1 = t_1, \quad Ht_2 = 0, \quad Ht_i = \mu_i t_i \qquad (i = 3, 4, \ldots),$$
$$At_1 = t_2, \quad At_2 = t_1, \quad At_i = \nu_i t_i \qquad (i = 3, 4, \ldots),$$

so that

$$AHt_1 = t_2, \quad AHt_2 = 0, \quad AHt_i = \nu_i \mu_i t_i \qquad (i = 3, 4, \ldots).$$

To find the characteristic functions of $K(x,y)$, belonging to characteristic values $\neq 0$, we write $AHf=\lambda f$ for $f=\sum_{i=1}^{\infty} a_i t_i$ and $\lambda \neq 0$. From this we derive

$$a_1 t_2 + \sum_3 \nu_i \mu_i a_i t_i = \sum_1 \lambda a_i t_i;$$

hence $a_1 = a_2 = 0$ and $\nu_i \mu_i a_i = \lambda a_i$ $(i=3,4,\ldots)$. Since $\nu_i \mu_i \neq \nu_k \mu_k$ for $i \neq k$ we have therefore $\lambda = \nu_k \mu_k$ for a certain value of k $(\geqq 3)$ and $a_i = 0$ for $i \neq k$, which shows that the functions $a_k t_k(x)$ $(k \geqq 3)$ are the only characteristic functions with characteristic values $\neq 0$. Making them H-normal, we obtain $a_k = \mu_k^{-1/2}$, so that, for $i=1,2,\ldots,$

$$\lambda_i = \nu_{i+2}\,\mu_{i+2},$$

$$\psi_i(x) = \mu_{i+2}^{-1/2}\, t_{i+2}(x),$$

$$\chi_i(x) = H\psi_i = \mu_{i+2}^{-1/2}\, H t_{i+2} = \mu_{i+2}^{1/2}\, t_{i+2}(x),$$

hence

$$\Sigma \lambda_i \psi_i(x)\,\overline{\chi_i(y)} = \sum_3 \nu_i \mu_i t_i(x)\, t_i(y) = K(x,y) - t_2(x)\, t_1(y)$$

or

$$K(x,y) - p(x,y) = \Sigma \lambda_i \psi_i(x)\,\overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, where

$$p(x,y) = t_2(x)\, t_1(y).$$

Evidently

$$\int_{\triangle} H(x,z)\, p(z,y)\, dz = 0,$$

as required by Theorem 9.

Furthermore, by Theorem 4,

$$\int_{\triangle} K(x,y)\, f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x) + \tilde{p}(x),$$

where $a_i = \int_{\triangle} f(x)\,\overline{\chi_i(x)}\, dx$ and $H\tilde{p} = \int_{\triangle} H(x,y)\,\tilde{p}(y)\, dy = 0$. Taking $f(x) = t_1(x)$, we have

$$\int_{\triangle} K(x,y)\, f(y)\, dy = K t_1 = AH t_1 = t_2(x)$$

and

$$a_i = (t_1, \chi_i) = \mu_{i+2}^{1/2}(t_1, t_{i+2}) = 0;$$

hence

$$t_2(x) = \tilde{p}(x).$$

Evidently $H\tilde{p} = H t_2 = 0$, as required.

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Fourth communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 15. *The star body* $|(x_1^2+x_2^2)\,x_3|<1$ *in* R_3.

By another theorem of H. DAVENPORT [16]), the star body

$$K: \qquad |(x_1^2+x_2^2)\,x_3|\leqslant 1$$

is of determinant

$$\triangle(K)=\sqrt{(23)}/2.$$

Let α, β be the two *complex conjugate* roots and γ the *real* root of

$$t^3-t-1=0.$$

Then the lattice Λ_0 defined by

$$\Lambda_0: \quad x_1+x_2 i=u_1+\alpha u_2+\alpha^2 u_3, x_1-x_2 i=u_1+\beta u_2+\beta^2 u_3, x_3=u_1+\gamma u_2+\gamma^2 u_3$$
$$(u_1, u_2, u_3=0, \mp 1, \mp 1, \ldots)$$

is a critical lattice of K, and every other critical lattice Λ of K may be written in the form [17]) $\Lambda=\Omega\Lambda_0$ where Ω is one of the automorphisms

$$\Omega: \qquad x_1=t_1 x_1'-t_2 x_2', \quad \mp x_2=t_2 x_1'+t_1 x_2', \quad x_3=t_3 x_3'$$

of K, and where t_1, t_2, t_3 are real numbers such that

$$(t_1^2+t_2^2)\,t_3=\mp 1.$$

Theorem N: *The star body* K: $(x_1^2+x_2^2)\,|x_3|\leq 1$ *is boundedly reducible.*

Proof: It again suffices to show that Λ_0 is a strongly critical lattice of K. Now Λ_0 contains the point

$$P_0=(1, 0, 1) \qquad (u_1=1, u_2=u_3=0)$$

on the boundary of K, and it is evident that every neighbouring point can be transformed, by one of the automorphisms Ω near to the unit automorphism, into a point collinear with O and P_0.

Hence the assertion is proved if we can show the following result: *There exists a bounded star body* K^* *contained in* K *such that*

$$d(\Lambda^*)\geqslant d(\Lambda_0)$$

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16) Proc. Lond. Math. Soc. **45** (2), 98—125 (1939).

17) This result is proved, in slightly different form, in L. J. MORDELL, Journ. Lond. Math. Soc. **17**, 107—115 (1942).

for every K^*-admissible lattice Λ^* which is (i) *sufficiently near to Λ_0, and which* (ii) *contains a point P^* arbitrarily near to P_0 and collinear with O and P_0.*

Now every latice Λ^* near to Λ_0 can be written in the form

$$\Lambda^*: \quad x_1 = v_1 + \frac{\alpha+\beta}{2} v_2 + \frac{\alpha^2+\beta^2}{2} v_3, \quad x_2 = \frac{\alpha-\beta}{2i} v_2 + \frac{\alpha^2-\beta^2}{2i} v_3,$$

$$x_3 = v_1 + \gamma v_2 + \gamma^2 v_3,$$

with

$$\begin{aligned} v_1 &= u_1 + (a_{11} u_1 + a_{12} u_2 + a_{13} u_3), \\ v_2 &= u_2 + (a_{21} u_1 + a_{22} u_2 + a_{23} u_3), \qquad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots), \\ v_3 &= u_3 + (a_{31} u_1 + a_{32} u_2 + a_{33} u_3), \end{aligned}$$

where the coefficients a_{hk} are real numbers such that

$$a = \max_{h,k=1,2,3} |a_{hk}|$$

is less than any given constant. The point P^* corresponding to P_0 is $P^* = (x_1^*, x_2^*, x_3^*)$ where

$$x_1^* = 1 + a_{11} + \frac{\alpha+\beta}{2} a_{21} + \frac{\alpha^2+\beta^2}{2} a_{31}, \quad x_2 = \frac{\alpha-\beta}{2i} a_{21} + \frac{\alpha^2-\beta^2}{2i} a_{31},$$

$$x_3 = 1 + a_{11} + \gamma a_{21} + \gamma^2 a_{31},$$

and so P^* is collinear with O and P_0 if and only if

$$(a): \qquad a_{21} = a_{31} = 0,$$

because the three points

$$(1, 0, 1), \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2i}, \gamma\right), \left(\frac{\alpha^2+\beta^2}{2}, \frac{\alpha^2-\beta^2}{2i}, \gamma^2\right)$$

are linearly independent. We consider from now on only lattices satisfying (a).

Put for shortness,

$$\begin{aligned} S(U) &= (u_1 + \alpha u_2 + \alpha^2 u_3)(u_1 + \beta u_2 + \beta^2 u_3)(u_1 + \gamma u_2 + \gamma^2 u_3) = \\ &= (u_1^3 + u_2^3 + u_3^3) + u_3^2 u_1 + (-u_2 u_3^2 + 2 u_3 u_1^2 - u_1 u_2^2) - 3 u_1 u_2 u_3, \end{aligned}$$

so that

$$(x_1^2 + x_2^2) x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. For the corresponding point of Λ^*,

$$(x_1^2 + x_2^2) x_3 = S(V),$$

or on replacing $V = (v_1, v_2, v_3)$ by its expression in U,

$$(x_1^2 + x_2^2) x_3 = S(U) + T(U).$$

Here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^3) + (B_1 u_2^2 u_3 + B_2 u_3^2 u_1 + B_3 u_1^2 u_2) + \\ + (C_1 u_2 u_3^2 + C_2 u_3 u_1^2 + C_3 u_1 u_2^2) + D u_1 u_2 u_3,$$

with the coefficients.

$$\begin{aligned}
A_1 &= 3a_{11} + O(a^2),\\
A_2 &= 3a_{22} - a_{12} + O(a^2),\\
A_3 &= 3a_{33} - a_{23} + a_{13} + O(a^2),\\
B_1 &= -3a_{12} + 3a_{23} - 2a_{32} - a_{13} + O(a^2),\\
B_2 &= a_{11} + 2a_{33} - 3a_{23} + 4a_{13} + O(a^2).\\
B_3 &= 3a_{12} + 3a_{32} + O(a^2),\\
C_1 &= -a_{22} - 2a_{33} + a_{12} + 3a_{32} - 3a_{13} + O(a^2),\\
C_2 &= 4a_{11} + 2a_{33} + 3a_{13} + O(a^2),\\
C_3 &= -a_{11} - 2a_{22} - 3a_{32} + O(a^2),\\
D &= -3a_{11} - 3a_{22} - 3a_{33} + 4a_{12} - 2a_{23} + 2a_{32} + O(a^2).
\end{aligned}$$

In all these formulae, the O-term consists of the products of two or three of the a_{hk}. If

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|),$$

then these formulae imply, in particular, that

$$A = O(a).$$

On solving for the coefficients a_{hk}, we find that

$$\begin{aligned}
3a_{11} &= A_1 + O(a^2),\\
69a_{22} &= 2A_1 + 27A_2 + 9B_3 + 6C_3 + O(a^2),\\
69a_{33} &= -A_1 + 27A_3 - 9B_2 + 3C_2 + O(a^2),\\
23a_{12} &= 2A_1 + 4A_2 + 9B_3 + 6C_3 + O(a^2),\\
23a_{23} &= -11A_1 - 2A_3 - 7B_2 + 10C_2 + O(a^2),\\
23a_{32} &= -3A_1 - 6A_2 - 2B_3 - 9C_3 + O(a^2),\\
23a_{13} &= -10A_1 - 6A_3 + 2B_2 + 7C_2 + O(a^2),
\end{aligned}$$

and also obtain the three identities,

$$\begin{aligned}
B_1 &= -A_1 - B_2 - B_3 + C_2 + O(a^2),\\
C_1 &= A_1 - A_2 - C_2 - C_3 + O(a^2),\\
D &= -A_2 - A_3 + B_2 + B_3 - C_2 + O(a^2),
\end{aligned}$$

and the inequality,

$$a = O(A), \quad O(a^2) = O(A^2).$$

So far, the star body K^* has not yet been defined. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqq 3, \quad |u_2| \leqq 3, \quad |u_3| \leqq 3$$

belong to K^t. Then the ten points of Λ_0 given by

$$U=(1,0,0),\ (0,1,0),\ (0,0,1),\ (0,1,1),\ (-1,0,1),\ (1,1,0),$$
$$(0,-1,1),\ (2,0,-1),\ (-1,1,0),\ (1,1,1),$$

satisfy the equation,

$$S(U)=1.$$

If Λ^* is K^*-admissible, then the points of Λ^* belonging to the same U cannot be *inner* points of $K^*=K^t$. The numbers

$$\alpha_1,\alpha_2,\alpha_3,\ \beta_1,\beta_2,\beta_3,\ \gamma_1,\gamma_2,\gamma_3,\ \delta$$

defined by

$$T(1,0,0)=\alpha_1,\ T(0,1,1)=\beta_1,\ T(0,-1,1)=\gamma_1,$$
$$T(0,1,0)=\alpha_2,\ T(-1,0,1)=\beta_2,\ T(2,0,-1)=\gamma_2,\ T(1,1,1)=\delta,$$
$$T(0,0,1)=\alpha_3,\ T(1,1,0)=\beta_3,\ T(-1,1,0)=\gamma_3,$$

are then non-negative since

$$(x_1^2+x_2^2)\,x_3=S(U)+T(U)=1+T(U)\geqq 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{aligned}
\alpha_1 &= A_1,\\
\alpha_2 &= A_2,\\
\alpha_3 &= A_3,\\
\beta_1 &= A_2+A_3+B_1+C_1,\\
\beta_2 &= -A_1+A_3-B_2+C_2,\\
\beta_3 &= A_1+A_2+B_3+C_3,\\
\gamma_1 &= -A_2+A_3+B_1-C_1,\\
\gamma_2 &= 8A_1-A_3+2B_2-4C_2,\\
\gamma_3 &= -A_1+A_2+B_3-C_3,\\
\delta &= A_1+A_2+A_3+B_1+B_2+B_3+C_1+C_2+C_3+D,
\end{aligned}$$

and conversely,

$$\begin{aligned}
A_1 &= \alpha_1,\\
A_2 &= \alpha_2,\\
A_3 &= \alpha_3,\\
B_1 &= -\alpha_3+\tfrac{1}{2}\beta_1+\tfrac{1}{2}\gamma_1,\\
B_2 &= 2\alpha_1+\tfrac{3}{2}\alpha_3-2\beta_2-\tfrac{1}{2}\gamma_2,\\
B_3 &= -\alpha_2+\tfrac{1}{2}\beta_3+\tfrac{1}{2}\gamma_3,\\
C_1 &= -\alpha_2+\tfrac{1}{2}\beta_1-\tfrac{1}{2}\gamma_1,\\
C_2 &= 3\alpha_1+\tfrac{1}{2}\alpha_3-\beta_2-\tfrac{1}{2}\gamma_2,\\
C_3 &= -\alpha_1+\tfrac{1}{2}\beta_3-\tfrac{1}{2}\gamma_3,\\
D &= -5\alpha_1+\alpha_2-2\alpha_3-\beta_1+3\beta_2-\beta_3+\gamma_2+\delta.
\end{aligned}$$

We deduce from these formulae that

$$\begin{aligned}
\beta_1 &= -\alpha_1 + \alpha_2 - \tfrac{1}{2}\alpha_3 + 2\beta_2 - \beta_3 + \tfrac{1}{2}\gamma_2 + O(a^2),\\
\gamma_1 &= \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\gamma_2 - \gamma_3 + O(a^2),\\
\delta &= 3\alpha_1 - 2\alpha_2 + \tfrac{3}{2}\alpha_3 - 2\beta_2 + \tfrac{1}{2}\beta_3 - \tfrac{1}{2}\gamma_2 + \tfrac{1}{2}\gamma_3 + O(a^2).
\end{aligned}$$

Hence, if

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then all three numbers a, A, α are of the same order,

$$\alpha = O(a) = O(A), \quad O(a^2) = O(\alpha^2), \quad O(A^2) = O(\alpha^2).$$

Finally, the lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0)\begin{vmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+a_{22} & a_{23} \\ 0 & a_{32} & 1+a_{33} \end{vmatrix} = d(\Lambda_0)(1+\sigma),$$

and here

$$\begin{aligned}
\sigma &= a_{11} + a_{22} + a_{33} + O(a^2),\\
&= \tfrac{1}{23}(8A_1 + 9A_2 + 9A_3 - 3B_2 + 3B_3 + C_2 + 2C_3) + O(A^2),\\
&= \tfrac{1}{23}(3\alpha_1 + 6\alpha_2 + 5\alpha_3 + 5\beta_2 + \tfrac{5}{2}\beta_3 + \gamma_2 + \tfrac{1}{2}\gamma_3) + O(\alpha^2).
\end{aligned}$$

We find therefore, just as in the last proof, that

either

$$\sigma > 0, \quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0, \quad d(\Lambda^*) = d(\Lambda_0),$$

and that the second case can hold only if $\alpha = a = A = 0$, that is, if Λ^* coincides with Λ_0; whence the assertion.

§ 16. *Some further examples.*

I have applied the method of the last paragraphs to three further star bodies in R_3 and R_4. From the well-known results of A. OPPENHEIM [18]) on the minima of the indefinite quadratic forms in three and four variables, I have so deduced that

the star body K_1: $|x_1^2 + x_2^2 - x_3^2| \leqq 1$ in R_3 with $\triangle(K_1) = \sqrt{\tfrac{8}{2}}$,

the star body K_2: $x_1^2 + x_2^2 + x_3^2 - x_4^2| \leqq 1$ in R_4 with $\triangle(K_2) = \sqrt{\tfrac{7}{4}}$,

and the star body K_3: $|x_1^2 + x_2^2 - x_3^2 - x_4^2| \leqq 1$ in R_4 with $\triangle(K_3) = \tfrac{3}{2}$,

are each one boundedly reducible. As before, Theorem L is the basis of the proof; since no new ideas are used, I omit this proof.

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[18]) See L. E. DICKSON, Studies in the theory of numbers (Chicago 1930), chapters 8 and 9.

In all these examples of boundedly reducible star bodies, it would be of great interest to obtain irreducible star bodies of equal determinants contained in them.

§ 17. *Applications.*

The following theorems show that the preceeding results can be useful for other purposes.

Theorem O: *There exists a positive constant c with the following property: If α_1, α_2 are real numbers, and t is a positive number, then there exist integers u_1, u_2, u_3 not all zero such that*

$$\{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\} |u_3| \leqq \frac{2}{\sqrt{23}},$$

$$(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2 \leqq \frac{c}{t}, \quad |u_3| \leqq t.$$

Hence if α_1, α_2 are irrational, then there are arbitrarily large integers u_1, u_2, u_3 such that [19])

$$\left(\frac{u_1}{u_3} - \alpha_1\right)^2 + \left(\frac{u_2}{u_3} - \alpha_2\right)^2 \leqq \frac{2}{\sqrt{23}\,|u_3|^3}.$$

Proof: By Theorem N, a positive number r exists such that

K^*: $$(x_1^2 + x_2^2)|x_3| \leqq 1, \quad x_1^2 + x_2^2 + x_3^2 \leqq r^2$$

is of the same determinant as

K: $$(x_1^2 + x_2^2)|x_3| \leqq 1,$$

namely $\Delta(K) = \Delta(K^*) = \sqrt{(23)}/2$. On applying the transformation

Ω: $$x_1 = \tau x_1', \; x_2 = \tau x_2', \; x_3 = \tau^{-2} x_3' \quad (\tau > 0),$$

of K, we find that

K_τ^*: $$(x_1^2 + x_2^2)|x_3| \leqq 1, \quad \tau^2 (x_1^2 + x_2^2) + \tau^{-4} x_3^2 \leqq r^2$$

is likewise of determinant $\Delta(K_\tau^*) = \Delta(K) = \sqrt{(23)}/2$. Let Λ be the lattice

$$\Lambda: \; x_1 = u_1 - \alpha_1 u_3, \; x_2 = u_2 - \alpha_2 u_3, \; x_3 = \frac{\sqrt{23}}{2} u_3 \quad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots).$$

Since $d(\Lambda) = \sqrt{(23)}/2$, at least one point $P \neq O$ of Λ lies inside or on the boundary of K_τ^*; let this be the point belonging to the integers u_1, u_2, u_3 not all zero. Then

$$\{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\} |u_3| \leqq \frac{2}{\sqrt{23}},$$

$$r^2 \{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\} + \frac{23}{4r^4} u_3^2 \leqq r^2,$$

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[19]) A slightly less exact result is proved in a joint paper by DAVENPORT and myself, in DUKE Math. Journal 13, 105—111 (1946).

hence

$$(u_1 - a_1 u_3)^2 + (u_2 - a_2 u_3)^2 \leqq \frac{\tau^2}{\tau^2}, \quad |u_3| \leqq \frac{2\tau^2 \tau}{\sqrt{23}}.$$

If now

$$c = \frac{2\tau^3}{\sqrt{23}}, \quad \tau = \left(\frac{23\, t^2}{4\tau^2}\right)^{1/4},$$

then

$$(u_1 - a_1 u_3)^2 + (u_2 - a_2 u_3)^2 \leqq \frac{c}{t}, \quad |u_3| \leqq t,$$

as asserted. — Assume next that a_1 is irrational and that t tends to infinity. Then u_3 is different from zero for sufficiently large t, and since $|u_1 - a_1 u_3|$ tends to zero, $|u_3|$ tends to infinity.

In a similar way, Theorem M leads to the following result:

Theorem P: *There exists a positive constant γ with the following property: If β_1, β_2 are real numbers and t is a positive number, then integers v_1, v_2, v_3 not all zero exist such that*

$$|v_1 v_2 (\beta_1 v_1 + \beta_2 v_2 + v_3)| \leqq \tfrac{1}{7},$$

$$|v_1| \leqq t, \quad |v_2| \leqq t, \quad |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqq \frac{\gamma}{t^2}.$$

Assume further that β_1, β_2 have the following stronger properties: (i) *β_1, β_2, 1 are linearly independent over the rational field.* (ii) *When the integers v, v', v'' tend to infinity in any way, then*

$$\lim v^2 |\beta_1 v + v'| = \infty, \quad \lim v^2 |\beta_2 v + v''| = \infty.$$

Under these conditions, there exist an infinity of triples of integers v_1, v_2, v_3 all different from zero such that

$$0 < |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqq \frac{1}{7|v_1 v_2|}.$$

The results on boundedly reducible star bodies are also of use for obtaining asymptotic formulae for the determinants of certain star bodies depending on a parameter [20]). For instance, it is easy to deduce from Theorem M that when $\alpha > 0$ tends to zero, then the star body

K_1: $|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha \leqq 1$ is of determinant $\triangle(K_1) = \frac{1}{7}\, e^{-2/\alpha}(1 + O(\alpha))$,

and the star body

K_2: $(|x_1|^\alpha + |x_2|^\alpha)|x_3|^{\alpha/2} \leqq 1$ is of determinant $\triangle(K_2) = \frac{1}{7}\, e^{-1/\alpha}(1 + O(\alpha))$.

I remark, finally, that the just given examples of boundedly reducible star bodies in R_3 and R_4 are all automorphic, and even satisfy the stronger

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[20]) For a special case, see my paper Proc. Cambr. Phil. Soc. **40**, 116—120 (1944), in particular the proof of Theorem 4.

conditions of Theorem 23 of Part I. This suggests that the following problem has an affirmative answer:

Problem 10: *Is it true that every automorphic star body is boundedly reducible if it satisfies the conditions of Theorem 23 of Part I?*

§ 18. *An addition to Theorem 9 of Part I.*

In Theorem 9 of Part I, $\Delta(K)$ was proved to depend continuously on K if K varied in a rather restricted way. To conclude this Part II, we prove a more general continuity property of $\Delta(K)$.

Theorem Q: *Let* $F(X)$ *and* $F_r(X)$ $(r=1, 2, 3, \ldots)$ *be distance functions in* R_n *such that*

$$\lim_{r\to\infty} F_r(X) = F(X)$$

uniformly in X *on the unit sphere* $|X|=1$ [21]*; and let the star bodies*

$$K\colon\ F(X)\leqq 1, \quad \text{and} \quad K_r\colon\ F_r(X)\leqq 1 \quad (r=1,2,3,\ldots)$$

be of the finite type. Then

$$\liminf_{r\to\infty} \Delta(K_r) \geqq \Delta(K).$$

Proof: Let $\varepsilon > 0$ be arbitrarily small. By the Corollary to Theorem 10 of Part I, there exists a positive number t such that the determinant of the star body

$$K^t\colon \qquad F(X)\leqq 1, \qquad |X|\leqq t$$

satisfies the inequalities,

$$(1-\varepsilon)\,\Delta(K) \leqq \Delta(K^t) \leqq \Delta(K).$$

There is further an integer $r_0 = r_0(\varepsilon)$ such that

$$F_r(X)\leqq 1+\varepsilon \text{ for the points } X \text{ of } K^t \qquad \text{if } r\geqq r_0;$$

hence K^t is contained in $(1+\varepsilon)K_r$ if $r \geq r_0$. This implies

$$\Delta(K^t)\leqq (1+\varepsilon)^n\,\Delta(K_r) \qquad \text{if } r\geqq r_0,$$

whence

$$\Delta(K_r)\geqq (1+\varepsilon)^{-n}\,\Delta(K^t)\geqq \frac{1-\varepsilon}{(1+\varepsilon)^n}\,\Delta(K) \qquad \text{if } r\geqq r_0.$$

For $\varepsilon \to 0$, the assertion is obtained.

In the result

$$\liminf_{r\to\infty} \Delta(K_r) \geqq \Delta(K)$$

of Theorem Q, the sign "$\geq$" cannot always be replaced by the equality sign, as the following example shows.

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21) This implies the uniform convergence in every bounded set.

Theorem R: *For every dimension n, there exist star bodies K and K_r ($r = 1, 2, 3, \ldots$) in R_n satisfying the hypothesis of Theorem Q, but such that*

$$\lim_{r\to\infty} \Delta(K_r) \text{ exists and is greater then } \Delta(K).$$

Proof: Denote by $c > 0$ a constant which is so large that the sphere

$$H: \qquad |X| \leqq c$$

is of greater determinant than the star body

$$K: \qquad F(X) \leqq 1, \qquad \text{where} \quad F(X) = |x_1 x_2 \ldots x_n|^{1/n};$$

denote further by $r = 1, 2, 3, \ldots$ the sequence of all positive integers. The distance function

$$F_r(X) = \min\left\{F(X), \frac{1}{c}\left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2\right)^{1/2}\right\}$$

defines a star body K_r: $F_r(X) \leq 1$ which contains K and is easily seen to be of the finite type. The automorphisms of K,

$$\Omega_r: \qquad x_1 = r\, x_1', \ldots, \; x_{n-1} = r\, x_{n-1}', \; x_n = r^{-(n-1)} x_n',$$

change K_r into K_1; hence

$$\Delta(K_1) = \Delta(K_2) = \Delta(K_3) = \ldots = \lim_{r\to\infty} \Delta(K_r).$$

On the other hand,

$$\Delta(K_r) \geqq \Delta(H) > \Delta(K)$$

since K_r contains H; hence

$$\lim_{r\to\infty} \Delta(K_r) > \Delta(K).$$

Consider now $F_r(X)$ on the unit sphere $|X| = 1$. Here

$$F(X) \leqq 1, \text{ and } \frac{1}{c}\left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2\right)^{1/2} \geqq \frac{r^{n-1}|x_n|}{c},$$

and so

$$F_r(X) = F(X) \text{ unless } |x_n| \leqq c\, r^{-(n-1)}.$$

If further

$$|x_n| \leqq c\, r^{-(n-1)}, \quad |X| = 1,$$

then

$$F(X) \leqq (c\, r^{-(n-1)})^{1/n}, \quad 0 \leqq F_r(X) \leqq F(X),$$

whence

$$|F_r(X) - F(X)| \leqq F(X) \leqq (c\, r^{-(n-1)})^{1/n},$$

and here the right-hand side tends to zero uniformly in X, as asserted.

Theorem Q leaves many interesting questions unsolved. For instance, the star domain

$$K_\lambda: \qquad |(x_1^2-x_2^2)(x_1^2-\lambda x_2^2)| \leqslant 1$$

is easily proved to be of the finite type; is $\triangle(K_\lambda)$ a continuous function of λ?

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Mathematics. — *On the G-function.* IV. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 27, 1946.)

§ 10. Second expansion formula.

Theorem 2. Assumptions: *k, l, m, n, p and q are integers with*

$$q \geqq 1, \quad 0 \leqq l-1 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq k \leqq q; \quad . \quad . \quad (111)$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions (1), (99) *and* (100); *r is an arbitrary integer which satisfies the inequality*

$$r \geqq \operatorname{Max}(0, k+l-m-n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = A^{m,n-l+1}_{k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{k}(s)\, G_{p,q}^{k,l-1,n}(z\, e^{(k+l-m-n-2s-1)\pi i}) \\ + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n-2r)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (113)$$

Proof. We may distinguish three cases:

First case:

$$1 \leqq l \leqq n \leqq p \leqq q, \quad k+l-n \leqq m \leqq k \leqq q, \quad r \geqq 0.$$

Formula (113) can be established by induction. If $r = 0$, then (113) reduces to (102) with $\lambda = 0$. We may therefore suppose $r \geqq 1$ and assume that (113) with $r-1$ instead of r has yet been proved.

Now we have by (58)

$$G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n-2r+2)\pi i} \,||\, a_t) = e^{2\pi i a_t} G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n-2r)\pi i} \,||\, a_t) - 2\pi i\, e^{\pi i a_t} G_{p,q}^{k,l-1,n}(z\, e^{(k+l-m-n-2r+1)\pi i}).$$

If this is substituted on the right-hand side of (113) with $r-1$ instead of r, the sum $\sum_{t=1}^{n-l+1}$ not only gives the corresponding sum in (113) but also

$$-2\pi i\, G_{p,q}^{k,l-1,n}(z\, e^{(k+l-m-n-2r+1)\pi i}) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r-1)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)$$

and this expression may by means of (59) be reduced to

$$A^{m,n-l+1}_{k}\, \Omega^{m,n-l+1}_{k}(r-1)\, G_{p,q}^{k,l-1,n}(z\, e^{(k+l-m-n-2r+1)\pi i}),$$

since $\Omega^{m,n-l+1}_{k}(k+l-m-n-r) = 0$, because of $k+l-m-n-r \leqq -1$.

It follows therefore that the sum $\sum_{s=0}^{r-2}$ on the right-hand side of (113) with $r-1$ instead of r reduces to the sum $\sum_{s=0}^{r-1}$ in (113). Thus the first case is finished.

Second case:

$$1 \leqq l \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq k+l-n,\ r \geqq k+l-m-n.$$

This case may also be proved by induction. Owing to the first case formula (113) is true if $m = k+l-n$. We may therefore suppose $0 \leqq m \leqq k+l-n-1$ and assume that (113) with $m+1$ instead of m has yet been proved.

Now it follows from (113) with $m+1$ instead of m, $z\,e^{-\pi i}$ instead of z, $r-1$ instead of r and s replaced by $s-1$

$$e^{\pi i b_{m+1}} G_{p,q}^{m+1,n}(z e^{-\pi i}) = e^{\pi i b_{m+1}} A_k^{m+1,n-l+1} \sum_{s=1}^{r-1} \Omega_k^{m+1,n-l+1}(s-1)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} e^{\pi i (b_{m+1}-a_t)} \triangle_k^{m+1,n-l+1}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,||\, a_t).$$

We further have by (113) with $m+1$ instead of m and $z\,e^{\pi i}$ instead of z

$$e^{-\pi i b_{m+1}} G_{p,q}^{m+1,n}(z e^{\pi i}) = e^{-\pi i b_{m+1}} A_k^{m+1,n-l+1} \sum_{s=0}^{r-1} \Omega_k^{m+1,n-l+1}(s)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} e^{\pi i (a_t-b_{m+1})} \triangle_k^{m+1,n-l+1}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,||\, a_t).$$

From these two relations and (55) it appears

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = &-\frac{e^{-\pi i b_{m+1}}}{2\pi i} A_k^{m+1,n-l+1} \Omega_k^{m+1,n-l+1}(0)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-1)\pi i}) \\ &- \frac{e^{-\pi i b_{m+1}}}{2\pi i} A_k^{m+1,n-l+1} \sum_{s=1}^{r-1} \{\Omega_k^{m+1,n-l+1}(s) - e^{2\pi i b_{m+1}} \Omega_k^{m+1,n-l+1}(s-1)\} \times \\ &\qquad \times G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i}) \\ &+ \frac{1}{\pi} \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \sin(b_{m+1}-a_t)\pi \,.\, \triangle_k^{m+1,n-l+1}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (114)$$

Now it is obvious on account of the definition of the coefficients A

$$-\frac{e^{-\pi i b_{m+1}}}{2\pi i} A_k^{m+1,n-l+1} = A_k^{m,n-l+1}.$$

Moreover we find without difficulty in view of the definition of the coefficients Ω

$$\Omega_k^{m+1,n-l+1}(0) = \Omega_k^{m,n-l+1}(0)$$

and

$$\Omega^{m+1,n-\frac{l+1}{k}}(s) - e^{2\pi i b_{m+1}}\,\Omega^{m+1,n-\frac{l+1}{k}}(s-1) = \Omega^{m,n-\frac{l+1}{k}}(s).$$

Finally it follows from the definition of the coefficients $\triangle$ that

$$\frac{1}{\pi}\sin(b_{m+1}-a_t)\,\pi\,.\,\triangle^{m+1,n-\frac{l+1}{k}}(t) = \triangle^{m,n-\frac{l+1}{k}}(t).$$

Formula (114) is therefore equivalent to (113). So the second case is also finished.

Third case:

$$q \geqq 1,\ n = l-1,\quad 0 \leqq l-1 \leqq p \leqq q,\quad 0 \leqq m \leqq k \leqq q,\quad r \geqq 1+k-m.$$

From the definition of the function G we easily deduce

$$G^{m,l-1}_{p,q}\left(z\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right) = G^{m,l}_{p+1,q+1}\left(z\left|\begin{matrix}\alpha,a_1,\ldots,a_p\\ b_1,\ldots,b_k,\alpha,b_{k+1},\ldots,b_q\end{matrix}\right.\right); \quad (115)$$

herein is α an arbitrary number.

To the function $G^{m,l}_{p+1,q+1}(z)$ on the right-hand side of this relation we may apply (113) with $n = l$, $k+1$ instead of k, $p+1$ instead of p and $q+1$ instead of q. Now it is clear, on account of the definitions of the coefficients A, Ω and $\triangle$ and the function G, that in the particular case under consideration ($\alpha, a_1, \ldots, a_p$ instead of $a_1, a_2, \ldots, a_{p+1}$ and $b_1, \ldots, b_k, \alpha, b_{k+1}, \ldots, b_q$ instead of $b_1, \ldots, b_{q+1}$)

$$A^{m.1}_{k+1} = A^{m,0}_{k}\ ,\ \Omega^{m,1}_{k+1}(s) = \Omega^{m,0}_{k}(s),$$

$$G^{k+1,l-1,l}_{p+1,q+1}(\zeta) = G^{k,l-1,l-1}_{p,q}(\zeta)$$

and

$$\triangle^{m,1}_{k+1}(1) = 0.$$

We therefore get (113) with $n = l-1$ when we apply (113) to the right-hand side of (115).

With this the theorem has been completely proved.

§ 11. Third expansion formula.

Theorem 3. Assumptions: *k, l, m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq l-1 \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq k \leqq q \text{ and } m+n \leqq k+l;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions (1), (99) *and* (100); *r is an arbitrary integer which satisfies the inequality*

$$0 \leqq r \leqq k+l-m-n.$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & A^{m,n-l+1}_{k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{k}(s)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i}) \\ & + \bar{A}^{m,n-l+1}_{k} \sum_{\tau=0}^{k+l-m-n-r-1} \bar{\Omega}^{m,n-l+1}_{k}(\tau)\, G_{p,q}^{k,l-1,n}(z e^{(m+n-k-l+2\tau+1)\pi i}) \\ & + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (116)$$

Proof. The theorem can be established by induction. The formula is true if $r = k+l-m-n$, since (116) with $r = k+l-m-n$ is equivalent to (113) with $r = k+l-m-n$. We may therefore suppose $0 \leqq r \leqq k+l-m-n-1$ and assume that (116) with $r+1$ instead of r has yet been proved. Now it follows from (57), if $n \geqq l$,

$$\begin{aligned} G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r-2)\pi i} \,||\, a_t) = {} & e^{-2\pi i a_t}\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,||\, a_t) \\ & + 2\pi i\, e^{-\pi i a_t}\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2r-1)\pi i}). \end{aligned}$$

If this is substituted on the right-hand side of (116) with $r+1$ instead of r, the sum $\sum_{t=1}^{n-l+1}$ not only yields the sum $\sum_{t=1}^{n-l+1}$ in (116) but besides

$$2\pi i\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2r-1)\pi i}) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r+1)\pi i a_t} \triangle^{m,n-l+1}_{k}(t) \quad [36]$$

and this expression is by (59) equal to

$$\begin{aligned} -G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2r-1)\pi i}) \{ & A^{m,n-l+1}_{k}\, \Omega^{m,n-l+1}_{k}(r) \\ & - \bar{A}^{m,n-l+1}_{k}\, \bar{\Omega}^{m,n-l+1}_{k}(k+l-m-n-r-1) \}. \end{aligned}$$

The sums $\sum_{s=0}^{r}$ and $\sum_{\tau=0}^{k+l-m-n-r-2}$ on the right-hand side of (116) with $r+1$ instead of r reduce therefore to the sums $\sum_{s=0}^{r-1}$, respect. $\sum_{\tau=0}^{k+l-m-n-r-1}$ in (116). So the theorem is established.

§ 12. Extension of theorem 3.

In the same manner as formula (113) we may prove the formula conjugate to (113)

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & \bar{A}^{m,n-l+1}_{k} \sum_{s=0}^{r-1} \bar{\Omega}^{m,n-l+1}_{k}(s)\, G_{p,q}^{k,l-1,n}(z e^{(m+n-k-l+2s+1)\pi i}) \\ & + \sum_{t=1}^{n-l+1} e^{(k+l-m-n-2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z e^{(m+n-k-l+2r)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (117)$$

[36] This is still true if $n = l-1$, since the sums $\sum_{t=1}^{n-l+1}$ then vanish.

This relation holds, provided that the conditions (111), (112), (1), (99) and (100) are satisfied.

We now replace r by $k+l-m-n-r$ and s by τ. Then formula (117) reduces to

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = \overline{A}^{m,n-l+1}_{k} \sum_{\tau=0}^{k+l-m-n-r-1} \overline{\Omega}^{m,n-l+1}_{k}(\tau)\, G_{p,q}^{k,l-1,n}\left(z e^{(m+n-k-l+2\tau+1)\pi i}\right) \\ + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\; G_{p,q}^{k,l,n}\left(z\, e^{(k+l-m-n-2r)\pi i} \,||\, a_t\right); \end{aligned}\right\} \quad (118)$$

herein is r an arbitrary integer which satisfies the inequality

$$r \leqq \text{Min}\,(0, k+l-m-n).$$

We may now show that formula (116) holds under conditions which are much more general than those of theorem 3. Indeed, I will prove:

Theorem 4. *Suppose that k, l, m, n, p and q are integers which satisfy the conditions* (111); *further that the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions* (1), (99) *and* (100); *finally that r is an arbitrary integer* (*positive, negative or zero*).

Then formula (116) *is valid.*

Proof. Observing that $\Omega^{m,n-l+1}_{k}(s)$ and $\overline{\Omega}^{m,n-l+1}_{k}(s)$ vanish for $s = -1, -2, -3, \ldots$, we may distinguish six cases [37]:

First case: $m+n \geqq k+l$, $r \geqq 0$. Formula (116) reduces to (113).

Second case: $m+n \geqq k+l$, $k+l-m-n \leqq r \leqq 0$. Formula (116) reduces to (102) with $\lambda = -r$.

Third case: $m+n \geqq k+l$, $r \leqq k+l-m-n$. Formula (116) reduces to (118).

Fourth case: $m+n \leqq k+l$, $r \geqq k+l-m-n$. Formula (116) reduces to (113).

Fifth case: $m+n \leqq k+l$, $0 \leqq r \leqq k+l-m-n$. This is the case of theorem 3.

Sixth case: $m+n \leqq k+l$, $r \leqq 0$. Formula (116) reduces to (118).

§ 13. Some more lemmas.

Lemma 19. *Suppose that k, l, p, q, $\varkappa$ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ \varkappa \geqq 1,\ 0 \leqq \nu \leqq k \leqq q \quad \text{and} \quad l+\nu-1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions

$$a_j - b_h \neq 1, 2, 3, \ldots \; (j = \nu+1, \ldots, l+\nu-1;\ h = 1, \ldots, k), \quad (119)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \; (j = 1, \ldots, \nu;\ t = 1, \ldots, \nu;\ j \neq t). \quad (120)$$

---

[37] Comp. also definition 4.

Then the following formula holds [38]):

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &- \sum_{h=1}^{k-\nu+\varkappa-1} \Omega_{k}^{0,\nu}(h)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{-2h\pi i}) \\ &- \frac{1}{A_{k}^{0,\nu}} \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-1)\pi i a_\sigma}\, \triangle_{k}^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \parallel a_\sigma). \end{aligned}\right\} \quad (121)$$

Proof. If we put $m = 0$, $n = l + \nu - 1$, $r = k - \nu + \varkappa$ and $z = \zeta\, e^{(\nu-k)\pi i}$ in (113) and suppose that $\nu \leqq k$, then we find (121), because of (5) and (50).

Lemma 20. *Suppose that k, l, p, q, $\varkappa$, λ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 1 \leqq \lambda \leqq \varkappa,\ 0 \leqq \nu \leqq k \leqq q \quad \text{and} \quad l + \nu - 1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &\sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi_{\nu,k}^{k,0}(h;\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ &- \frac{1}{A_{k}^{0,\nu}} \sum_{\tau=0}^{\lambda-1} \Phi_{\nu,k}^{k,0}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma}\, \triangle_{k}^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \parallel a_\sigma). \end{aligned}\right\} \quad (122)$$

Proof. From (74) (with $\lambda = 1$) and (73) (with $\lambda = 0$) it follows

$$\Phi_{\nu,k}^{k,0}(h;1) = -\,\Omega^{0,\nu}{}_{k}(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

We further have by (74) if $\lambda \neq 1 - h$

$$\Phi_{\nu,k}^{k,0}(h;\lambda) = -\sum_{\tau=0}^{\lambda-2} \Phi_{\nu,k}^{k,0}(1;\tau)\, \Omega^{0,\nu}{}_{k}(h+\lambda-\tau-1) - \Phi_{\nu,k}^{k,0}(1;\lambda-1)\, \Omega^{0,\nu}{}_{k}(h);$$

in view of (74) we find therefore if $\lambda \neq 1 - h$

$$\Phi_{\nu,k}^{k,0}(h;\lambda) = \Phi_{\nu,k}^{k,0}(h+1;\lambda-1) - \Phi_{\nu,k}^{k,0}(1;\lambda-1)\, \Omega^{0,\nu}{}_{k}(h). \quad . \quad (124)$$

From (123) and (73) (with $\lambda = 0$) it appears that (122) with $\lambda = 1$ reduces to (121). Hence we may suppose $2 \leqq \lambda \leqq \varkappa$ and assume that (122) with $\lambda - 1$ instead of λ has already been proved.

Now formula (122) with $\lambda - 1$ instead of λ may be written in the following way

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &\Phi_{\nu,k}^{k,0}(1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2\lambda+2)\pi i}) \\ &+ \sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi_{\nu,k}^{k,0}(h+1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ - \frac{1}{A_{k}^{0,\nu}} \sum_{\tau=0}^{\lambda-2} &\Phi_{\nu,k}^{k,0}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma}\, \triangle_{k}^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \parallel a_\sigma). \end{aligned}\right\} \quad (125)$$

[38]) The products $\triangle_{k}^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w \parallel a_\sigma)$ on the right of (121) must be defined by a limiting process when $a_\sigma - b_h = 1, 2, 3, \ldots$ $(1 \leqq h \leqq k)$; comp. the Remark at the end of § 9.

The first term on the right-hand side of this relation is because of (121) with $\zeta e^{(-2\lambda+2)\pi i}$ instead of ζ and $\varkappa - \lambda + 1$ instead of $\varkappa$ equal to

$$\Phi_{\nu,k}^{k,0}(1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2\lambda+2)\pi i})$$

$$= -\sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi_{\nu,k}^{k,0}(1;\lambda-1)\, \Omega_k^{0,\nu}(h)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i})$$

$$-\frac{\Phi_{\nu,k}^{k,0}(1;\lambda-1)}{A_k^{0,\nu}} \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\lambda+1)\pi i a_\sigma} \triangle_k^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \| a_\sigma).$$

If this is substituted on the right-hand side of (125), then (125) reduces in virtue of (124) to (122), so that the lemma has been proved.

Lemma 21. *Suppose that k, l, p, q, r and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ r \geqq 1,\ 0 \leqq \nu \leqq k \leqq q \text{ and } l+\nu-1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) &= \sum_{h=1}^{k-\nu} \Phi_{\nu,k}^{k,0}(h;r)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2r+2)\pi i}) \\ &- \frac{1}{A_k^{0,\nu}} \sum_{\sigma=1}^{\nu} e^{(k-\nu+1)\pi i a_\sigma}\, \Theta_\nu^{k,0}(\sigma;r-1)\, \triangle_k^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2r+1)\pi i} \| a_\sigma). \end{aligned}\right\} \quad (126)$$

Proof. From (80) it follows

$$\sum_{\tau=0}^{r-1} e^{(2r-2\tau-2)\pi i a_\sigma}\, \Phi_{\nu,k}^{k,0}(1;\tau) = \Theta_\nu^{k,0}(\sigma;r-1).$$

We therefore find (126) if we put $\varkappa = \lambda = r$ in (122).

Remark. Formula (122) is also valid if the following conditions are satisfied: k, l, p, q, $\varkappa$, λ and ν are integers with

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq k \leqq q,\ \nu \geqq 0,\ l+\nu-1 \leqq p \leqq q,\ \lambda \leqq 0 \text{ and } \varkappa \geqq 1+\nu-k;$$

the numbers $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the condition (119).

For, if $\lambda \leqq 0$, the sum $\sum_{h=1}^{k-\nu+\varkappa-\lambda}$ on the right-hand side of (122) is because of (73) and (75) equal to $G_{p,q}^{k,l-1,l+\nu-1}(\zeta)$; since $\Phi_{\nu,k}^{k,0}(1;\tau) = 0$ for $\tau < 0$, the sum $\sum_{\tau=0}^{\lambda-1}$ is zero for $\lambda \leqq 0$ (comp. definition 4).

Similarly formula (126) is also true under the following conditions: k, l, p, q, r and ν are integers with

$$l \geqq 1,\ 1 \leqq 1+\nu \leqq k \leqq q,\ 1+\nu-k \leqq r \leqq 0 \text{ and } l+\nu-1 \leqq p \leqq q;$$

the numbers $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the condition (119).

Lemma 22. *Suppose that k, l, m, n, p, q and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,\ 0 \leqq n-l+1 \leqq \nu \leqq k \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{s=0}^{\lambda-1} \Omega_k^{m,n-l+1}(s)\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{-2s\pi i}) \\ &= \sum_{h=1}^{k-\nu} \{\Phi_{\nu,k}^{m,n-l+1}(h;\lambda) - \Omega_k^{m,n-l+1}(h+\lambda-1)\}\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(-2h-2\lambda+2)\pi i}) \\ &- \frac{1}{A_k^{0,\nu}} \sum_{\tau=1}^{\nu} e^{(k-\nu+1)\pi i a_\tau}\, \Theta_\nu^{m,n-l+1}(\sigma;\lambda-1)\, \triangle_k^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w\, e^{(2\nu-2k-2\lambda+1)\pi i} \mid a_\tau). \end{aligned}\right\} \quad (127)$$

P r o o f. We first suppose $\lambda \leqq 0$. Then the left-hand side of (127) vanishes since $\Omega_k^{m,n-l+1}(s) = 0$ for $s < 0$. Because of $\Phi_{\nu,k}^{m,n-l+1}(1;\tau) = 0$ if $\tau < 0$, it appears from (71)

$$\Phi_{\nu,k}^{m,n-l+1}(h;\lambda) - \Omega_k^{m,n-l+1}(h+\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

We further have by (54)

$$\Theta_\nu^{m,n-l+1}(\sigma;\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

Hence formula (127) is certainly true if $\lambda \leqq 0$.

We now consider the case with $\lambda > 0$. Because of (77) we have

$$\sum_{s=0}^{\lambda-1} \Omega_k^{m,n-l+1}(s)\, \Phi_{\nu,k}^{k,0}(h;\lambda-s) = \Phi_{\nu,k}^{m,n-l+1}(h;\lambda) - \Omega_k^{m,n-l+1}(h+\lambda-1); \quad (128)$$

besides it follows from (79)

$$\sum_{s=0}^{\lambda-1} \Omega_k^{m,n-l+1}(s)\, \Theta_\nu^{k,0}(\sigma;\lambda-s-1) = \Theta_\nu^{m,n-l+1}(\sigma;\lambda-1) \quad (129)$$

If we replace in (126) ζ by $w\, e^{-2s\pi i}$ and r by $\lambda - s$ and use (128) and (129), we easily find (127).

Lemma 23. *Suppose that k, l, m, n, p, q, μ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,$$
$$0 \leqq n-l+1 \leqq \nu,\ 0 \leqq \mu \leqq k-\nu \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} & A_k^{m,n-l+1} \sum_{s=0}^{\lambda-1} \Omega_k^{m,n-l+1}(s)\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{-2s\pi i}) \\ = & A_k^{m,n-l+1} \sum_{h=1}^{k-\nu-\mu} \{\Phi_{\nu,k}^{m,n-l+1}(h;\lambda) - \Omega_k^{m,n-l+1}(h+\lambda-1)\}\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{(-2h-2\lambda+2)\pi i} \\ & - \bar{A}_k^{0,\nu} B_\nu^{m,n-l+1} \sum_{\varkappa=1}^{\mu} \Psi_{\nu,k}^{m,n-l+1}(\varkappa;\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\varkappa)\pi i}) \\ - & B_\nu^{m,n-l+1} \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+1)\pi i a_\sigma}\, \Theta_\nu^{m,n-l+1}(\sigma;\lambda-1)\, \Delta_k^{0,\nu}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,||\, a_\sigma). \end{aligned}\right\} \quad (130)$$

P r o o f. From the definitions 5 and 6 it follows

$$\frac{A_k^{m,n-l+1}}{A_k^{0,\nu}} = B_\nu^{m,n-l+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Formula (130) with $\mu = 0$ is therefore equivalent to (127). Hence we may suppose $1 \leqq \mu \leqq k-\nu$ and assume that (130) with $\mu - 1$ instead of μ has already been proved.

Now it follows from (57), if $\nu \geqq 1$,

$$\left.\begin{aligned} G_{p,q}^{k,l,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu-1)\pi i} \,||\, a_\sigma) = & e^{-2\pi i a_\sigma}\, G_{p,q}^{k,l,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,||\, a_\sigma) \\ & + 2\pi i e^{-\pi i a_\sigma}\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu)\pi i}). \end{aligned}\right\} \quad (132)$$

If this is substituted on the right-hand side of (130) with $\mu - 1$ instead of μ, the expression $-B_\nu^{m,n-l+1} \sum_{\sigma=1}^{\nu}$ not only yields the corresponding expression in (130) but besides

$$-2\pi i\, B_\nu^{m,n-l+1}\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times$$

$$\times \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+2)\pi i a_\sigma}\, \Theta_\nu^{m,n-l+1}(\sigma;\lambda-1)\, \Delta_k^{0,\nu}(\sigma) \quad [39])$$

and this expression is by virtue of (80) equal to

$$-2\pi i\, B_\nu^{m,n-l+1}\, G_{p,q}^{k,l-1,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times$$

$$\times \sum_{\tau=0}^{\lambda-1} \Phi_{\nu,k}^{m,n-l+1}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\lambda-2\mu-2\tau)\pi i a_\sigma}\, \Delta_k^{0,\nu}(\sigma)$$

$$= G_{p,q}^{k,l-1,l+\nu-1}(w e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \sum_{\tau=0}^{\lambda-1} \Phi_{\nu,k}^{m,n-l+1}(1;\tau) \times$$

$$\times \{A_k^{0,\nu} B_\nu^{m,n-l+1}\, \Omega_k^{0,\nu}(k-\nu+\lambda-\mu-\tau) - \bar{A}_k^{0,\nu} B_\nu^{m,n-l+1}\, \bar{\Omega}_k^{0,\nu}(\mu+\tau-\lambda)\}$$

(after (59)).

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[39]) This is still true for $\nu = 0$, since the sums $\sum_{\sigma=1}^{\nu}$ then vanish.

The right-hand side of this relation is on account of (131), (71) (with $h = k - \nu - \mu + 1$) and (81) equal to

$$G_{p,q}^{k,l-1;l+\nu-1}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times$$

$$\times [A_k^{m,n-l+1}\{\Omega_k^{m,n-l+1}(k-\nu+\lambda-\mu) - \Phi_{\nu,k}^{m,n-l+1}(k-\nu-\mu+1;\lambda)\}$$

$$- \overline{A}_k^{0,\nu} B_\nu^{m,n-l+1} \Psi_{\nu,k}^{m,n-l+1}(\mu;\lambda)].$$

It appears therefore that the sums $\sum_{h=1}^{k-\nu-\mu+1}$ and $\sum_{\varkappa=1}^{\mu-1}$ on the right-hand side of (130) with $\mu - 1$ instead of μ reduce by the substitution (132) to the corresponding sums $\sum_{h=1}^{k-\nu-\mu}$ and $\sum_{\varkappa=1}^{\mu}$ in (130). This establishes the lemma.

Mathematics. — *Ueber die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen.* By L. BERWALD †. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of May 25, 1946.)

Es gibt im Wesentlichen zwei Theorien der Parallelübertragung in Finslerschen Räumen [1]), von denen die eine auf E. NOETHER [2]) zurückgeht und 1925 vom Verfasser [3]) entwickelt wurde, während die andere von E. CARTAN [4]) (1933) stammt. Herr CARTAN ist wiederholt auf die Beziehungen zurückgekommen, die zwischen den beiden Theorien bestehen. Er bemerkt in dieser Hinsicht unter Anderem: „Il y aurait intérêt à établir effectivement les formules permettant de retrouver les tenseurs de l'une des théories par les procédés de l'autre théorie" [5]). Dieser Zusammenhang zwischen den wichtigsten Tensoren der beiden Theorien soll im Folgenden auseinandergesetzt werden.

1. Der Kürze halber benutzen wir die Bezeichnungen der Abhandlung [4]) von CARTAN. Wir ersetzen ferner die Ableitung $\frac{\partial \dots}{\partial x'^i}$ der Theorie des Verfassers durch die CARTANsche Ableitung

$$\dots|\!|_i = L\frac{\partial \dots}{\partial x'^i}. \quad \dots \quad (1)$$

Weiter erinnern wir an die Gleichungen

$$l^i = \frac{x'^i}{L}, \qquad l_i = \frac{\partial L}{\partial x'^i}, \qquad A_{ihj} = \tfrac{1}{2} g_{ih|\!|j}, \quad \dots \quad (2)$$

die Tensoren der CARTANschen Theorie durch solche der Theorie des Verfassers ausdrücken. Endlich ist der Tensor

$$g_{ik} = \frac{1}{2}\frac{\partial^2 (L^2)}{\partial x'^i \, \partial x'^k} \quad \dots \quad (2a)$$

beiden Theorien gemeinsam.

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[1]) Die Parallelübertragung von J. L. SYNGE, A generalisation of the Riemannian line-element. Trans. Amer. Math. Soc. 27 (1925), S. 61 und J. H. TAYLOR, A generalization of Levi-Civita's parallelism and the Frenet formulas, ebenda, S. 246, kommt hier nicht in Betracht, weil ihr keine Krümmungstheorie des Raumes entspricht.

[2]) E. NOETHER, Invarianten beliebiger Differentialausdrücke. Göttinger Nachr. 1918, S. 37.

[3]) Vgl. insbesondere L. BERWALD, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. Math. Zeitschrift, 25 (1926), S. 40.

[4]) E. CARTAN, Les espaces de Finsler. Actualités scientifiques et industrielles 79. Hermann et Cie., Paris 1934.

[5]) E. CARTAN, Les espaces de Finsler. Abh. a. d. Seminar f. Vektor- und Tensoranalysis. IV. Moskau und Leningrad 1937, S. 70; bes. S. 81.

Zunächst besprechen wir, wie die beiden *Parallelübertragungen* zusammenhängen. Die Parameter der Parallelübertragung des Verfassers sind die I^k_{ij}, die der Parallelübertragung von CARTAN (wenn das Stützelement parallel mitübertragen wird), die Γ^{*k}_{ij}. Herr CARTAN selbst hat bereits die grundlegende Beziehung

$$\boxed{I_{ihj} = \Gamma^*_{ihj} + A_{ihj|0} \quad ; \quad I_i{}^j{}_k = \Gamma^{*j}_{i\ k} + A_i{}^j{}_{k|0}} \quad . \quad . \quad . \quad (3)$$

aufgestellt [6]). Aus ihr folgt, dass das kovariante Differential des Einheitsvektors in beiden Theorien dasselbe ist.

Um auch umgekehrt die Γ^*_{ihj} durch geometrische Objekte auszudrücken, die in der Theorie des Verfassers auftreten, bezeichnen wir die kovariante Ableitung im Sinne des Verfassers durch einen Strichpunkt; z.B.

$$g_{ih;j} = \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ih}}{\partial x'^m}\frac{\partial I^m}{\partial x^j} - I_{ihj} - I_{hij}. \quad . \quad . \quad . \quad . \quad (4)$$

Dann gilt insbesondere

$$L_{;p} = 0, \quad I_{i;j} = 0. \text{ [7])}$$

Wegen der Identität $g_{ih|j} = 0$ und der Symmetrie der A_{ihj} ist

$$\tfrac{1}{2}\, g_{ih;j} = -A_{ihj|0}. \text{ [8])} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Die gesuchte inverse Relation lautet also

$$\boxed{\Gamma^*_{ihj} = I_{ihj} + \tfrac{1}{2}\, g_{ih;j}.} \text{ [9])}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Mittels der Gleichungen (3), (5), (6) können wir unmittelbar einige Tensoren der einen Theorie durch solche der anderen ausdrücken. Wegen der Symmetrie der $g_{ih||j}$, $g_{ih;j}$ [10]) ergibt sich

$$A_{ihj|k} = \tfrac{1}{2}\, g_{ih\,|j;k} - \tfrac{1}{4}\, g^{mp}\,(g_{ih\,|m}\, g_{jp;k} + g_{hj||m}\, g_{ip;k} + g_{ji|\,m}\, g_{hp;k}) \quad (7)$$

und umgekehrt

$$\tfrac{1}{2}\, g_{ih||j;k} = A_{ihj||k} - A_i{}^m{}_k\, A_{jmk|0} - A_h{}^m{}_j\, A_{imk|0} - A_j{}^m{}_i\, A_{hmk|0}. \quad (8)$$

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[6]) a.a.O. [4]), Nr. 17.

[7]) a.a.O. [3]), Formeln (25), (35).

[8]) Siehe J. A. SCHOUTEN und J. HAANTJES, Ueber die Festlegung von allgemeinen Massbestimmungen und Uebertragungen in Bezug auf ko- und kontravariante Vektordichten. Monatshefte f. Math. u. Phys. 43 (1936), S. 161; bes. Formel (94).

[9]) Eine äquivalente Gleichung bei A. KAWAGUCHI, Beziehung zwischen einer metrischen linearen Uebertragung und einer nicht-metrischen in einem allgemeinen metrischen Raume. Diese Proceedings 40 (1937), S. 596, Nr. 3.

[10]) Vgl. die erste Gleichung (9).

Ferner gelten die Formeln [11])

$$\left.\begin{aligned} g_{ih;k} &= L\, l_p \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k};\\ g_{ih;k||j} &= L\, g_{jp} \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k} + L^2 \frac{\partial^4 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k\, \partial x'^j};\\ g_{ih||j;k} - g_{ih;k||j} &= L\left(g_{ip} \frac{\partial^3 I^p}{\partial x'^h\, \partial x'^j\, \partial x'^k} + g_{hp} \frac{\partial^3 I^p}{\partial x'^j\, \partial x'^i\, \partial x'^k}\right). \end{aligned}\right\} \quad (9)$$

mit deren Hilfe der Tensor $A_{ihj|k}$ durch die Tensoren $L\frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k}$ und $L^2 \frac{\partial^4 I^p}{\partial x'^i \partial x'^h \partial x'^j\, \partial x'^k}$ dargestellt werden kann.

Um umgekehrt den Tensor $L \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^k\, \partial x'^h}$ durch Tensoren der CARTAN-schen Theorie auszudrücken, leite man zunächst die zweite Identität (3) nach x'^h ab und ersetze sodann $L \frac{\partial\, \Gamma^{*j}_{i\ k}}{\partial x'^h}$ mittels der Gleichung (44) der Arbeit [4]) von CARTAN durch die $A_i{}^j{}_k$ und ihre kovarianten Ableitungen. Ferner beachte man die Gleichungen [12]).

$$\left.\begin{aligned} A_{imk|0||h} &= A_{ikh||m|0} - l_m A_{ikh|0} + A_{ikm|h} - A_i{}^p{}_k A_{mph|0} - A_k{}^p{}_m A_{iph|0}\\ &\quad - A_m{}^p{}_i A_{kph|0},\\ A_i{}^j{}_{k|0||h} &= -2 A_m{}^j{}_h A_i{}^m{}_{k|0} + g^{mj} A_{imk|0||h}. \end{aligned}\right\} \quad (10)$$

Man erhält dann schliesslich

$$\boxed{\begin{aligned} L \frac{\partial^3 I^j}{\partial x'^i\, \partial x'^k\, \partial x'^h} &= g^{mj}(A_{ikh||m|0} - A_{ikh|m}) -\\ &- l^j A_{ikh|0} + A_i{}^j{}_{k|h} + A_k{}^j{}_{h|i} + A_h{}^j{}_{i|k}\\ &- 2(A_i{}^j{}_m A_k{}^m{}_{h|0} + A_k{}^j{}_m A_h{}^m{}_{i|0} + A_h{}^j{}_m A_i{}^m{}_{k|0}). \end{aligned}} \quad (11)$$

Wegen (7), (9), (11) sind die Identitäten

$$A_{ikh|m} = 0 \quad \text{und} \quad \frac{\partial^3 I^j}{\partial x'^i\, \partial x'^k\, \partial x'^h} = 0$$

äquivalent. Jede von ihnen kennzeichnet die „affin-zusammenhängenden Räume" des Verfassers [13]).

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[11]) a.a.O. [3]) Formeln (41), (37), (25).

[12]) Zur Ableitung der ersten Gleichung (10) führt man am Besten die Grössen

$$C_{ijh} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x'^h} = \frac{1}{4}\frac{\partial^3 (L^2)}{\partial x'^i\, \partial x'^j\, \partial x'^h},\quad C_{ijhk} = \frac{\partial C_{ijh}}{\partial x'^k}$$

ein. Vgl. a.a.O. [4]), S. 11 und 19.

[13]) L. BERWALD, a.a.O. [3]), S. 47; E. CARTAN, a.a.O. [4]), Nr. 41.

Endlich schreiben wir noch die Beziehung

$$l_{i\|j;k} = g_{ij;k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

an, die es gestattet, die $g_{ij;k}$ durch Ableitungen der l_i zu ersetzen.

2. Wir gehen jetzt zu den *Krümmungstensoren* über.

Der „Grundtensor der Krümmung" des Verfassers [14]), der durch

$$K^{j}_{.kh} = \frac{\partial^2 I^j}{\partial x'^k \, \partial x^h} - \frac{\partial^2 I^j}{\partial x'^h \, \partial x^k} - I_k{}^j{}_r \frac{\partial I^r}{\partial x'^h} + I_h{}^j{}_r \frac{\partial I^r}{\partial x'^k} \quad . \quad . \quad (13)$$

gegeben ist, tritt auch in der CARTANschen Theorie auf. Es ist nämlich

$$\boxed{K^{j}_{.kh} = L R_0{}^j{}_{kh}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Durch Ableitung des Tensors $K^{j}_{.kh}$ nach x'^i erhält man den Krümmungstensor $K_{i.kh}^{\ j}$ des Verfassers [14]):

$$K_{i.kh}^{\ j} = \frac{\partial K^{j}_{.kh}}{\partial x'^i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Aus (14), (15) folgt leicht

$$K_{i.kh}^{\ j} = l_i R_0{}^j{}_{kh} + R_0{}^j{}_{kh\|i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

und

$$K_{ijkh} = R_{ijkh} - 2 A_i{}^m{}_j R_{0mkh} + l^m R_{mjkh\|i} \quad . \quad . \quad . \quad . \quad (17)$$

Aus jeder der beiden Formeln (16), (17) ergibt sich

$$K_{0jkh} = R_{0jkh} \quad , \quad K_{i0kh} = R_{i0kh} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

wenn

$$l^m K_{mjkh} = K_{0jkh} \quad , \quad l^m K_{imkh} = K_{i0kh}$$

gesetzt wird. Ferner zeigt (17), dass der Tensor

$$T_{ijkh} = -(g_{ij;k;h} - g_{ij;h;k}) = \frac{\partial g_{ij}}{\partial x'^p} K^{p}_{.kh} + K_{ijkh} + K_{jikh}, \quad [15]) \quad (19)$$

der in der Theorie des Verfassers als „Tensor der Streckenkrümmung" auftritt, durch

$$T_{ijkh} = l^m (R_{mjkh\|i} + R_{mikh\|j}) - 2 A_i{}^m{}_j R_{0mkh} \quad . \quad . \quad . \quad (20)$$

gegeben ist.

Um umgekehrt den Krümmungstensor R_{ijkh} der CARTANschen Theorie durch Tensoren der Theorie des Verfassers auszudrücken, kann man von der Formel (XIX) der Abhandlung [4]) von CARTAN ausgehen. Mittels (6), (13), (14), (19) findet man durch eine Rechnung, die keine Schwierigkeit bietet:

$$\boxed{R_{ijkh} = \tfrac{1}{2}(K_{ijkh} - K_{jikh}) - \tfrac{1}{4} g^{mp} (g_{im;k} g_{jp;h} - g_{im;h} g_{jp;k}).} \quad (21)$$

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[14]) a.a.O. [3]), S. 46.

[15]) a.a.O. [3]), Formeln (36) und (54).

Aus $R_{ijkh} = 0$ folgt wegen (17) $K_{ijkh} = 0$, aber das Umgekehrte gilt im Allgemeinen nicht.

Die Gleichung (17) kann durch eine andere Formel ersetzt werden. Um sie abzuleiten, hat man die „BIANCHIsche Identität"

$$\left.\begin{aligned} R_{mjkh||l} + R_{jrkh}\, A_m{}^r{}_l - R_{mrkh}\, A_j{}^r{}_l + P_{mjhl|k} - P_{mjkl|h} \\ + P_{mjkr}\, A_h{}^r{}_{l|0} - P_{mjhr}\, A_k{}^r{}_{l|0} + S_{mjrl}\, R_0{}^r{}_{hk} = 0 \end{aligned}\right\} \quad (22)$$

der CARTANschen Theorie heranzuziehen und die Gleichungen [16])

$$A_0{}^r{}_i = 0 \quad , \quad S_{0jmq} = 0 \quad , \quad P_{0jkr} = A_{jkr|0} \quad . \quad . \quad . \quad (23)$$

zu berücksichtigen. Man erhält so

$$\boxed{\begin{aligned} K_{ijkh} = R_{ijkh} + A_{imk|0}\, A_j{}^m{}_{h|0} - A_{imh|0}\, A_j{}^m{}_{k|0} - A_i{}^m{}_j\, R_{0mkh} \\ + A_{ijk|0|h} - A_{ijh|0|k}. \end{aligned}} \quad (24)$$

Es ist also

$$\left.\begin{aligned} \tfrac{1}{2}\,(K_{ijkh} - K_{jikh}) &= R_{ijkh} + A_{imk\,0}\, A_j{}^m{}_{h|0} - A_{imh\,0}\, A_j{}^m{}_{k|0}, \\ \tfrac{1}{2}\, T_{ijkh} &= A_{ijk|0|h} - A_{ijh|0|k}. \end{aligned}\right\} \quad (25)$$

Schliesslich werde noch für den zweiten Krümmungstensor P_{ijkh} von CARTAN der Ausdruck

$$\boxed{P_{ijkh} = \tfrac{1}{2}\,(g_{kh;\,i||j} - g_{kh;\,j||i}) - \tfrac{1}{4}\, g^{mp}\,(g_{ik;\,m}\, g_{jh||p} - g_{jk;\,m}\, g_{ih||p})} \quad (26)$$

angegeben, den man erhält, wenn man von Gleichung (XVII) der Arbeit [4]) von CARTAN ausgeht und die Formeln (3), (7), (9) benutzt.

3. Die vorstehenden Formeln lassen sich vor Allem auf *die Finslerschen Räume* anwenden, *in denen die Parallelübertragung eines Linienelementes, wenn sein Zentrum einen gegebenen Weg beschreibt, die Winkelmetrik der Linienelemente* (*im Sinn von* CARTAN [17])) *ungeändert lässt. Diese Räume sind nach* CARTAN [17]) *durch* $A_{ihj|0} = 0$, also *durch das Zusammenfallen der beiden betrachteten Parallelübertragungen gekennzeichnet.* Sie sind für $n = 2$ mit den „LANDSBERGschen Räumen" des Verfassers [18]) identisch. Nach (5) und (9) können sie auch durch $g_{ih;\,j} = 0$ oder durch

$$l_p\, \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k} = 0$$

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[16]) a.a.O. [4]), Nr. 9, 10, 36, 37.

[17]) a.a.O. [4]), Nr. 11, 40.

[18]) L. BERWALD, Ueber zweidimensionale allgemeine metrische Räume. Journal f. d. reine u. angew. Mathematik 156 (1927), S. 191; bes. S. 208 f., und § 5; E. CARTAN, Sur un problème d'équivalence et la théorie des espaces métriques généralisés. Mathematica, Cluj 4 (1930), S. 114; bes. S. 133 f.

charakterisiert werden. Der Rang der Matrix

$$\left(\frac{\partial^3 I^1}{\partial x'^i \partial x'^h \partial x'^j}, \frac{\partial^3 I^2}{\partial x'^i \partial x'^h \partial x'^j}, \ldots \frac{\partial^3 I^n}{\partial x'^i \partial x'^h \partial x'^j}\right), (i, h, j = 1, 2 \ldots n)$$

von der wir nur eine Zeile angeschrieben haben, ist für diese Räume also höchstens $n - 1$. Der Ausdruck (26) zeigt ferner, *dass für die in Rede stehenden Räume der Krümmungstensor* P_{ijkh} *Null ist.* Endlich gilt für sie

$$R_{ijkh} = \tfrac{1}{2}(K_{ijkh} - K_{jikh}), K_{ijkh} = R_{ijkh} - A_i{}^m{}_j R_{0mkh}.$$

Die betrachteten Räume gehören zu der umfassenderen Klasse der *Finslerschen Räume, für die der Tensor* T_{ijkh} *verschwindet. Diese Räume* (*die „Räume ohne Streckenkrümmung"* in der Theorie des Verfassers) *sind durch*

$$A_{ijk|0|h} - A_{ijh|0|k} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

gekennzeichnet.

Applied Mechanics. — *Large distortions of circular rings and straight rods.* I. By A. VAN WIJNGAARDEN. (Nationaal Luchtvaartlaboratorium.) (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of May 25, 1946.)

1. *Introduction.* In a series of three papers that recently have appeared in these Proceedings [1]), BIEZENO and KOCH dealt with the problem of the distortions of a closed circular ring, loaded by an arbitrary equilibrium-system of forces in its own plane. At first sight, this problem might seem to be rather elementary, but the real aim of their investigation was to give a solution, valid also for loads of such a size, that the influence of the distortions on the distribution of the bending moments in the ring, that is neglected in the elementary theory, is appreciable. On the other hand, however, their treatment of the problem is still limited, a fact, the authors are of course fully aware of. Indeed, the authors use the differential-equation of the slightly curved beam to describe the distortions, an approximation, that can only be made, if the difference of the slope of the elastic line in its deformed and its undeformed shape is small. So, their solution. though it tends farther than the solution, obtained by the use of elementary means, f.i. CASTIGLIANO's theorem, is still limited to not too large distortions. Moreover, the distortions, as calculated by their method, appear as an infinite series of terms. As the calculation of the separate terms is rather tedious, we have, for practical reasons, to confine ourselves to a small number of terms. The authors f.i. use only two terms. So there might be an uncontrolable influence of the neglected terms.

It might be interesting therefore, to give an exact treatment of the same problem, partly for having the exact results for themselves, partly for providing a check to the approximate results. But it is obvious, that the exact solution of the general problem — with arbitrarily varying, continuous, tangential and normal loads — might be extremely difficult, if not impossible. As however both the particular problems, the authors deal with, viz. the principal one, treated in the third paper, which led the authors to their investigation, as well as the introducing problem, treated in the second paper, are problems, in which only discrete forces are applied at

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1) C. B. BIEZENO and J. J. KOCH, The generalized buckling problem of the circular ring, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 447—468 (1945).

The circular ring under the combined action of compressive and bending loads, ibid., **49**, 1—8 (1946).

On the non-linear deflection of a semi-circular ring, clamped at both ends, ibid., **49**, 139—145 (1946).

the ring, we too will confine ourselves here to such problems. It will be shown, that we can treat them in an exact way indeed. We shall apply our theory to the two problems, just mentioned, so as to provide the required check.

The method used in this paper, is principally based on the same ideas, as used already by some authors in the classical period of the theory of elasticity for elastica-problems. The method is not at all restricted to closed circular rings, and many other problems of large distortions can be solved by the same means. As an illustration, we shall give the exact solutions of two problems of a straight beam. The first of these problems has been treated allready approximately by SONNTAG [2]) and afterwards graphically by BIEZENO [3]).

2. *The distortion of a circular ring.* Let the problem be to determine the distortion of a part of a circular ring loaded only at its ends. At the one end P_0, the load consists of a couple M_0, a normal force N_0 and a shearing force D_0 (see fig. 1), and at the other end P_1, it consists of a

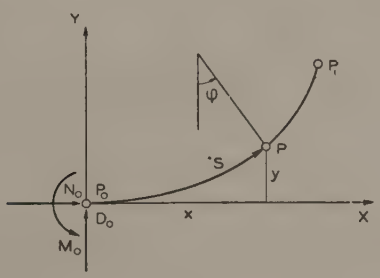

Fig. 1.

couple M_1, a normal force N_1 and a shearing force D_1, so that the ring is in equilibrium. A system of axes $X - Y$, with its origin in P_0 is introduced such that the X-axis coincides with the tangent in P_0 (in the distorted state) in the direction of P_1, and the Y-axis coincides with the normal in P_0 in that direction, in which (in the undistorted state) the centre of the circle lies. Then it follows from reasons of equilibrium, that the bending moment M (positive, if it decreases the curvature) in an arbitrary point $P(x, y)$ of the ring is:

$$M = M_0 - D_0 x + N_0 y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if r is the radius of the unbent ring, R the radius of curvature of the bent ring and EI the flexural rigidity, then the differential equation of

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[2]) R. SONNTAG, Der beiderseits gestützte, symmetrisch belastete gerade Stab mit endlicher Durchbiegung und seine Stabilität, Ing. Arch., **12**, 283—306 (1943).

[3]) C. B. BIEZENO, On a special case of bending, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **45**, 438—442 (1942).

the ring, if the distortions due to tension and shearing are neglected against those due to bending, is [4]):

$$M = EI\left(\frac{1}{r} - \frac{1}{R}\right) \ldots\ldots\ldots \quad (2)$$

We pass from the XY-system to a system of natural coordinates s and η, where s is the length of the arc P_0P, and φ is the angle between the Y-axis and the normal in P. As $1/R = d\varphi/ds$, $dx/ds = \cos\varphi$ and $dy/ds = \sin\varphi$, we find by combining (1) and (2) and differentiating with respect to s:

$$EI\frac{d^2\varphi}{ds^2} - D_0\cos\varphi + N_0\sin\varphi = 0. \ldots\ldots \quad (3)$$

If the following nondimensional quantities are introduced

$$\alpha = \frac{2\,D_0\,r^2}{EI},\ \tan\beta = \frac{D_0}{N_0} = \frac{1}{\nu},\ \sigma = \frac{s}{r},\ \xi = \frac{x}{r},\ \eta = \frac{y}{r} \quad \ldots \quad (4)$$

equation (3) can be written as:

$$\frac{d^2\varphi}{d\sigma^2} + \frac{\alpha}{2\sin\beta}\sin(\varphi - \beta) = 0. \ldots\ldots\ldots \quad (5)$$

After multiplication by $d\varphi/d\sigma$ integration of this equation leads to:

$$\left(\frac{d\varphi}{d\sigma}\right)^2 - \frac{\alpha}{\sin\beta}\cos(\varphi - \beta) = \frac{\alpha}{\sin\beta}\gamma, \ldots\ldots \quad (6)$$

where γ is a constant of integration. It follows that:

$$\frac{d\varphi}{d\sigma} = \pm\sqrt{\frac{\alpha}{\sin\beta}}\sqrt{\gamma + \cos(\varphi - \beta)}. \ldots\ldots \quad (7)$$

The ambigious sign is of much importance. If the load is small, the shape of the elastic line does not differ much from the original circle and there will be no inflexions. In this case, φ increases monotonously with s from zero to its final value. With higher loads, it may however occur that there are inflexions in the elastic line. Then for instance, φ decreases from zero to a lower limit φ_{min}, increases from that value to an upper limit φ_{max}, and decreases again to its final value. In this case successively the minus-, plus- and minus-sign must be used in (7). We shall deal with this question in more detail further on, but denote from now all integrations with respect to φ by the symbol $\int ! d\varphi$, as a warning that the appropiate sign should be attributed to the integrand.

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[4]) In the corresponding formula (19) of the first paper in note [1]) on page 450, the factor r^2 has to be deleted.

With these precautions, the solution of equation (7) can be given in the following form:

$$\left.\begin{aligned} \sigma &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} ! \frac{d\varphi'}{\sqrt{\gamma+\cos(\varphi'-\beta)}} \\ \xi &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} ! \frac{\cos\varphi'\, d\varphi'}{\sqrt{\gamma+\cos(\varphi'-\beta)}} \\ \eta &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} ! \frac{\sin\varphi'\, d\varphi'}{\sqrt{\gamma+\cos(\varphi'-\beta)}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the boundary conditions of each particular problem, β, γ and the position of the eventual inflexions, should be determined as functions of α.

3. *The reduction of the integrals, involved in eqs.* (8). We reduce the integrals, involved in eqs. (8) to elementary functions and the standard elliptic integrals:

$$\left.\begin{aligned} E(k,\varphi) &= \int_0^{\varphi} \sqrt{1-k^2\sin^2\psi}\, d\psi \quad , \quad \mathsf{E}(k) = E\left(k, \frac{\pi}{2}\right), \\ F(k,\varphi) &= \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \quad , \quad \mathsf{K}(k) = F\left(k, \frac{\pi}{2}\right), \end{aligned}\right\} \quad . \quad (9)$$

by substituting $\psi = \frac{\varphi'-\beta}{2}$ and $k = \sqrt{\frac{2}{1+\gamma}}$. The result is:

$$\left.\begin{aligned} \sigma &= \sqrt{\frac{2\sin\beta}{\alpha}}\, f_1(k,\varphi) \Big|_{-\beta/2}^{(\varphi-\beta)/2} \\ \xi &= \sqrt{\frac{2\sin\beta}{\alpha}}\, \{\cos\beta\, f_2(k,\varphi) + \sin\beta\, f_3(k,\varphi)\} \Big|_{-\beta/2}^{(\varphi-\beta)/2} \\ \eta &= \sqrt{\frac{2\sin\beta}{\alpha}}\, \{\sin\beta\, f_2(k,\varphi) - \cos\beta\, f_3(k,\varphi)\} \Big|_{-\beta/2}^{(\varphi-\beta)/2} \end{aligned}\right\} \quad . \quad (10)$$

where

$$\left.\begin{aligned} f_1(k,\varphi) &= k\, F(k,\varphi) \\ f_2(k,\varphi) &= \frac{2\,E(k,\varphi) - (2-k^2)\,F(k,\varphi)}{k} \\ f_3(k,\varphi) &= \frac{2}{k}\sqrt{1-k^2\sin^2\varphi} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

The elliptic integrals $E(k, \varphi)$ and $F(k, \varphi)$ are tabulated for the case of real k and $|k| \leqq 1$, and real φ and $0 \leqq \varphi \leqq \pi/2$. But these conditions are not always satisfied in our problems. We shall see that, on the contrary, γ also assumes values $\gamma < -1$, so that k is purely imaginary, as well as values $-1 < \gamma < 1$, so that k is real, but $|k| > 1$. Furthermore φ can have values $\varphi < 0$ as well as $\varphi > \pi/2$. We therefore have to introduce some simple transformations as to be able to use the tabulated values.

If $|k| > 1$, we find, by substituting $k \sin \psi = \sin \psi'$ in eqs. (9), that

$$\left.\begin{aligned} E(k, \varphi) &= \frac{1}{k^*} E(k^*, \varphi^*) - \frac{1-k^{*2}}{k^*} F(k^*, \varphi^*) \\ F(k, \varphi) &= k^* F(k^*, \varphi^*) \\ &\text{with } k^* = 1/k \text{ and } \varphi^* = \sin^{-1} k \sin \varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

So we have reached our aim, if, at least, $|k \sin \varphi| \leqq 1$; this holds certainly true in our case, as otherwise $E(k, \varphi)$ and $F(k, \varphi)$ would be complex, in contradiction with the fact that σ, ξ and η are real. From (12) we find for the functions $f_j(k, \varphi)$ if $|k| > 1$:

$$\left.\begin{aligned} f_1(k, \varphi) &= F(k^*, \varphi^*) \\ f_2(k, \varphi) &= 2E(k^*, \varphi^*) - F(k^*, \varphi^*) \\ f_3(k, \varphi) &= 2k^* \cos \varphi^* \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If k is purely imaginary, we substitute in eqs. (9): $\psi = \frac{\pi}{2} - \psi'$. Then we obtain the result:

$$\left.\begin{aligned} E(k, \varphi) &= \frac{1}{\sqrt{1-k^{**2}}} \{\mathsf{E}(k^{**}) - E(k^{**}, \varphi^{**})\} \\ F(k, \varphi) &= \sqrt{1-k^{**2}} \{\mathsf{K}(k^{**}) - F(k^{**}, \varphi^{**})\} \\ &\text{with } k^{**} = \frac{-ik}{\sqrt{1-k^2}} \text{ and } \varphi^{**} = \frac{\pi}{2} - \varphi \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

Now k^{**} is real and $|k^{**}| < 1$. So we find from (14), if k is purely imaginary

$$\left.\begin{aligned} f_1(k, \varphi) &= ik^{**} \{\mathsf{K}(k^{**}) - F(k^{**}, \varphi^{**})\} \\ f_2(k, \varphi) &= \frac{-i}{k^{**}} \{2\mathsf{E}(k^{**}) - (2-k^{**2})\mathsf{K}(k^{**}) - 2E(k^{**}, \varphi^{**}) + \\ &\qquad + (2-k^{**2}) F(k^{**}, \varphi^{**})\} \\ f_3(k, \varphi) &= \frac{-2i}{k^{**}} \sqrt{1-k^{**2} \sin^2 \varphi^{**}} \end{aligned}\right\} \quad . \quad (15)$$

If φ is real, but does not satisfy the condition $0 \leqq \varphi \leqq \pi/2$, we can always

reduce the integrals to the tabulated ones by means of the self-evident relations:

$$\left.\begin{aligned} E(k,\varphi) &= -E(k,-\varphi) \quad , \quad E(k,\varphi) = 2\,\mathsf{E}(k) - E(k,\pi-\varphi) \\ F(k,\varphi) &= -F(k,-\varphi) \quad , \quad F(k,\varphi) = 2\,\mathsf{K}(k) - F(k,\pi-\varphi) \end{aligned}\right\}. \qquad (16)$$

4. *Complications due to the inflexions of the elastic line.* We shall investigate now in more detail the question of the inflexions. From equation (7) we learn that for the extreme values φ_{ext} of φ, for which inflexions occur, holds:

$$\gamma + \cos(\varphi_{ext} - \beta) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

It is obvious that inflexions can only occur, if $-1 \leqq \gamma \leqq 1$, so if $k \geqq 1$. Therefore it recommends itself to express our results in terms of k^*. We verify easily that the values $\varphi_{\min}$ resp. $\varphi_{\max}$ corresponding to the "lower" resp. "upper" inflexions, are given by

$$\left.\begin{aligned} \varphi_{\min} &= \beta - 2\sin^{-1} k^* \\ \varphi_{\max} &= \beta + 2\sin^{-1} k^* \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Attention should be given to the fact that the quantities $\varphi_{\min}$ and $\varphi_{\max}$, as defined by (18), are only mathematical quantities without a physical sense, unless inflexions really occur. So with sufficiently small α, we are certain that both $\varphi_{\min}$ and $\varphi_{\max}$ are lying outside the range of φ from zero to its final value (the latter value varies in general with α) and that they have no meaning for the problem. But of course we can calculate them, and see how they vary with increasing values of α. Then we see for instance $\varphi_{\min}$ increasing with α, until it reaches a maximum value equal to zero, and then decreasing again. From that moment it has a physical sense.

The quantity $(\varphi - \beta)/2$, that occurs in the limits of integration, takes for these special values of φ the form:

$$\left.\begin{aligned} (\varphi_{\min} - \beta)/2 &= -\sin^{-1} k^* \\ (\varphi_{\max} - \beta)/2 &= \sin^{-1} k^* \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

We can now distinguish several different cases of integration according to the number and kind of inflexions. With a view to the applications that we shall make of our theory, we shall confine ourselves to those cases, in which no more than two inflexions occur. Then there are four possibilities. In case I, there is no inflexion at all in the part of the ring under consideration. In case II, there is one lower inflexion, in case III one upper inflexion, and in case IV, there are two inflexions, a lower and an upper one. In case I, φ' increases directly from 0 to φ. In case II, φ' first decreases from 0 to $\varphi_{\min}$ (where the integrand has the negative sign) and then it increases from $\varphi_{\min}$ to φ. In case III, φ' increases from 0 to $\varphi_{\max}$ and then decreases from $\varphi_{\max}$ to φ, where the integrand has the negative sign. In case IV, φ' decreases from 0 to $\varphi_{\min}$, where the integrand has the

negative sign, then increases from $\varphi_{\min}$ to $\varphi_{\max}$, and at last decreases again from $\varphi_{\max}$ to φ, where the integrand has the negative sign.

Now, we have to see, what are the consequences of these considerations with respect to the limits of integration in the eqs. (10). If we denote the three functions f_1, f_2 and f_3, for a moment by f, then we have in the different cases:

$$f(k,\varphi)\Big|_{-\beta/2}^{(\varphi-\beta)/2} = \left\{\begin{array}{ll} f\left(k,\frac{\varphi-\beta}{2}\right)-f\left(k,-\frac{\beta}{2}\right) & \text{in case I} \\ f\left(k,\frac{\varphi-\beta}{2}\right)+f\left(k,-\frac{\beta}{2}\right)-2f(k,-\sin^{-1}k^*) & \text{in case II} \\ -f\left(k,\frac{\varphi-\beta}{2}\right)-f\left(k,-\frac{\beta}{2}\right)+2f(k,\sin^{-1}k^*) & \text{in case III} \\ -f\left(k,\frac{\varphi-\beta}{2}\right)+f\left(k,-\frac{\beta}{2}\right)+2f(k,\sin^{-1}k^*)- & \\ \qquad -2f(k,-\sin^{-1}k^*) & \text{in case IV} \end{array}\right\}. \quad (20)$$

But the functions $f_1(k,\varphi)$ and $f_2(k,\varphi)$ are odd functions of φ, whereas $f_3(k,\varphi)$ is an even function of φ. Moreover, the values assumed by the functions f for $\varphi = \sin^{-1}k^*$ are very special ones. Indeed, we have:

$$\left.\begin{array}{l} f_1(k,\sin^{-1}k^*) = \mathsf{K}(k^*) \\ f_2(k,\sin^{-1}k^*) = 2\,\mathsf{E}(k^*)-\mathsf{K}(k^*) \\ f_3(k,\sin^{-1}k^*) = 0 \end{array}\right\}. \quad \ldots\ldots \quad (21)$$

So we find at last:

$$f_1(k,\varphi)\Big|_{-\beta/2}^{(\varphi-\beta)/2} = \left\{\begin{array}{ll} f_1\left(k,\frac{\varphi-\beta}{2}\right)+f_1\left(k,\frac{\beta}{2}\right) & \text{in case I} \\ f_1\left(k,\frac{\varphi-\beta}{2}\right)-f_1\left(k,\frac{\beta}{2}\right)+2\mathsf{K}(k^*) & \text{in case II} \\ -f_1\left(k,\frac{\varphi-\beta}{2}\right)+f_1\left(k,\frac{\beta}{2}\right)+2\mathsf{K}(k^*) & \text{in case III} \\ -f_1\left(k,\frac{\varphi-\beta}{2}\right)-f_1\left(k,\frac{\beta}{2}\right)+4\mathsf{K}(k^*) & \text{in case IV} \end{array}\right\} \quad \ldots \quad (22)$$

$$f_2(k,\varphi)\Big|_{-\beta/2}^{(\varphi-\beta)/2} = \left\{\begin{array}{ll} f_2\left(k,\frac{\varphi-\beta}{2}\right)+f_2\left(k,\frac{\beta}{2}\right) & \text{in case I} \\ f_2\left(k,\frac{\varphi-\beta}{2}\right)-f_2\left(k,\frac{\beta}{2}\right)+4\mathsf{E}(k^*)-2\mathsf{K}(k^*) & \text{in case II} \\ -f_2\left(k,\frac{\varphi-\beta}{2}\right)+f_2\left(k,\frac{\beta}{2}\right)+4\mathsf{E}(k^*)-2\mathsf{K}(k^*) & \text{in case III} \\ -f_2\left(k,\frac{\varphi-\beta}{2}\right)-f_2\left(k,\frac{\beta}{2}\right)+8\mathsf{E}(k^*)-4\mathsf{K}(k^*) & \text{in case IV} \end{array}\right\} \quad (23)$$

$$f_3(k,\varphi)\Big|_{-\beta/2}^{(\varphi-\beta)/2} = \left\{\begin{array}{l} f_3\left(k,\frac{\varphi-\beta}{2}\right) - f_3\left(k,\frac{\beta}{2}\right) \text{ in case I} \\ f_3\left(k,\frac{\varphi-\beta}{2}\right) + f_3\left(k,\frac{\beta}{2}\right) \text{ in case II} \\ -f_3\left(k,\frac{\varphi-\beta}{2}\right) - f_3\left(k,\frac{\beta}{2}\right) \text{ in case III} \\ -f_3\left(k,\frac{\varphi-\beta}{2}\right) + f_3\left(k,\frac{\beta}{2}\right) \text{ in case IV} \end{array}\right\}. \quad (24)$$

In the equations of this and the preceding section we have all means at hand, by which our problems can be solved. Only in the case, that $\alpha \ll 1$ some difficulties remain which will be considered in the following section.

5. *Expansions for small α.* A look at the equations (8) suffices to see, that difficulties arise for very small values of α. Indeed, β will generally not be small for small α, as it depends on the ratio of the normal and the shearing force. So, with small α, γ will be very large, and the fractions in (10) will all have the indefinite form 0/0. To overcome this difficulty, we develop the integrals in (8) in suitable series, keeping in mind, that (α being small) certainly no inflexions are present. The integrals, therefore, can be taken directly between the limits 0 and φ.

We introduce the variable $\delta = \sqrt{\frac{\sin\beta}{\gamma}}$ which is small with α. From (8) we find:

$$\left.\begin{aligned} \sigma &= \frac{\delta}{\sqrt{\alpha}} \int_0^{\varphi} \frac{d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \\ \xi &= \frac{\delta}{\sqrt{\alpha}} \int_0^{\varphi} \frac{\cos\varphi'\,d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \\ \eta &= \frac{\delta}{\sqrt{\alpha}} \int_0^{\varphi} \frac{\sin\varphi'\,d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \end{aligned}\right\} \quad \ldots\ldots \quad (25)$$

As $\cot\beta = \nu$, we have:

$$\left.\begin{aligned} \left\{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2\right\}^{-1/2} &= 1-\tfrac{1}{2}\,\delta^2(\nu\cos\varphi+\sin\varphi)+ \\ &+\tfrac{3}{8}\,\delta^4(\nu^2\cos^2\varphi+2\nu\cos\varphi\sin\varphi+\sin^2\varphi)-\ldots \end{aligned}\right\} \quad (26)$$

If this expression is substituted into (25), we have only to evaluate elementary integrals. For the applications in view the case $\varphi = \pi/2$ is particularly interesting. It leads to:

$$\left.\begin{aligned}
\sigma(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{\frac{\pi}{2} - \frac{1}{2}\delta^2(\nu+1) + \frac{3}{8}\delta^4\left(\frac{\pi}{4}\nu^2 + \nu + \frac{\pi}{4}\right) - \dots\right\}\\
\xi(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{1 - \frac{1}{2}\delta^2\left(\frac{\pi}{4}\nu + \frac{1}{2}\right) + \frac{3}{8}\delta^4\left(\frac{2}{3}\nu^2 + \frac{2}{3}\nu + \frac{1}{3}\right) - \dots\right\}\\
\eta(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{1 - \frac{1}{2}\delta^2\left(\frac{\nu}{2} + \frac{\pi}{4}\right) + \frac{3}{8}\delta^4\left(\frac{1}{3}\nu^2 + \frac{2}{3}\nu + \frac{2}{3}\right) - \dots\right\}
\end{aligned}\right\}. \quad (27)$$

Next we develop both δ and ν with respect to α:

$$\left.\begin{aligned}
\delta &= \delta_0\sqrt{a} + \delta_1 a\sqrt{a} + \delta_2 a^2\sqrt{a} + \dots\\
\nu &= \nu_0 + \nu_1 a + \nu_2 a^2 + \dots
\end{aligned}\right\}. \quad . \quad . \quad . \quad (28)$$

Then it follows from (27):

$$\left.\begin{aligned}
\sigma(\pi/2) &= \frac{\pi}{2}\delta_0 + \left(\frac{\pi}{2}\delta_1 - \frac{1+\nu_0}{2}\delta_0^3\right)a +\\
&\quad\left\{\frac{\pi}{2}\delta_2 - \frac{3(1+\nu_0)}{2}\delta_0^2\delta_1 + \frac{3}{8}\left(\frac{\pi}{4}\nu_0^2 + \nu_0 + \frac{\pi}{4}\right)\delta_0^5 - \frac{\nu_1}{2}\delta_0^3\right\}a^2 + \dots\\
\xi(\pi/2) &= \delta_0 + \left(\delta_1 - \frac{\pi\nu_0 + 2}{8}\delta_0^3\right)a +\\
&\quad\left\{\delta_2 - \frac{3(\pi\nu_0+2)}{8}\delta_0^2\delta_1 + \frac{2\nu_0^2 + 2\nu_0 + 1}{8}\delta_0^5 - \frac{\pi}{8}\nu_1\delta_0^3\right\}a^2 + \dots\\
\eta(\pi/2) &= \delta_0 + \left(\delta_1 - \frac{2\nu_0+\pi}{8}\delta_0^3\right)a +\\
&\quad\left\{\delta_2 - \frac{3(2\nu_0+\pi)}{8}\delta_0^2\delta_1 + \frac{\nu_0^2 + 2\nu_0 + 2}{8}\delta_0^5 - \frac{\nu_1}{4}\delta_0^3\right\}a^2 + \dots
\end{aligned}\right\}. \quad . \quad . \quad (29)$$

Applications of these formulas will be given later on.

Applied Mechanics. — *Large distortions of circular rings and straight rods.* II. By A. VAN WIJNGAARDEN. (Nationaal Luchtvaartlaboratorium.) (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of May 25, 1946.)

6. *The closed circular ring, loaded by two diametral forces.* We shall apply our theory-first to the case of a closed circular ring, loaded diametrically by two radial forces P, the first example, treated by BIEZENO and KOCH in their second paper [5]). We attribute the positive sign to P if it is a compressive force and accordingly we attribute the positive sign to the radial deflection u, if it is directed inwards. We write for shortness $\lambda = u/r$. The forces P may be applied at the points $\sigma = 0$ and $\sigma = \pi$ respectively. From reasons of symmetry it follows immediately, that for these points $\varphi = 0$ and $\varphi = \pi$ respectively, and that to the point $\sigma = \pi/2$ corresponds $\varphi = \pi/2$. Further it is evident that $D_0 = P_0/2$, and $N_0 = 0$. So $\nu = 0$ and $\beta = \pi/2$.

If we substitute $\varphi = \pi/2$ in the eqs. (10), they become:

$$\left.\begin{aligned} \sqrt{\alpha} &= \frac{2\sqrt{2}}{\pi} f_1(k, \varphi) \Big|_{-\pi/4}^{0} \\ \lambda(0) &= 1 - \sqrt{\frac{2}{\alpha}} f_2(k, \varphi) \Big|_{-\pi/4}^{0} \\ \lambda\left(\frac{\pi}{2}\right) &= 1 - \sqrt{\frac{2}{\alpha}} f_3(k, \varphi) \Big|_{-\pi/4}^{0} \end{aligned}\right\} \quad \ldots\ldots \quad (30)$$

Assuming a number of values of k, we calculate from the first of the equations (30) the corresponding values of α, and then from the second and third equation the corresponding of $\lambda(0)$ and $\lambda\left(\frac{\pi}{2}\right)$. For small positive values of α, γ is large and $k < 1$. For $\gamma = 1$ we have $k = 1$, and for smaller values of γ, i.e. larger values of α, evidently k is > 1, so that eqs. (13) must be used to calculate the elliptic integrals. To $\gamma = 0$ corresponds $k = \frac{1}{2}\sqrt{2}$ and consequently $\varphi_{\min} = 0$. A lower inflexion occurs, and we use the eqs. (22), (23) and (24) for case II. For negative values of α, γ is < -1, and k is imaginary; so we use eqs. (15). A survey of the values of k, k^*, k^{**} and $\varphi_{\min}$ as functions of $\lambda(0)$, which itself varies within the limits $1 - \pi/2$

[5]) Compare also R. SONNTAG, Die Kreisringfeder, Ing. Arch., **13**, 380—397 (1943). The author solves the same problem, at least, as far as no inflexions occur. His method is exact but somewhat akward and unnecessarily intricate.

and $1 + \pi/2$, is given in fig. 2. The dotted part of the graph for $\varphi_{\min}$ is that one, where $\varphi_{\min}$, though real, has no physical sense.

For very small positive or negative values of α, we use the expansions of number **5.** As $\nu = 0$, it follows from (28) that $\nu_0 = \nu_1 = \nu_2 = \ldots = 0$. Furthermore $\sigma\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$; so from the first of the eqs. (29) it follows that $\delta_0 = 1$, $\delta_1 = 1/\pi$ and $\delta_2 = 3/\pi^2 - {}^3/_{16}$, and further:

$$\left.\begin{aligned} \lambda(0) &= \left(\frac{\pi}{8} - \frac{1}{\pi}\right)\alpha + \left(\frac{5}{16} - \frac{3}{\pi^2}\right)\alpha^2 \ldots = 0{,}074389\,\alpha + 0{,}008536\,\alpha^2 \ldots \\ \lambda\left(\frac{\pi}{2}\right) &= -\left(\frac{1}{\pi} - \frac{1}{4}\right)\alpha - \left(-\frac{1}{16} - \frac{3}{4\pi} + \frac{3}{\pi^2}\right)\alpha^2 \ldots = -0{,}068310\,\alpha - 0{,}002731\,\alpha^2 \ldots \end{aligned}\right\} \quad (31)$$

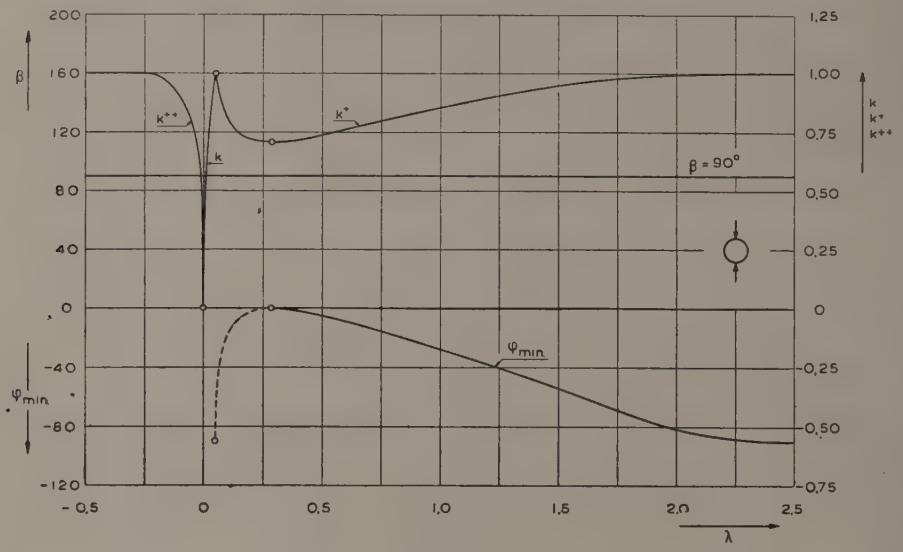

Fig. 2.

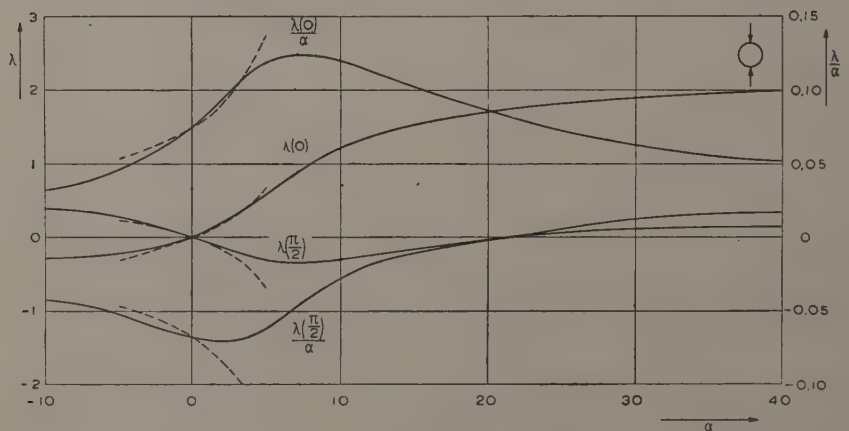

Fig. 3.

These exact developments can be compared with those, derived from the corresponding expressions of BIEZENO and KOCH. The coefficients of α are of course the same, as this term is the "elementary" one; as a matter of fact the authors did not calculate this term separately, but took it from the elementary theory. For the coefficients of α^2 they find 0,006289 and $-0{,}006425$ respectively.

In fig. 3, $\lambda(0)$, $\lambda(\pi/2)$, $\lambda(0)/\alpha$ and $\lambda(\pi/2)/\alpha$ are given as functions of α. The drawn lines give the exact values. The dotted ones represent the approximate values of BIEZENO and KOCH.

7. *The semicircular ring, loaded by a radial force.* We consider the problem of a semicircular ring, clamped at both ends, and loaded in its middle by a radial force P. We attribute signs as in number **6**. Now the shearing force D_0 for $\sigma = 0$ ($\varphi = 0$) is again $P/2$, but the normal force N_0 is not zero, and, moreover, unknown. On the other hand, apart from the condition $\sigma = \pi/2$ for $\varphi = \pi/2$, also the condition $\xi = 1$ for $\varphi = \pi/2$ has to be fulfilled. So, the eqs (10) take the form:

$$\left.\begin{aligned} \sqrt{\frac{\alpha}{2\sin\beta}} &= \frac{2}{\pi} f_1(k,\varphi)\Big|_{\beta/2}^{\pi/4-\beta/2} \\ \sqrt{\frac{\alpha}{2\sin\beta}} &= \{\cos\beta\, f_2(k,\varphi) + \sin\beta\, f_3(k,\varphi)\}\Big|_{-\beta/2}^{\pi/4-\beta/2} \\ \lambda(0) &= 1 - \sqrt{\frac{2\sin\beta}{2}}\{\sin\beta\, f_2(k,\varphi) - \cos\beta\, f_3(k,\varphi)\}\Big|_{-\beta/2}^{\pi/4-\beta/2} \end{aligned}\right\}. \quad (32)$$

In this case we first take a certain value of k and together with this value, some different values of β. Then we calculate the difference of the expressions at the right hand side of the first and the second equation (32). By interpolation we determine that value of β, for which this difference is zero, (as it should be, because both expressions must be equal to the same quantity at the left hand side). If in this way we have found a pair k, β, we calculate α, and from the third equation $\lambda(0)$. The physical interpretation of this proces is, that the ring of section **6** is considered, but now loaded by a second pair of diametral forces at right angles to the first pair, the magnitude of which is determined in such a way, that the displacement of the points $\sigma = \pm\pi/2$ is zero.

To start with the process, we solve first the problem for small α, by means of the results of number **5**. If we substitute in the first of the eqs. (29): $\sigma(\pi/2) = \pi/2$, and in the second one: $\xi(\pi/2) = 1$, the values of δ_k and ν_k can be calculated. The first values are:

$$\left.\begin{aligned}
&\delta_0 = 1\\
&\delta_1 = \frac{\pi-2}{\pi^2-8} = 0{,}610606\\
&\delta_2 = \frac{-3\pi^6+124\pi^4-256\pi^3-640\pi^2+1792\pi-512}{16(\pi^2-8)^3} = 0{,}556041\\
&\nu_0 = \frac{8-2\pi}{\pi^2-8} = 0{,}918277\\
&\nu_1 = \frac{-\pi^5-8\pi^4+76\pi^3-64\pi^2-448\pi+768}{2(\pi^2-8)^3} = 0{,}007376
\end{aligned}\right\} . \quad (33)$$

With these values, we find from the third of the eqs. (29):

$$\left.\begin{aligned}
&\lambda(0) = \frac{\pi^3-20\pi+32}{8(\pi^2-8)}\alpha +\\
&+\frac{-5\pi^6+6\pi^5+204\pi^4-680\pi^3-576\pi^2+4608\pi-4608}{16(\pi^2-8)^3}\alpha^2-\ldots =\\
&\qquad\qquad 0{,}0116618\,\alpha + 0{,}0007136\,\alpha^2 - \ldots
\end{aligned}\right\} . \quad (34)$$

This can be compared with the result of BIEZENO and KOCH, where in stead of the coefficient 0,0007136 appears 0,00047.

The value of β for $\alpha = 0$ proves to be 47,4397°, so that we have a preliminary idea, what the value of β amounts to for small values of α. The numerical work is much more complicated than in the case of the foregoing section. The elastic behaviour of the ring appears however to be highly interesting. For $\lambda(0) = 0{,}0974$ and $\alpha = 5{,}5206$ a lower inflexion appears, followed (for $\lambda(0) = 0{,}1236$ and $\alpha = 7{,}1617$) by an upper in-

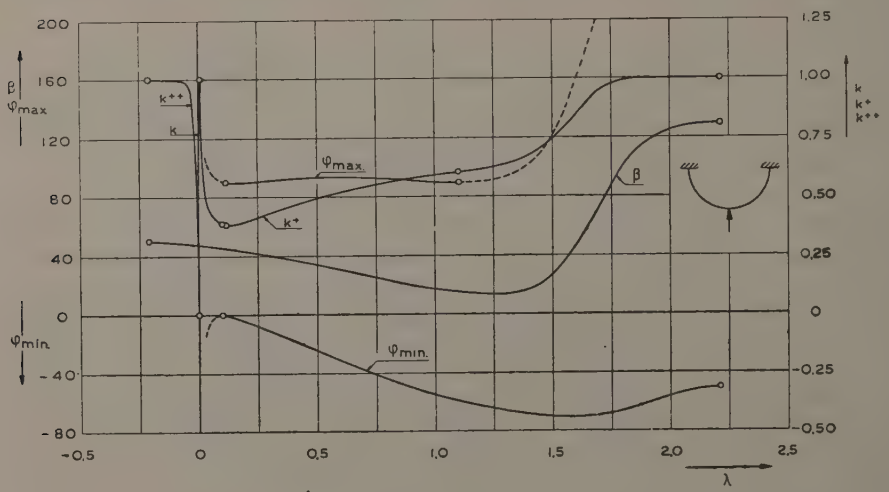

Fig. 4.

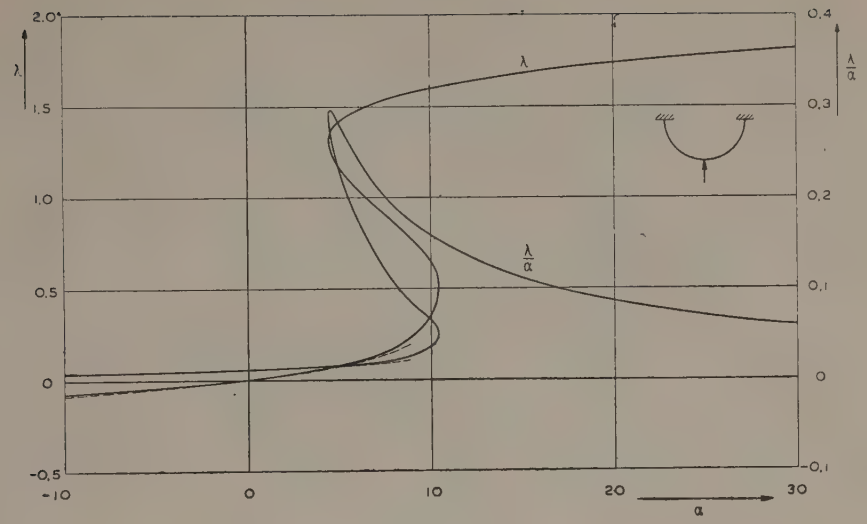
λ
λ/α
2.0
1.5
1.0
0.5
0
-0.5
0.4
0.3
0.2
0.1
0
-0.1
-10
0
10
20
30
α

Fig. 5.

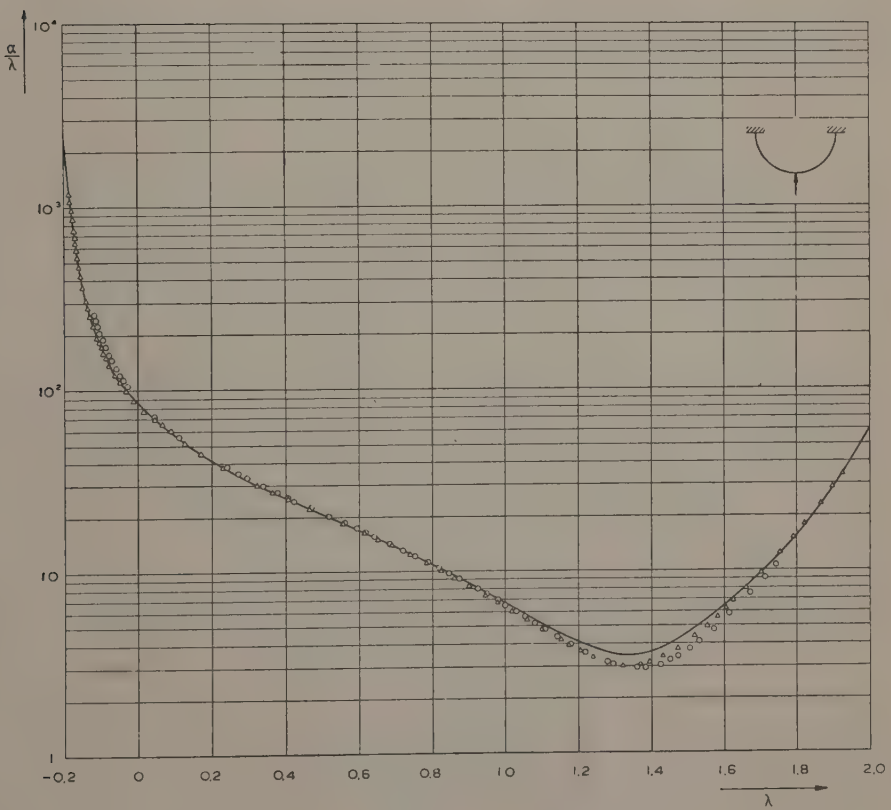
α/λ
10⁴
10³
10²
10
1
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
λ

Fig. 6.

flexion. In the mean time the flexability, as defined by the ratio of displacement and loading force $(\lambda/\alpha)'$ has increased extraordinarily. If $\lambda(0) = 0{,}48$, α assumes a maximum value 10,45. With increasing λ, α then decreases. This region therefore is an unstable one. For $\lambda(0) = 1{,}1000$ and $\alpha = 5{,}5637$ the upper inflexion disappears again. For $\lambda(0) = 1{,}27$, α reaches its minimum value 4,45 and up from here it increases rapidly, to become infinite for $\lambda(0) = 1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. For negative values of $\lambda(0)$, $|\alpha|$ increases rapidly with $|\lambda|$, to become infinite for $\lambda(0) = 1 - \frac{1}{2}\sqrt{\pi^2 - 4}$.

In fig. 4, the values of β, k (or k^*, k^{**}), $\varphi_{\min}$ and $\varphi_{\max}$ are given as functions of $\lambda(0)$, which itself varies within the limits $1 - \frac{1}{2}\sqrt{\pi^2 - 4}$ and $1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. The dotted parts of the lines for $\varphi_{\min}$ and $\varphi_{\max}$ represent again the regions, where these quantities have no physical sense.

In fig. 5 the quantities $\lambda(0)$ and $\lambda(0)/\alpha$ are given as functions of α. The drawn lines are the exact results from our theory; the dotted lines represent the approximate results of BIEZENO and KOCH.

Some experiments were made on two semicircular rings, made of steel strip, with a width of 30 mm and a thickness of 0,2 mm and 0,3 mm respectively. The radius of the ring was $r = 300$ mm. The results are plotted in fig. 6. The drawn line is the theoretical curve and the small triangles and circles represent the experimental values.

8. *The straight beam, supported in two points and loaded by a force in its middle.* We shall now give two other applications of our theory, dealing with a straight beam, supported in two points, and loaded by a normal force P in its middle.

As a first possibility, we assume, that the beam is free to move over its supports, which are supposed to remain at a fixed distance. The length of the deflected beam between the supports is then larger than this distance and increases with the deflection. In this simple example the load P again shows a maximum P_{cr} at a critical deflection u_{cr}, as was allready shown by SONNTAG.

The standard of length, which in the foregoing examples we choose equal to the radius r of the ring, is now taken equal to l, viz. the half distance of the supports (see fig. 7). These supports are supposed to exert on the

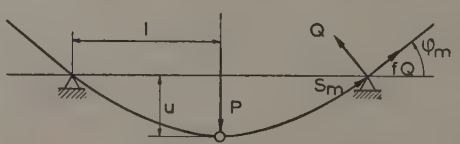

Fig. 7.

beam a normal reaction force Q and a tangential force, due to friction, fQ, where $f = tg\,\varphi_f$ is the coefficient of friction. The sign of f must be chosen in accordance with the fact that the load P is reached in an increasing or

decreasing way. The slope of the beam at the supports may be φ_m. Then it follows from considerations of symmetry and equilibrium, that $D_0 = -P/2$ and $N_0 = \frac{1}{2}P \tan(\varphi_m - \varphi_f)$, so that $\beta = -\frac{1}{2}\pi + \varphi_m - \varphi_f$. As for $\varphi = \varphi_m$, we have $d\varphi/d\sigma = 0$, it follows from equation (7) that $\gamma = -\sin \varphi_f$ and consequently $k = 1/\sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f) \geqq 1$. We therefore have to use the transformation (13) with $k^* = \sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f)$.

We know, that for $\varphi = \varphi_m$, $\xi = 1$, and so we can find from the second equation (10) for a number of values of φ_m the corresponding α and from this we calculate the relative lengthening $\mu = \sigma_m - 1$, and the deflection $\lambda = u/l$ with the aid of the first and third equation (10).

The calculations have been actually performed for the case that no friction was present ($\varphi_f = 0$) and no inflexions occurred in the elastic line. The results are plotted in fig. 8. The approximate results of SONNTAG and

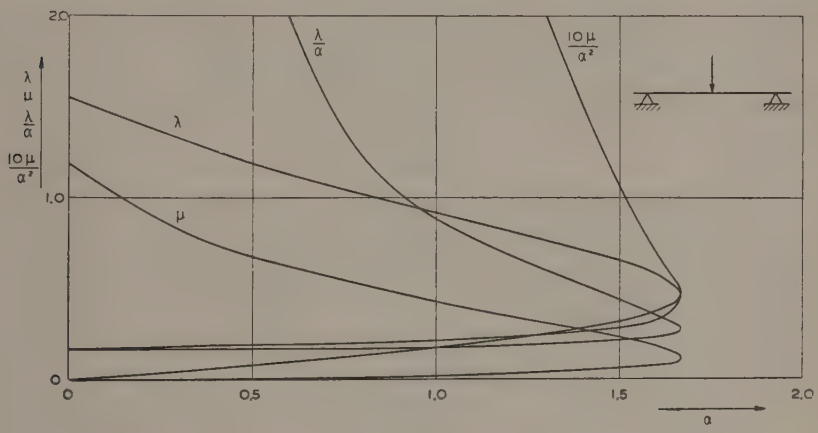

Fig. 8.

those of BIEZENO are very well confirmed. Of course we can give more accurate figures. So we find for the critical load $P_{cr} = 1{,}667\ EI/l^2$, and the corresponding values $\varphi_{mcr} = 38{,}30°$, $u_{cr} = 0{,}4766\ l$, and $s_{mcr} = 1{,}125\ l$. SONNTAG's values are 1,70; 39°, and 0,48 respectively, the critical length being not explicitly given.

As a last example, we assume that the beam does not move over its supports, so that its length between the supports remains constant ($2l$), but that on the other hand, one of the supports is free to move towards the other (see fig. 9). We suppose this movement not to be influenced by

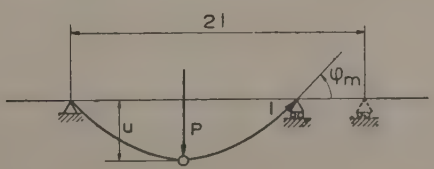

Fig. 9.

friction. The considerations are similar to those in the foregoing example. For $\varphi = \varphi_m$ we now have $\sigma = 1$. So we find α from the first equation (10) and calculate the deflection $\lambda = u/l$ and the contraction $\mu = 1 - \xi_m$ from the second and the third equation. The results are plotted in fig. 10.

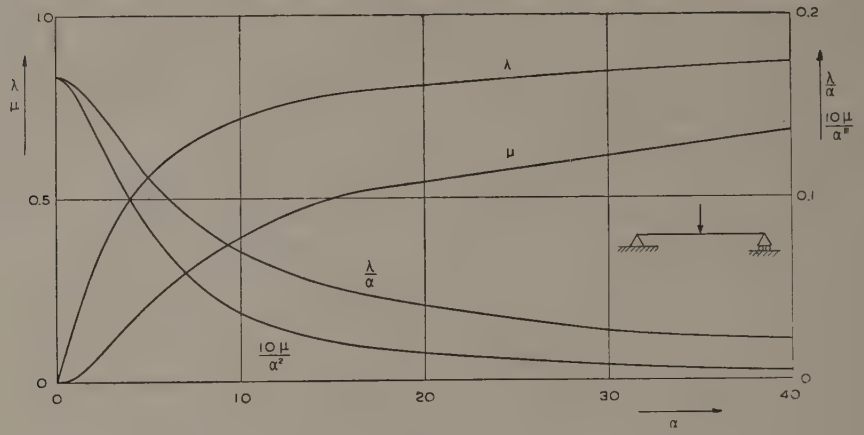

Fig. 10.

Anatomy. — *On the dependence of the weight of the brain on the $^2/_9$, resp. the $^5/_9$ power of the weight of the body.* By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of May 25, 1946.)

It has appeared from DUBOIS' investigations that the weight of the brain E_i with individuals of the same sex and the same kind is proportional to the $^2/_9$ power of the weight of the body P_i,

$$E_i\,(:)\,P_i^{2/9}$$

and with average individuals of closely related kinds (Cat, Lynx, Poema, Panther and Lion; Mouse and Rat; Cavia and Rabbit) the weight of the brain E_g is proportional to the $^5/_9$ power of the weight of the body P_g.

$$E_g\,(:)\,P_g^{5/9}$$

This phenomenon might be explained, if we could assume that the average weight (resp. volume) of every cell-element, $\bar{e}$, from which the brain has been built, is proportional to the $^2/_9$ power of the weight of the body and the number of these elements with individuals of the same sex and the same kind is constant, while with related kinds of animals this number varies according to the $^1/_3$ power of the weight of the body (resp. according to the length of the body) of the average individual of the kind.

From countings of cells by SUGITA, CONKLIN, DONALDSON and LEVI it has become very obvious that the number of neural-cells, N_i, with individuals of the same kind is constant,

$$N_i = \text{constant},$$

while it could be shown by me, with data of my own and those taken from VAN ERP TAALMAN KIP and CATTANEO that the number of neural-cells with closely related kinds, N_g, is proportional to the length of the spinal marrow, resp. the $^1/_3$ power of the weight of the body, P_g, of the average individual of the kind,

$$N_g\,(:)\,P_g^{1/3}$$

It remains for us to make it plausible that the weight (volume) of the average cell-element, $\bar{e}$, is proportional to the $^2/_9$ power of the weight of the body.

It is a difficult matter, however, to determine the volume of a cell-element. We must understand by it the volume of a neurone with the offshoots belonging to it (axis-cylinder and dendrites), glia and intercellular substance.

Fortunately we can reach our goal in an indirect manner. For it has appeared, by weighing and measuring, that the weight of the brain E is proportional to the surface of the cortex S,

$$E(:)S,$$

(BRODMANN; DAVISON, KRAUS and WEIL; ARIËNS KAPPERS; BRUMMELKAMP.)

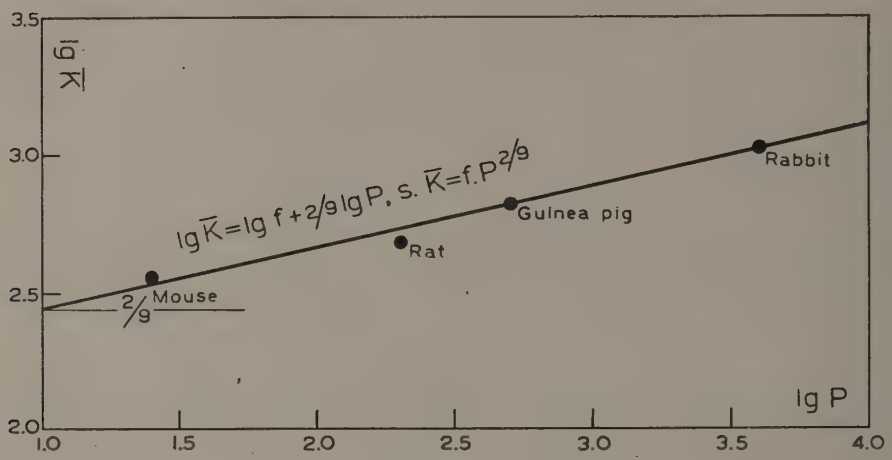

Fig. 1.

Further, it is known [1]) that the sum of all nucleus-volumes, K_z, found in columns of the same base, going perpendicularly through the cortex from pia to marrow, in a cortex not irregularly built by curving or otherwise, is nearly constant in one and the same individual and with individuals of closely related kinds (BRUMMELKAMP; BRUMMELKAMP and VAN VEEN),

$$K_z = \text{constant}.$$

As the total nucleus-volume of the whole cortex, K_t, is equal to the product of the surface of the cortex and the total nucleus-volume per column, it follows from this that with closely related kinds and within the same kind the total nucleus-volume of the cortex is equal to the weight of the brain,

$$K_t(:)E.$$

For the total nucleus-volume we can also write the product of the number of nuclei, N, and the average nucleus-volume $\bar{K}$,

$$K_t = N \,.\, \bar{K},$$

[1]) It should be mentioned here that Dr. VAN ALPHEN does not share our opinion in this respect. To this I may add that I measured again the in cytotectonic respect very different cortex of the area calcarina and praecentralis, and found my former conclusions to be right.

so that also

$$E(:)N.\bar{K}.$$

It goes without saying that the weight of the brain is equal to the product of the number of elements, N, and the average weight of one element, $\bar{e}$,

$$E = N.\bar{e},$$

so that it must also be true that

$$\bar{e}(:)\bar{K}.$$

If we can show that $\bar{K}$ is proportional to the $^2/_9$ power of the weight of the body, this will also have to obtain for $\bar{e}$.

I have examined this for the mouse, the rat, the cavia and the rabbit. The weight of the body of these animals amounted to resp. 25 gr., 200 gr., 500 gr., and 4000 gr. The average nucleus-volume, $\bar{K}$, was determined in the following way. Going through the cortex from pia to marrow, the largest outline of all the nuclei of the neural-cells that became visible was drawn in Nissl-preparations 20 μ thick, with the help of the microscope (ocular Leitz 8; objective Leitz, oil-immersion 16, fluorite-system) and the camera lucida of Abbe. If we consider the nuclei as ellipsoids of revolution (Bok), we can establish from the dimensions of the long and the short axis, a and b, because $I = {}^1/_6 \pi a . b^2$, the volume of each nucleus separately and by summation and succeeding division come to the average nucleus-volume, $\bar{K}$; see table. Every average is the result of more than 2000 nucleus-measurements.

	Average nucleus-volume $\bar{K}$	Weight of the body P
Mouse	361	25
Rat	496	200
Cavia	644	500
Rabbit	1.081	4.000

If we draw in the diagram $\lg \bar{K}$ against $\lg P$, the points of relation appear to lie on a right line with a direction $^2/_9$, from which follows immediately that for these kinds of animals it can be said, at least approximately, that

$$\bar{K}(:)P^{2/9}$$

from which follows also, in connection with the preceding, that

$$\bar{e}(:)P^{2/9}$$

or in words, that the weight (volume) of the average cell-element, $\bar{e}$, is indeed proportional to the $^2/_9$ power of the weight of the body, P.

Two questions arise now. The first is: whence this proportion between the number of brain-cells, N, and the $^1/_3$ power of the weight of the body, resp. first power of the length of the body? We shall try to find an answer to this.

It has been known since SPEMANN that the differentiation of the neural plate is dependent on the chorda dorsalis lying beneath the ectoderm. KINGSBURY, BRUNS, LEHMANN a.o. showed that this dependence is of a quantitative kind and that the differentiation only extends over a distance over which direct contact is present between the chorda and the ectoderm. RAVEN could make it plausible that the transferring of a chemical agens plays an important part here. So there is a direct proportion between the length of the chorda dorsalis and that of the neural plate. It is reasonable to suppose that with closely related kinds always the same fraction of the available number of neural cells appears as matrix of the great brain. The consequence of this is a proportion between the number of these matrix-cells (and, with equal division-frequency, of the number of brain-cells) and the number of cells that build up the neural plate. For the same reason, because of the longitudinal arrangement of the cells of the neural plate the length of this plate will also be proportional to the number of brain-cells. As, however, on the other hand the length of the neural plate is determined by the length of the chorda dorsalis and this itself determines the length of the stem of the body, resp. the length of the body, we can also understand that there is a direct proportion between the number of cells of the neural plate and the length of the body, resp. the $^1/_3$ power of the weight of the body and therefore also of the number of brain-cells and the above-mentioned body-quantities.

Secondly, we ask ourselves whether a plausible explanation can be given of the fact that with closely related kinds the average nucleus-volume $\bar{K}$ is proportional to the $^2/_9$ power of the weight of the body, resp. the $^2/_3$ power of the length of the body. The following consideration may be useful. Neurones with long offshoots have a large cell-volume; inversely, neurones with short offshoots have a small cell-volume. Long offshoots belong to a great length of the body, short offshoots of homologous cells to a small length of the body. Probably the volume of homologous neural cells varies directly proportional to the length of its offshoots, resp. to the length of the body of the individual in question.

From measurements of LEVI, taken on the spinal ganglia and the cells of the ventral horn of a great number of kinds of animals, varying much in length of body, it can at any rate be shown convincingly that the volume of these cells, C, is proportional to the $^1/_3$ power of the weight of the body, resp. the first power of the length of the body, L,

$$C\,(:)\,L.$$

Bok showed that in one and the same individual the $^2/_3$ power of the cell-volume is proportional to the nucleus-volume,

$$C^{2/3}(:)K.$$

If a general significance must be ascribed to the two relations, it also follows that

$$\bar{K}(:)L^{2/3}, \text{ resp. } P^{2/9}.$$

Apart from these considerations the fact remains, established by measurement and weighing, that with the above-mentioned kinds of animals the number of brain-cells (N) is proportional to the first power of the stem-length (c.q. the $^1/_3$ power of the weight of the body) and that the average volume (c.q. weight), $\bar{e}$, of a cell-element is proportional to the $^2/_9$ power of the weight of the body, from which follows that the weight of the brain ($E = N \,.\, \bar{e}$) is proportional to the $^5/_9$ power of the weight of the body in average individuals of related kind and porportional to the $^2/_9$ power individuals of the same kind.

Summary.

An explanation is given of the dependence of the weight of the brain on the $^2/_9$ power of the weight of the body in individuals of one and the same kind and on the $^5/_9$ power of the weight of the body with the average individuals of closely related kinds.

It appears that with closely related kinds the number of cell-elements from which the brain has been built is proportional to the (stem-)length, resp. the $^1/_3$ power of the weight of the body and is constant with individuals of one and the same kind. It is shown in an indirect way that the average size of a cell-element is proportional to the $^2/_9$ power of the weight of the body. It follows immediately from this that with individuals of one and the same kind the weight of the brain is dependent on the $^2/_9$ power of the weight of the body and with the average individuals of closely related kinds on the $^5/_9$ power of the weight of the body. We should understand by the volume of a cell-element the volume of a neurone with the offshoots belonging to it, glia and intercellular substance. We mean by the volume of a neural cell the volume of the body of a neural cell without its offshoots.

The proportion between the number of cell-elements and the $^1/_3$ power of the weight of the body, resp. the (stem-)length becomes plausible when we point to the direct interdependence of the length of the chorda (the future body-stem) and the neural plate (of which a certain fraction, probably constant with closely related kinds, becomes matrix of the brain).

The dependence of a cell-element on the $^2/_3$ power of the stem-length, resp. the $^2/_9$ power of the weight of the body, is plausible when we consider, on the one hand that the average size of a cell-element is proportional to the average nucleus-size of the neural cell belonging to it and on the other

hand that the average nucleus-size is probably proportional to the $^2/_3$ power of the average nucleus-volume, which itself is again directly proportional to the length of the offshoots belonging to it and is thus also proportional to the length of the body-stem; the average nucleus-size will then be dependent on the $^2/_3$ power of the stem-length, resp. of the $^2/_9$ power of the weight of the body. This last fact could be confirmed experimentally for the Mouse, the Rat, the Cavia and the Rabbit.

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Palæontology. — *The evolution of the skeleton of Rhinoceros sondaïcus Desmarest.* By D. A. HOOIJER. (Communicated by Prof. S. T. BOK.)

(Communicated at the meeting of May 25, 1946.)

In an earlier paper (HOOIJER, 1946) I described the fossil remains of *Rhinoceros sondaïcus* Desmarest from the Pleistocene of Java collected by EUG. DUBOIS some fifty years ago. I had many recent skulls and also four skeletons of the species for comparison. The fossil teeth and many of the limb and foot bones proved to be larger than the recent, with the exception of the humerus, femur and tibia which present smaller dimensions than the recent. I did not, then, especially emphasize this point, and merely stated (l.c., p. 76) that the femora might belong to some small variety. There is nothing peculiar in the fact that fossil bones and teeth of a still living species average larger than the recent; on the contrary this is a common thing to students of Pleistocene and prehistoric Mammals. Many animals have diminished in size, both on continents and on islands, since the Pleistocene, and the purpose of the present paper is to make it evident that the reduction in size may affect different parts of the skeleton to a different degree. This is shown by the rhinoceros material I have worked upon, and the explanation will be offered below.

Let us turn now to the facts. In the following table I give the observed ranges and means for the dimensions of upper toothrow and limb and foot bones of recent and fossil *Rhinoceros sondaïcus* Desmarest, extracted from my paper of 1946. Much more convenient than to compute many indices it is to follow SIMPSON (1941) in constructing ratio diagrams of the dimensions of the different bones. This principle has been fully explained by SIMPSON (l.c., pp. 23—25), but a short explanation will not be out of place here.

Rhinoceros sondaïcus Desmarest.

Length of	Recent		Fossil	
	Range	Mean	Range	Mean
pd^1—M^3	242—255	249	267—272	270
Humerus	3[illegible]—0	392	386—389	388
Radius	318—329	324	343—359	349
Metacarpal III	170—173	171	187	187
Femur	440—495	474	438—476	459
Tibia	323—335	328	320—337	330
Metatarsal III	150—155	153	165	165

First the direct measurements are converted to their logarithms, and then the differences are calculated from some one standard, for which I selected

the logarithms of the means of the measurements found in the fossil specimens. They are set in a straight vertical line, the larger observations fall to the right of this line, and the smaller to the left. The more nearly the line, connecting the means of the corresponding values in some other material, approaches a vertical line, the closer the similarity in proportions throughout the parts measured. It goes without saying that this will do regardless of absolute size which is ignored here; the differences between the logarithms represent the logarithms of the ratios. Size is of no importance; I have shown that the fossil remains are doubtless specifically identical with the recent Javan rhinoceros.

A glance at fig. 1 will show that the humerus and the femur, and, to a lesser extent also the tibia, have disproportional dimensions in the recent skeletons as compared to that of their forerunner in the Pleistocene. The fossil animal had the radius, tibia and distal limb segments longer relative to humerus and femur than the recent. Why should fore arm and manus, leg and pes have become shortened in the course of time?

The explanation presents itself immediately. It is exactly the same trend of evolution observed in some phyla of the brontotheres (Titanotheres) of North America, viz., the transformation from a mediportal to a graviportal type (OSBORN, 1929, especially Chapter IX). Humerus and femur lengthen, radius, tibia, and metapodials shorten when passing from swift-moving to slow-moving animals. In our example the tibia is shortened to a lesser degree than the radius, and the metatarsal seems to abbreviate less than the metacarpal. I was desirous to know whether the other recent rhinoceroses present proportions throughout the parts of their skeleton similar either to the recent or to the fossil Javan rhinoceros or not. Skeletons of *Dicerorhinus sumatrensis* (Fischer) and *Diceros bicornis* (L.) are in the Leiden Museum, and I took the measurements of *Rhinoceros unicornis* L. from CUVIER (1822) (with a slight correction for the length of the humerus which he measured in another way than I did). The ratio diagram, with the same standard of comparison as that in fig. 1, is given in fig. 2. It shows that the proportions for *unicornis* are more like the recent *sondaïcus* than those of the African *bicornis*, which has the maxima in the fore arm and leg instead of in the proximal limb segments. *D. sumatrensis* has an especially short radius but for the rest comes nearest to the fossil *sondaïcus* in the comparative proportions of its limb segments. It must be kept in mind that these similarities in ratios imply no genetic relations but represent only parallelisms in adaptation to speed and weight.

D. sumatrensis is regarded by OSBORN (1898) and others as the most primitive among the living species of rhinoceroses. The subfossil humerus from Sumatra described and figured by me (HOOIJER, 1946, pp. 26—27, pl. X fig. 6) constitutes all we know of the early history of the postcranial skeleton of the Sumatran form. When plotting this specimen against the log difference scale in the diagram, the point is seen to fall much to the right of the standard line, while the recent specimens all remain to the

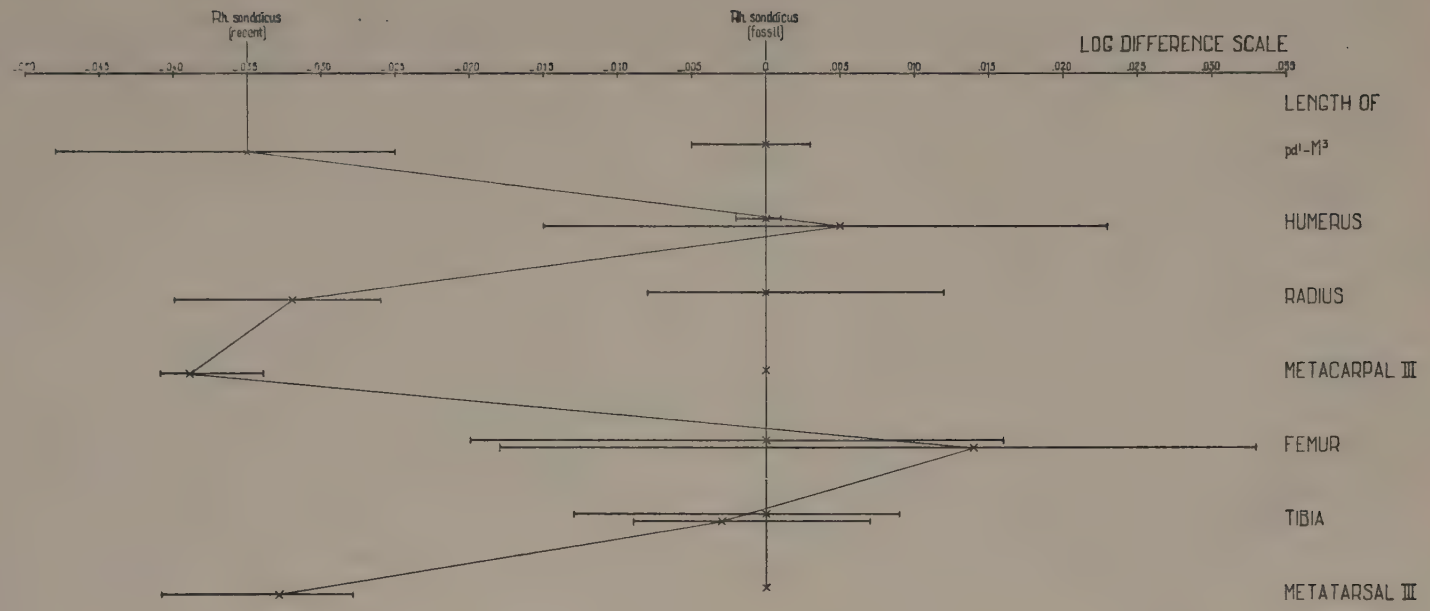

Fig. 1.

Ratio diagram comparing dimensions of recent and fossil teeth and bones of *Rhinoceros sondaïcus* Desmarest. The means of the proportions found in the fossil specimens are taken as the standard of comparison and are set in a straight vertical line. The observed ranges are represented as horizontal lines, with a cross at the mean value.

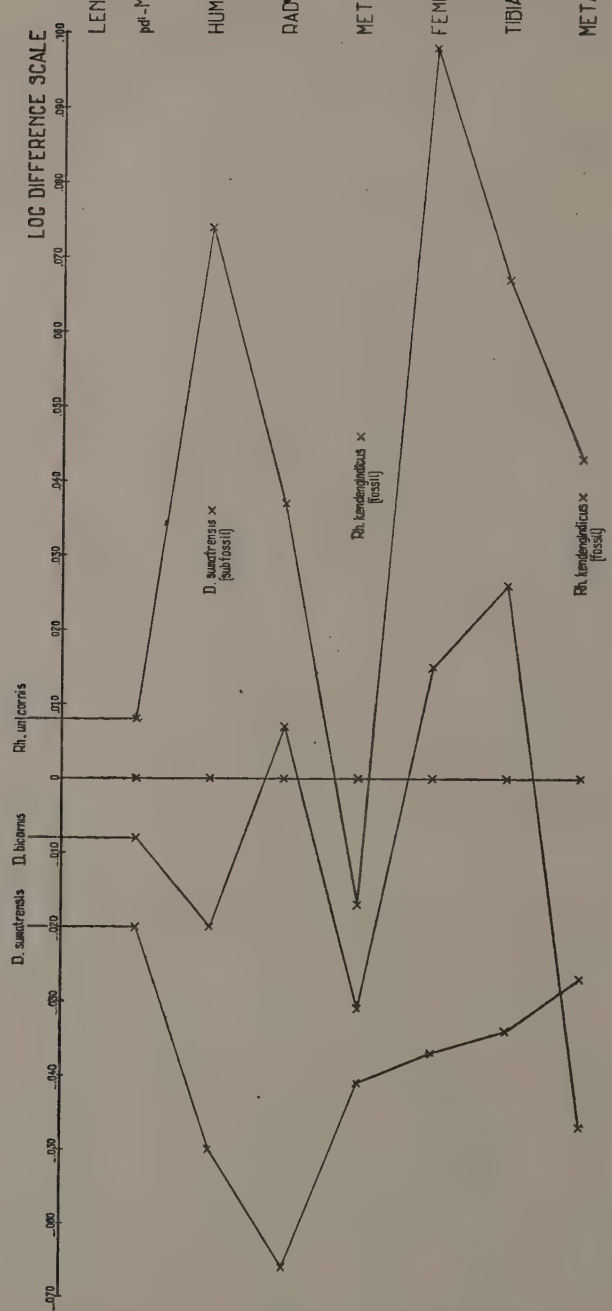

Fig. 2.

Ratio diagram comparing dimensions of *Dicerorhinus sumatrensis* (Fischer), *Diceros bicornis* (L.) and *Rhinoceros unicornis* L. Same standard of comparison as in fig. 1. Isolated crosses represent the subfossil humerus of *D. sumatrensis* (Fischer) from Sumatra and the metapodials of *Rh. kendengindicus* Dubois from the Pleistocene of Java in the Dubois Collection.

left of it. The teeth were larger too, but it is certain that the Sumatran rhino had a different story than the Javan in which latter the humerus remained of equal size or rather lengthened since the Pleistocene.

Several bones have been found associated with teeth in a cave deposit in Sarawak, Borneo. They might very well belong to *sumatrensis* (see HOOIJER, l.c., p. 10), but this is uncertain until the specimens will turn up again in the British Museum collection.

Apart from an uncertain astragalus from the Narbada beds (HOOIJER, l.c., p. 83) we know nothing about the post-cranial skeleton of *Rh. unicornis* L. in prehistoric or Pleistocene times. From Java I have described and figured two complete metapodials as belonging to a species, *Rh. kendengindicus* Dubois, which is distinguished from *unicornis* only by its less hypsodont teeth, more molariform premolars, and the upper molars being comparatively narrower posteriorly. The post-cranial remains comprise fragments of humerus and femur of the same size as recent *unicornis*, and a third metacarpal and metatarsal which are likewise larger than the corresponding bones in recent *sondaïcus*. The latter have been plotted in the diagram; they are not so much different in size as the corresponding bones in *unicornis* and fall near a single vertical line with the Sumatran humerus. It would be very interesting to know how the other parts of the skeleton of *kendengindicus* are; the teeth only indicate that the species combines progressive and primitive characters relative to *unicornis* which also dates from the Pleistocene.

Thus *sondaïcus* is the only Asiatic species of rhinoceros which is represented by a fair amount of material, which enables us to follow its history since the end of the Tertiary. The species is now very near complete extinction; probably less than seventy of this, one of the rarest and most famous of the large Mammals (HARPER, 1945, p. 381), are in existence at the present day (LOCH, 1937, p. 146). Recently COLBERT (1942) has postulated that *sondaïcus* (of which he examined only recent skulls) truly is a persisting primitive form and anatomically may be regarded as at about a lower Pleistocene or perhaps an upper Pliocene stage of development; it is, he says, a true living fossil. These conclusions are based on the comparison with skulls of the lower Pliocene genus *Gaindatherium*, of the Pleistocene *Rh. sivalensis* and *Rh. sinensis*, and of the recent *Rh. unicornis*. *Rh. sondaïcus* is shown to be intermediate in its skull characters between *Gaindatherium* and the more advanced Pleistocene and recent species of *Rhinoceros* mentioned above. COLBERT states that every distinguishing character shows an advance in the Indian rhinoceros over its expression in the Javan form, and surmises that the same holds for the post-cranial skeleton. This I am able to confirm, but matters turn out to be more complicated than first supposed.

When we trace *sondaïcus* back into the Pleistocene its limb and foot structure becomes more different from that of a graviportal type as the recent *unicornis* and is transformed to a type like *sumatrensis*, the only

recent species of rhinoceros which is regarded as mediportal (OSBORN, 1929, pp. 749, 780). In its progression into the graviportal or slow-moving type *sondaïcus* is not so advanced as the recent Indian rhinoceros, as shown also in the following table of indices which I have computed to enable the direct comparison with the tables of OSBORN (l.c., pp. 735—739). Of

	Tibio-femoral	Metatarso-femoral	Radio humeral	Metacarpo-humeral
Rh. unicornis L.	67	32	83	39
Rh. sondaïcus Desm. recent	6	32	83	44
Rh. sondaïcus Desm. fossil	72	36	90	48
D. sumatrensis (Fischer)	72	37	89	50

course this series does not represent a phylogenetic sequence but only a sequence of adaptive types.

The present contribution shows the mode of evolution known up to now only from certain series of species in successive stages to be found also within a species. Within one and the same species, for, as I have shown (HOOIJER, 1946) tooth for tooth and bone for bone the Pleistocene *Rhinoceros sondaïcus* Desmarest is identical with the living Javan rhinoceros. Should subspecific names be required, the Pleistocene form must be named *Rh. sondaïcus sivasondaicus* Dubois, and the recent *Rh. sondaïcus sondaïcus* Desmarest.

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Comparative Physiology. — *The presence of α- and β-amylase in the saliva of man and in the digestive juice of Helix pomatia.* I. By L. ANKER and H. J. VONK. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

In 1889 WIJSMAN showed that malt-extract contains two enzymes causing cleavage of starch. These are now called α-amylase and β-amylase, according to KUHN. These names were introduced by KUHN, because with the activity of α-amylase the maltose first is formed in the α-form, with that of the β-amylase in the β-form [1]). We know chiefly also through WIJSMAN that with the activity of α-amylase the residual product is not coloured by iodine, while with that of β-amylase the residual product is coloured with iodine. For the separation of these enzymes WIJSMAN used the separation-method of two substances discovered by BEIJERINCK, by means of a gelatine-plate often used in bacteriology. When he brought a drop of malt-extract on such a plate which contained soluble starch, after a few days, by colouring with a solution of iodine in potassium-iodide, he found at the place of the drop an uncoloured field surrounded by a purple ring, round which a very narrow clear-blue ring formed the transition to the dark blue colour of the rest of the medium. This observation led him to suppose that at the height of the purple ring, by a quicker diffusion, there was only one enzyme, which decomposes starch, not only into maltose, but also into a product that still gives a purple colour with iodine. He proved this, because part of this ring not yet coloured with iodine, transferred to another plate, there only gave a diffusion-field coloured purple by iodine. This enzyme is the β-amylase and the coloured disintegration-product is erythro-granulose which is decomposed further by α-amylase into products (chiefly maltose) that can no longer be coloured with iodine.

In the same way in 1934 GIESBERGER examined, among other things, the saliva of man. He found a weak purple ring round the uncoloured field and thought that by this he had demonstrated the presence of β-amylase beside that of the α-amylase, chiefly present (as appeared from the big uncoloured field). However he could not carry out the critical experiment, the transferring of part of the ring to another plate, the ring being too narrow for this purpose.

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[1]) This is shown with the polarimeter. After some time the normal equilibrium mixture of 36 % α-maltose and 64 % β-maltose arises. KUHN found that pancreasamylase (at least, chiefly) is an α-amylase. Some experiments with the polarimetric method will be discussed in a following communication.

In the same year, however, PURR, with a method which makes use of the connection between the degree of saccharification and the colouring with iodine, confirmed the current opinion that saliva-amylase is the purest α-type which is known; in this investigation he also showed that pancreas-juice of the pig contains β-amylase, but in an inactive form, which he could activate with vitamin C.

In the following year GIESBERGER published a rectification to his work of 1934. For when repeating the diffusion-experiments, he had only succeeded a few times in getting a purple ring. In these cases also the starch in the rest of the plate was not coloured dark-blue, but purple-blue. In most of the cases he found a clear blue ring round the colourless diffusion-field. Because of these new results he thought he could not maintain his original statement that saliva-amylase is a mixture of two enzymes. The clear blue ring would mean that as a consequence of a weak enzyme-concentration, unchanged starch is still to be found there, while the purple ring would point to a product of disintegration already present in the starch and not to an activity of β-amylase from the saliva.

That this ultimate conclusion is insufficiently founded may appear from the following theoretical considerations and from experiments taken to test them.

GIESBERGER does not mention a p_H at which he made his experiments, so that we may suppose that he worked at the acidity of the gelatine-starch plate. In such weakly acid surroundings (as appeared from our measurements 5,8) according to the p_H-optimumcurves of OHLSSON and those of VAN KLINKENBERG, the activity of β-malt-amylase has already been reduced by the p_H which is high for this enzyme, while α-malt-amylase acts about optimally here [2]). For the amylases of pancreas-juice and saliva (the former of which is doubtless and the latter probably α-amylase) the optimum lies between p_H 6.2 and 6.8 (dependent on the buffer used). Though the situation of the p_H-optima of enzymes is influenced by additional mixtures (the optimum for saliva-amylase does lie a little higher than for that of malt), it is not unreasonable to suppose that also for a β-amylase if present in the saliva or the pancreas-juice, the optimum will lie lower than that of the α-amylase. If this supposition is right, we may expect that under the circumstances of GIESBERGER's experiments the activity of a β-amylase if present in a very small quantity, becomes less clearly visible; for the few residual products of this disintegration that can be coloured with iodine will be changed relatively quickly into the products that are not coloured any more, by the α-amylase.

A comparison of the p_H-optimumcurves of α- and β-amylases will no more make us expect a purple ring at a higher p_H. On the other hand, the possibility of discovering β-amylase in the saliva at p_H-values lower than

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[2]) The optimum for α-malt-amylase lies at p_H 5.75, that for β-malt-amylase at p_H 4.9. For the former enzyme the optimum-curve runs steeply upward, for the latter it is flatter.

those of an unbuffered gelatine-plate, increases, in the first place because this enzyme becomes then more active and secondly because the resulting residual products will be disintegrated more slowly by α-amylase. Anyhow, it is wrong to conclude that β-amylase is absent, if the p_H was not sufficiently varied when the experiments were made.

The following experiments confirm that these expectations are reasonable (Table I). The experiments 1—3 of this table are repetitions of those made by GIESBERGER. The plates [3]) remained resp. 2, 3 and 4 days in the thermostat (we shall say more presently about the importance of this indication of time). In these experiments the purple ring was also absent; the white diffusion-field was surrounded by a clear blue ring. By this a reproduction of GIESBERGER's last results was obtained. In experiment 4 a weak NaOH-solution was added to the saliva and in experiment 5 a weak HCl-solution. In the letter case the colourless field was indeed surrounded by a narrow purple ring. Round this there lay a clear blue ring, about as broad, the presence of which can also be known in advance, as the residual products formed when the β-amylatic disintegration takes place can be coloured blue with iodine a considerable time after the beginning of the enzyme-activity (SAMEC, HANES) and only become purple via blue-violet and violet towards the time that the disintegration-limit is reached. So this result of experiment 5 is a first argument for β-amylase being present in the saliva. In order to be as certain as possible that this β-amylase did not come from food-remnants or other pollutions of the mouth we always used saliva which had been produced before breakfast after thorough cleansing of the mouth-cavity.

TABLE I.

Experiment	Digestive juice	Diffusiontime in days	Result
1	Saliva	2	—
2	„	3	—
3	„	4	—
4	„ with NaOH	2	—
5	„ „ HCl	2	+

GIESBERGER is right in not ascribing the purple ring found by him in some cases, to the activity of β-amylase, because in those cases the unchanged soluble starch also had a purple-blue colour in the plate, which might point to disintegration-products already being present. But he is wrong in taking the blue ring for the place where there is starch still unchanged as a consequence of a weak concentration of α-amylase. For if one lets the diffusion-experiments take place on an alkaline medium (at p_H 8

[3]) They contain 8 % gelatine (powder-gelatine Twee Torens, Delft) and 0.25 % soluble starch (amylum solubile of KAHLBAUM) without addition of buffer. For the preparation cf. the dissertation of VAN KLINKENBERG.

to 9) the uncoloured field is immediately surrounded by the blue of the unchanged starch.

It follows from this that α-amylase by itself (the β-amylase of saliva is probably also inactive at this p_H), does not cause a clear blue ring round the uncoloured diffusion-field; in other words the blue ring is an indicator for the presence of β-amylase.

This inequality of the diffusion-field appearing after colouring with iodine, which must be based on the fact that the α- and the β-amylase affect the starch-molecule differently, is in complete accordance with the characteristics that have been given of these enzymatic changes. The α-amylatic disintegration is characterised by a quickly disappearing colouring with iodine, because according to modern views this enzyme causes the starch-molecule to break up into fragments that can no longer be coloured, already in the first stage of the development. With the β-amylatic disintegration the colouring blue with iodine continues to exist a long time and only changes into purple when the disintegration-limit is reached. For this enzyme separates one maltose-molecule each time from the extremities of the starch-molecule, so that the disintegration is more gradual than that caused by the α-amylase.

WIJSMAN also mentions a blue ring round the purple one. In view of what was known at that time of enzymatic starch-disintegration it can be understood that he takes this ring for the place where the amylum, by the β-amylase, is made more suitable to be coloured by iodine.

At present, according to us, the most likely explanation of the phenomena is this:

a) a colourless diffusion-field, surrounded immediately by the blue of the unaffected starch, points to exclusive presence or activity of α-amylase,

b) a colourless diffusion-field, surrounded by a clear blue ring, points to a low concentration or little activity of β-amylase, present in the solution of α-amylase.

c) a colourless diffusion-field, surrounded by a purple and a blue ring, points to a higher concentration or a higher activity of β-amylase present in the solution of α-amylase.

In order to determine the p_H where the ring is clearest the experiments 6—17 were made (Table II) [4]). When in the following we speak of a ring, we mean by this the purple ring. From this series of experiments appears that the ring is clearest at $p_H = 4{,}5—4{,}7$ both for saliva and for the digestive juice of HELIX.

This confirms the supposition expressed above that β-amylase will be demonstrated best at a weakly acid reaction.

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[4]) The plates contain 8 % gelatine and 0.25 % soluble starch as in the experiments of table I. They were here buffered with potassium-biphtalate and NaOH (according to CLARK and LUBS) in various proportions.

A factor which may have a considerable influence on the results of the experiments is the time during which one allows the diffusion to proceed. In order to get the rings sharply divided from one another, it appeared necessary to let the diffusion take place at a low temperature (in a thermostat at 3° C) viz. during two days. Also at room-temperature one can make the phenomenon appear, but then iodine should be added sooner, as the diffusion takes a quicker course at a higher temperature. It is easy to see that when the period of diffusion too short, there is no sufficient differentiation. But also if one makes the period too long, the phenomenon becomes unclear. This is illustrated by the experiments 6—11 of Table II. It follows,

TABLE II.

Experiment	Digestive juice	Diffusiontime in days	Results					
			3.8	4.5	4.7	4.9	5.1	5.3
6	Saliva	2		+	+	+?	+?	—
7	„	3		+?	+?	—	—	—
8	„	4		+?	+?	—	—	—
9	dig. juice Helix	2		++	++	++	++	+?
10	„	3		+	+	+	+	+?
11	„	4		+	+	+	+	+?
12	„	2	+		++			
13	„	2	+		++		+	
14	„	2		++			++	
15	„	2		++	+	+	—	
16	Saliva with 60 % aethanol	2	+	+	+?			
17	Saliva (filtered)	2	+?	+	—			

— = no purple ring.
\+ = purple ring.
++ = broad purple ring.
+? = hardly visible purple ring.

among other things, from these experiments

a) that the diffusion at 3° C should not be allowed to take longer than two days,

b) that the digestive juice of *Helix pomatia* gives a clearer blue ring than the saliva of man, from which follows that the former probably contains more β-amylase.

When these experiments were carried out, care was taken that the gelatine-plates were of exactly the same thickness (equally large Petri-dishes with an equal number of cm^3 of gelatine). As it was often very difficult to judge the breadth of the ring (so that mostly we had our observations checked by two persons), it is important to compare the rings at different p_H-values only when this requirement that the plates are equally thick, is met. The p_H values were measured with the Hellige-comparator. In the whole studied p_H-region the degree of acidity of liquid gelatine

appeared to lie 0,2 higher than that of solid gelatine. The influence of saliva and digestive juice of Helix (the p_H of which appeared to lie resp. at 6,8 and 5,3) on the degree of acidity of the plates was not examined.

In view of the results of PURR, who, it appeared, could activate the β-amylase from the pancreas of pigs by the addition of vitamin C, we examined what influence this vitamin has on the ring. A tablet of "Cebion" of E. MERCK, Darmstadt, was dissolved in 2, 4 and 6 cm^3 of water. Such a tablet contains 0,05 gr. of crystalline ascorbic acid. To one drop of saliva one drop of vitamin-solution was added. In all the three cases the ring was clearer than in the controls. If the vitamin C was brought into the gelatine-media in equally strong concentrations, it appeared to have no influence. These experiments took place at p_H 4,5. In an experiment following upon this at p_H 6,4 a broad purple-blue ring arose with gradual transitions, both in- and outside. No opinion can as yet be offered on the interpretation of this phenomenon.

Finally, in imitation of WIJSMAN, the enzyme-solution (the saliva) was heated for *ten* minutes at 70° C in order to render (analogous to WIJSMAN's experiments) the β-amylase, if present, inactive. It appeared that the activity of the α-amylase was also so strongly reduced by this treatment that a light-blue field at the place of the drop was the only visible result. The diffusion-field of a saliva-solution heated for *five* minutes appeared to agree much with that on an alkaline medium (so only activity of α-amylase). This is a new argument for the opinion, formed because of the diffusion-experiments, that in the saliva beside the α-amylase a β-amylase is present, the latter of which is rendered inactive by 5 minutes' heating, while the former remains intact. The transition between the colourless field and the blue of the uneffected starch was somewhat less sharp than in the experiments on an alkaline plate, but there was hardly any blue ring left.

The facts stated in the preceding experiments form together a very strong indication for the presence of a small quantity of β-amylase in the saliva.

Summary.

According to the diffusion-method introduced by WIJSMAN in enzymology we examined whether β-amylase was present in the saliva of man and in digestive juice of *Helix pomatia*. The following results:

1) the presence of a purple and a clear-blue ring round the colourless diffusion-field in acid surroundings (optimal at $p_H = 4,5$) and their absence in alkaline surroundings,

2) the broadening of the purple ring by the addition of vitamin C,

3) the absence of the rings after 3 minutes' heating at 70° are all in favour of the supposition that β-amylase occurs beside the chiefly present α-amylase in the saliva of man, while a corresponding, even clearer diffusion-picture at a lower p_H points to the presence of β-amylase beside the chiefly present α-amylase in the digestive juice of *Helix pomatia*.

The presence of β-amylase in the digestive juices can have a certain biological significance. For the starch is decomposed further and especially more quickly by the α-amylase, which is chiefly present, in the presence of β-amylase. The activity of vitamin C to render β-amylase active might play a part here, as vitamin C is always present in the intestinal canal of herbivorous animals.

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Comparative Physiology. — *The determination of the coagulation-time of bloodplasma by means of apparatuses of* LINDERSTRØM–LANG *adapted to this purpose.* By H. J. VONK, A. STOLK and C. H. NUYTEN. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

Numerous different methods have been indicated to determine the blood-coagulation-time. A simple one is the sucking up of blood in a long glass capillary, which is put horizontally (if possible in a thermostat), after which every other minute small pieces of this capillary are broken off. When with this breaking off a fibrin-thread arises, the coagulation is assumed to be completed (SABRAZÉS, 1906). This method is rather primitive. Even though the capillaries are put in horizontal tubes which run through a water-thermostat, they must be touched and again and again taken out of their surroundings with a constant temperature. An advantage of this method is that a small quantity of blood is sufficient.

One can also pull a white horse-hair, which has been cleaned with alcohol and ether, through a glass capillary filled with blood. If the hair comes out no longer white but red, the point where the coagulation begins has been reached. After total coagulation the hair again comes out white (VIERORDT, 1878). There are about as many advantages and drawbacks as with the preceding method.

The blowing out of blood from glass capillaries can also be used to determine the coagulation-time (WRIGHT, 1893). Also the rising in them which of course does not take place after coagulation (LÖWENTHAL).

FONIO has given a short summary of these and other methods [1]). Later WOLVIUS [2]), FESTEN [3]) and PROOST [4]) have worked out a very useful method, based on the absorption of light by the coagulating blood sample. To determine the light-absorption WOLVIUS used an extinction-meter of MOLL. FESTEN and PROOST a Pot-galvanometer of Cambridge combined with a photo-cell. The only drawback of the method according to FESTEN and PROOST seems to me that the sample is not kept at a constant temperature. An advantage is that the blood is not set in motion, as is the case with so many other methods. FESTEN and PROOST do not give figures

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[1]) A. FONIO, Die Gerinnung des Blutes, Hndb. d. norm. u. path. Physiologie, VI, 1 p. 307—411, especially p. 358—364 (1928).

[2]) J. WOLVIUS, Diss. Utrecht 1923 (Een objectieve methode ter bepaling van het verloop der bloedstolling).

[3]) H. FESTEN, Nederl. Tijdschr. v. Geneesk. **83**, I, 396 (1939).

[4]) J. B. PROOST, Diss. Utrecht 1941 (Het phosphatiden- en cholesterinegehalte van het serum tijdens senium en puerperium en het eventueel verband met den stollingstijd van het bloed).

from which it appears that the method can be reproduced on one and the same blood-sample. A certain drawback of WOLVIUS' method is that a very expensive extinction-meter is necessary for it.

In the method which we are now going to describe, use is made of part of the apparatuses which LINDERSTRØM–LANG [5]) indicated for the carrying out of his micro-titrations. Here he uses a small test-tube, the content of which is about 1 or 2 cm^3. The sharply drawn-out point of the microburette is put in the titration-liquid. The solution with which the titration is done is driven out of the microburette by screwing up a mercuric pile. As with this way of titration the liquid in the vessel cannot be shaken, LINDERSTRØM–LANG brings about the mixture by making a small glass ball, in which a little Ferrum reductum (or a small piece of iron wire) has been melted, jump up and down by means of an electromagnet provided with a mercury-interruptor. It seemed to us that this method of stirring could also be made to serve the determination of the coagulation-time of blood, at least of recalcified oxalate-plasma or of oxalate-plasma with thrombine etc. The titration-vessel may then be filled with oxalate-plasma, which can be made to coagulate by adding a solution of calcium-chloride or with a mixture of oxalate-plasma and bloodserum (which still contains a remnant of thrombine). On the latter mixture the p_H optimum of the coagulation can very well be determined, when buffers or HCl and NaOH are added.

Every 30 or 60 seconds, by switching on the magnet, we try to see whether the little ball can still be set in motion [6]). If this is no longer the case, the coagulation is ended. Permanent motion of the ball is not desirable, as this has a strong influence on the coagulation-time and often no gel arises either, but a flock of fibrin, so that the coagulation-time cannot be determined.

The figure on page 687 shows how the experiments are arranged. It represents a horizontal section of the apparatus at A, a vertical one at B. The electromagnet (1) with nucleus (2) has been fixed on a little table (3) which, by means of a serrated path (4) and screw (5) can be moved up and down and adjusted at the right height. A second table (6) which can also be put higher and lower by means of a serrated path (7) and screw (8) carries a cork-plate (9) on which are two plates of asbestos screwed together (10 and 11) between which a heating-element (12) has been fixed (13 and 14 are its supply- and outlet-wires). On the upper asbestos plate stands a *thin* crystallizing-dish (15) (Jena-glass) with a

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[5]) K. LINDERSTRØM–LANG u. H. HOLTER, Zs. physiol. Chem. **201**, 9 (1931).

[6]) Before use the small glass balls should be boiled with diluted nitric acid, after which in the used acid one reacts with KCNS to iron. If no colouring red occurs, the balls can be used.

[7]) The mercury-interruptor and the switch in front of it have been left out in the drawing.

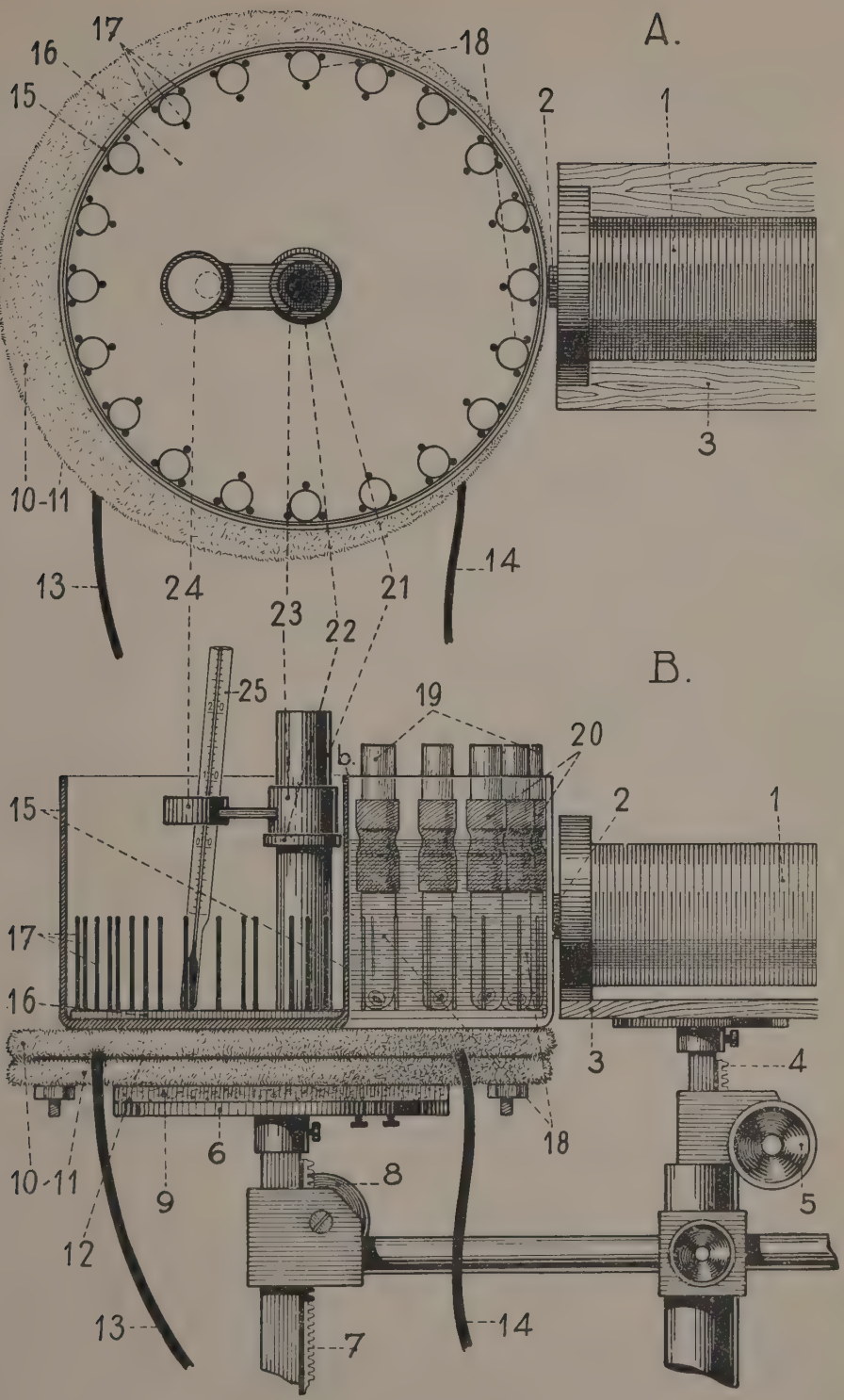
[8]) In fig. *B* on the right of *b* the crystallizing-dish is represented as filled with water, on the left of *b* empty and with the wall taken out.

height of about 6 cm and a diameter of 10 cm [8]). This can be shifted or turned with the hand. On the bottom of this crystallizing-dish there is a brass plate (16), on which along the edge twenty times three small brass bars (17) have been fixed, between which 20 glass tubes (18) can be put. These glass tubes (the titration-vessels of LINDERSTRØM–LANG) contain the liquid of which the coagulation-time must be examined. They are closed by glass stoppers (19), which are fastened on the tubes by pieces of rubber-tube (20). In the vertical section the small glass ball with iron nucleus is drawn in the tubes on the right of *b*. On the middle of the brass plate (16) a brass bar (21) has been attached, which serves the purpose of removing the brass plate easily from the dish. On this bar there is a stud (22), on which rests a loose ring (23), to which a second ring (24) has been fastened, which serves as holder for a thermometer (25). The dish can be turned with the hand, so that every time another tube comes to stand before the magnet-nucleus. The heating-element can be regulated by switching some carbon filament lamps of 5, 10 and 16 candle-units. Thus at a room-temperature that does not vary too much the temperature of the bath can be easily kept constant for an hour within a limit of 0.5°.

With the apparatuses of LINDERSTRØM–LANG adapted in this way the coagulation-time of bloodserum for experiments can be determined to the nearest 1 to 0,5 minutes. The technics are less satisfactory for the determination of the coagulation-time of blood itself, as the glass ball is not clearly visible. This also obtains, however, for the (up to now most exact) experiments of WOLVIUS, FESTEN and PROOST, who also used oxalate-plasma instead of blood, to which at a certain moment a certain quantity of pure calcium-chloride was added. Many experiments, which show the various phenomena on which the classical blood-coaguation theory is founded, in a simple way were carried out by us successfully in this manner.

We must state most emphatically that the test-tubes should be cleaned very carefully indeed. The coagulates should be removed from them mechanically and then the tubes should be boiled in strong nitric acid or aqua regia. After this they are rinsed many times with tap-water and finally a few times with distilled water and dried. It appeared that the inconstant values found at the beginning of the experiments were entirely due to insufficient cleaning of the tubes. As is well-known, the character of the surface strongly influences the blood-coagulation time. To this and to traces of enzyme, possibly adsorbed to the wall in former experiments, the occurring of such-like irregularities must be ascribed.

In order to investigate the accuracy of the method we determined the coagulation-time of one and the same test-liquid several times. This was mostly made up of a mixture of 1 cm^3 of oxalate-plasma and 1 cm^3 of serum (to which the same quantity of oxalate had been added). Then the thrombine is present in the serum, the fibrinogen in the oxalate-plasma. The addition of oxalate to the serum takes away the calcium in it. When joined



Apparatus to determine the blood-coagulation time.
Explanation in the text.

to the plasma this might otherwise cause the formation of new thrombine, so that irregularities might arise.

The following results were obtained. A coagulation-time of 11 min. was found in 7 experiments at 23° for a mixture of 1 cm^3 of oxalate-plasma and 1 cm^3 of serum. 8 min. was found 6 times for 2 cm^3 of plasma and 3 cm^3 of serum. Differences below 1 minute could not be observed in this way (switching of the magnet every other minute). A few times the differences appeared to be greater than 1 min., but this was the case with a long coagulation-time (serum and plasma, brought together in a proportion 2 to 3, were both 1 day old, so that the activity of the thrombine becomes considerably weaker, probably by destruction of the enzyme). Thus we found: 58¾, 60¾, 58¾, 59¾ and 59¾ min. Average 59,5 min. The mean error of a single determination was then (rounded off) ¾ min., that of the average ⅜ min. In spite of the somewhat larger dispersion, the determination, as for percentage, becomes more exact when the time is longer.

In experiments with serum and oxalate-plasma joined together, in various proportions of the quantities of the components, a minimal coagulation-time is found. The cause of this is the following. When there is a constant quantity of oxalate-plasm, at first the coagulation-time will become shorter as the quantity of added serum increases, because the quantity of the enzyme is enlarged. As it is only the plasma that contains fibrinogen, the concentration of the fibrinogen decreases as the quantity of added serum increases, so that the formation of a gel will be delayed and finally this will not appear at all. The following experiment shows this:

No.	Quantity of ox. plasma	Quantity of serum	Temp.	Coag. time
1	1 cm^3	0.5 cm^3	22°	50 min.
2	1 cm^3	0.7 cm^3	22°	30 min.
3	1 cm^3	0.9 cm^3	22°	26 min.
4	1 cm^3	1.0 cm^3	22°	22 min.
5	1 cm^3	1.5 cm^3	22°	18 min.
6	1 cm^3	2.0 cm^3	22°	14 min.
7	1 cm^3	2.5 cm^3	22°	13 min.
8	1 cm^3	3.0 cm^3	22°	15 min.
9	1 cm^3	3.5 cm^3	22°	15 min.
10	1 cm^3	4.0 cm^3	22°	16 min.

Tube no. 7 shows the minimal coagulation-time. It appeared however, that after 24 hours the coagulate could be shaken out of the tube without difficulty, while this was least easy with the coagulates in number 3 and 4. So we shall be able to determine the coagulation-time in the most exact manner when the proportion of the mixture is 1 : 1. For the more solid the congelation that has been formed, the more unmistakably shall we be able to determine the moment when the motion of the ball ceases.

The method of WOLVIUS, where the absorption-curve is completely fixed and measured, is undoubtedly the most exact. But it requires very expensive apparatuses. A further advantage is that it can also be carried out at a constant temperature and that the liquid is not set in motion, except when we mix with calciumchloride. The methods of FESTEN and PROOST also have the latter advantage. But they don't work at a constant temperature, which is a drawback especially for a longer period of time, while our method does work at a constant temperature. As for accuracy, they are about equal to our method. The motion of the glass ball every other minute has with our method a certain influence on the coagulation. On the other hand, a special mixture of the liquids by a spatula, needed for the optical methods, is superfluous here. This drawback is removed, however, for comparative experiments, when the coagulation-times do not differ too much. (If one wishes to determine the absolute coagulation-time by means of our method, one can make three similar experiments. In the first tube one lets the ball move every other minute. Only when the liquid in this tube has coagulated one lets the ball in number 2 jump every other minute. When the coagulation of this tube is completed, then that in no. 3. In this way one approximates the true coagulation time.)

Further, our method has the advantage that a great number of determinations can be made at the same time. So those who have the titration-apparatuses of LINDERSTRØM–LANG at their disposal, can make them very suitable in a very simple way, for the determining of the blood-coagulation time. The description of this method seemed important to us, as in many modern investigations (e.g. of ASTRUP c.s. [9]) rather subjective methods are still used to determine this coagulation-time (observation of the first fibrinous flock or moving of the tube with the hand).

Summary.

A method is described to determine the blood-coagulation-time, founded on an application of the way in which in the micro-titrations of LINDERSTRØM–LANG the mixture of the liquids takes place.

[9]) ASTRUP, GALSMAR, VOLKERT, Acta physiol. Scand. 8 (1944).

Zoology. — *Over het oriëntatieprobleem bij vogels* [1]). (Preliminary communication.) By S. DIJKGRAAF. (Communicated by Prof. W. H. ARISZ.)

(Communicated at the meeting of April 27, 1946.)

Inleiding.

Het is bekend, dat vogels over merkwaardige oriëntatievermogens beschikken. Dit volgt zoowel uit het gedrag van trekvogels, als uit transportproeven met postduiven en in het wild levende vogelsoorten. In het laatste geval werden de dieren meestal in den broedtijd gevangen, zoo snel mogelijk naar elders vervoerd, geringd en weer vrijgelaten. Een zeker percentage van de vogels bleek dan op het nest terug te keeren („homing"), ook in die gevallen, waarin het als vrijwel zeker kon worden beschouwd, dat de dieren het gebied van vrijlating tevoren nooit hadden bezocht. De vraag dringt zich op; hoe weet een vogel vanaf een willekeurig punt in *onbekend* terrein zijn nest terug te vinden?

Er doen zich tér verklaring drie mogelijkheden voor:

1. De vogel vliegt zoolang zoekend rond, tot hij — al of niet — ergens zijn „woongebied" raakt (het gebied dus, dat hem door ervaring bekend is), waarna hij verder rechtstreeks naar het nest kan terugkeeren.
2. De vogel kent het gebied van vrijlating weliswaar niet door individueele ervaring, doch bezit er een zekere *aangeboren* kennis van.
3. De vogel richt zich op den terugweg naar prikkels, die op den heenweg, tijdens het transport naar het punt van vrijlating, op hem inwerkten.

Wat de eerste mogelijkheid betreft kan men zeggen, dat terugkeer wel in *vele* gevallen op deze wijze tot stand kan zijn gekomen, echter *niet in alle*. Het percentage terugkeerende vogels is vaak zóó hoog, de tijd tusschen vrijlating en aankomst op het nest zóó kort, dat men wel genoodzaakt is een min of meer rechtstreeks naar het woon- of broedgebied gerichte vlucht aan te nemen. Deze opvatting wordt bevestigd door het feit, dat *alle* vindplaatsen van vogels, die op den terugweg verongelukten, in de buurt van de rechte verbindingslijn bleken te liggen.

De tweede mogelijkheid doet aan als een uit de lucht gegrepen veronderstelling. Toch is iets dergelijks bij trekvogels gerealiseerd. Wanneer in sommige gevallen de jonge vogels zonder geleide van oudere soortgenooten voor de eerste maal trekken en daarbij een voor de soort specifieke route volgen is het duidelijk, dat de dieren een zekere aangeboren „kennis" van den te volgen weg en zijn kenmerken moeten bezitten. Bij „homing"-proeven met

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[1]) "On the Orientation Problem in Birds." With a summary in English, French and German.

trekvogels werd intusschen ook dan vaak terugkeer op het nest geconstateerd, wanneer het gebied van vrijlating ver buiten de normale trekroute lag. Dat ook in deze gevallen en bij de transportproeven met standvogels een aangeboren kennis van het terrein een rol zou hebben gespeeld, lijkt weinig waarschijnlijk [2]).

Tenslotte de derde mogelijkheid: oriëntatie tijdens het transport. In de meeste gevallen werden de dieren op zoodanige wijze vervoerd, dat visueele oriëntatie — bij vogels ongetwijfeld van overheerschend belang — reeds bij voorbaat uitgeschakeld was. Een normaal functionneeren van den zgn. draaiïngszin (gelocaliseerd in de booggangen van het labyrinth) werd in vele gevallen opzettelijk verhinderd, doordat men de vogels tijdens het vervoer liet roteeren. Het terugkeerpercentage werd daardoor niet beïnvloed. Evenmin was dat het geval, wanneer de vogels op de heenreis in narcose werden gehouden (KLUYVER 1935).

Van de drie genoemde mogelijkheden ter verklaring van „homing" vanuit onbekend gebied blijkt dus de eerste slechts in een deel der gevallen in aanmerking te komen, terwijl voor de tweede vermoedelijk hetzelfde geldt; de derde mogelijkheid is in geen enkel geval gerealiseerd. Dit leidt ertoe, een vierde mogelijkheid te overwegen: het bestaan van speciale, „onbekende" zintuigelijke vermogens.

Onder den indruk van de verrassende „homing"-prestaties van wilde vogelsoorten en het gedrag van trekvogels onder bepaalde omstandigheden (gericht vliegen bij duisternis, in dichte mist, boven open zee) waren de meeste ornithologen reeds lang tot de overtuiging gekomen, dat de oriëntatie van vogels op grond van de bekende zintuigen alléén niet bevredigend verklaard kan worden (STRESEMANN 1935, RÜPPELL 1936/1937, VAN OORDT 1936/1943, KNIERIEM 1942 e.a.). Het phenomeen werd op verschillende wijze aangeduid. SCHÜZ (1931) sprak van „Empfindung für die geographische Lage", BIERENS DE HAAN (1934) onderscheidt „kompaszin" en „plaatszin" (waarvan hij alleen de eerste gerealiseerd acht), VAN OORDT (1943) spreekt van „richtingszin" (bij trek) en „oriëntatiezin" (bij „homing"). Wij kunnen hier op deze termen en hun beteekenis niet nader ingaan.

Het vraagstuk boeit behalve de ornithologen uiteraard ook de zintuigphysiologen. Zij zijn over het algemeen minder geneigd het bestaan van nieuwe, „onbekende" zintuigvermogens aan te nemen (zie b.v. KOEHLER 1942; GRIFFIN 1944). Niettemin staan ook zij voor de moeilijkheid, het gedrag van de vogels bevredigend te verklaren. Tot dusver is er ondanks een groot aantal experimenteele gegevens nog geen algemeen aanvaarde oplossing van het „oriëntatieprobleem" gevonden. Dit is voornamelijk een

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[2]) Het blijft niettemin denkbaar, en men heeft deze verklaringsmogelijkheid dan ook m.i. ten onrechte tot dusver geheel buiten beschouwing gelaten. Evenals bij trekvogels de trekroute, zouden bij standvogels bepaalde (constante) kenmerken van het landschap, waarin de soort voorkomt, de dieren aangeboren „bekend" kunnen zijn (b.v. bergketens, groote rivieren, kustlijnen, vegetatietypen enz.).

gevolg van het feit, dat bij proeven en waarnemingen in de vrije natuur allerlei niet of moeilijk te controleeren factoren een rol kunnen spelen, zoodat het gedrag van de vogels verschillend kan worden uitgelegd en het trekken van scherp bepaalde conclusies meestal onmogelijk wordt gemaakt.

Een en ander bracht mij op het denkbeeld, het probleem langs geheel anderen weg te benaderen, namelijk door een onderzoek in het laboratorium, waarbij men de experimenteele omstandigheden volkomen beheerscht. De hieronder beschreven proeven vormen een eerste poging in

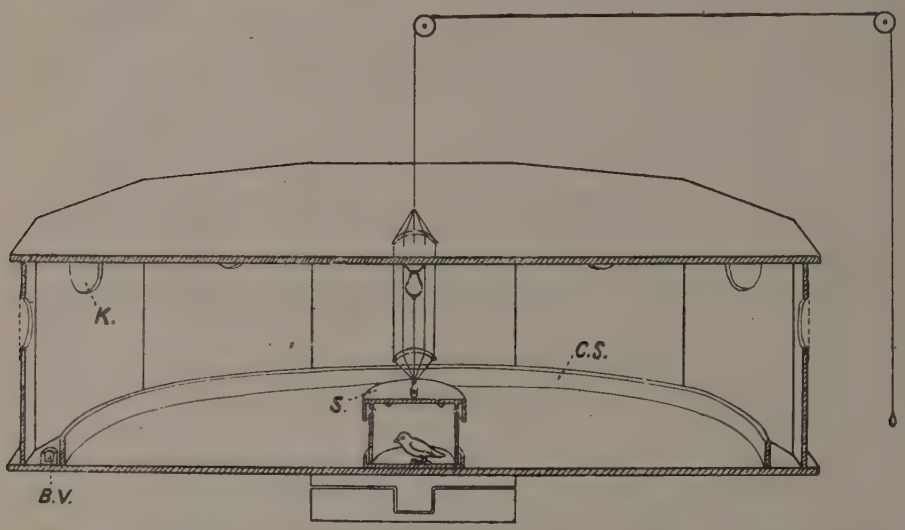

Fig. 1. Toestel voor de richtingdressuur. Het toestel is mediaan doorgesneden; slechts één helft is afgebeeld. De middellijn van het bodemvlak bedroeg 120 cm. B.V. bakje met voedsel; C.S. cirkelvormig schot (reep linoleum); K. kijkgat; S. stolp.

deze richting. Er werd getracht het bestaan van den zoo vaak gepostuleerden „richtingszin" met behulp van de dressuurmethode exact aan te toonen. Een positief resultaat leek nauwelijks te verwachten, zou echter omgekeerd van zoo groot belang zijn en zóóveel perspectieven openen voor een voortgezette analyse, dat de poging mij wel gerechtvaardigd scheen. De proeven werden verricht gedurende de maanden November 1941—Maart 1942, waarna het onderzoek ten gevolge van de tijdsomstandigheden vóórtijdig moest worden afgebroken. Daar het onzeker is, wanneer hervatting mogelijk zal zijn, worden de verkregen gegevens hieronder voorloopig medegedeeld.

Methode van onderzoek.

Het voor de richtingdressuur ontworpen toestel bestond uit een draaibaar opgestelde twaalfzijdige kist, die van binnen homogeen grijs was geverfd, teneinde visueele oriëntatie tegen te gaan (fig. 1). Het toestel was in een donkere kamer opgesteld, die tevens vrij goed tegen geluiden van buiten af geïsoleerd was. Het werd van binnen belicht door een melkglasbol (40 W),

die in het midden van de bovenzijde was gemonteerd. In de twaalf zijwanden waren door vitragestof afgesloten kijkgaten aangebracht, waardoorheen het dier ongemerkt kon worden gadegeslagen. De vogels werden één voor één in het proeftoestel gebracht en na de dressuurvoedering weer in hun eigen kooi teruggezet. Tijdens de overbrenging werd de vogel door een stolp met luchtsluis omgeven, zoodat hij ongestoord kon ademen, doch niet naar buiten kon kijken. Deze stolp werd midden op de bodem van het dressuurtoestel geplaatst, het toestel gesloten, waarna de stolp van buiten af door middel van een dun touw omhoog kon worden geheschen (vgl. fig. 1). De vogel kreeg daardoor gelegenheid, zich naar den wand van het toestel te begeven, waar een bakje met het begeerde voedsel was opgesteld, voor de vogel verborgen achter een 4 cm hoog, cirkelvormig rondom de bodem loopend schot. Het bakje werd steeds aan de zuidzijde van het toestel geplaatst en nu nagegaan, of de proefvogel zou leeren, vanaf het centrum ineens de juiste richting — het Zuiden — te kiezen.

Het dressuurtoestel was optisch zooveel mogelijk homogeen gehouden; teneinde visueele oriëntatie volkomen uit te schakelen werd het toestel tusschen twee dressuurvoederingen telkens in een anderen stand gebracht, zoodat op onregelmatig wisselende wijze beurtelings alle twaalf zijwanden de zuidzijde vormden en eventueele optische kenmerken het dier niet van nut konden zijn. Ook de plaats van den waarnemer werd met het oog op mogelijke geluidsprikkels (ritselen van kleeding e.d.) op onregelmatige wijze gevariëerd. Tenslotte werd de vogel tijdens het transport van zijn kooi naar het proeftoestel (een weg van ± 20 m door het gebouw) opzettelijk willekeurig gedraaid.

Als proefdieren fungeerden een groenling (*Chloris chloris* L.), een sijsje (*Chrysomitris spinus* L.) en — met het oog op de proeven van RÜPPELL — een spreeuw (*Sturnus vulgaris* L.) [3]). Het voedsel bestond bij de dressuurvoederingen, die meestal tweemaal daags plaats vonden, uit de resp. lievelingskostjes (hennepzaad, koolzaad en meelwormen). Er werd verder voor gezorgd, dat de dieren bij de proeven voldoende hongerig waren.

Resultaat der dressuurproeven.

1. Om de dieren aan de situatie in het proeftoestel te wennen, werd het cirkelvormige schot aanvankelijk weggelaten, zoodat het bakje met voedsel zichtbaar was. De dieren gingen dan steeds rechtstreeks op het voedsel af. Zij deden dit ook na plaatsing van een schot van 1 of 2 cm hoogte, waarachter het bakje nog gedeeltelijk zichtbaar was. Zoodra door gebruik van een hooger schot het bakje geheel aan het gezicht onttrokken was liepen de vogels naar een willekeurig punt van het schot en gluurden er overheen, gingen er vervolgens opzitten en bewogen zich huppend in één richting op het schot voort, tot zij het bakje in het oog kregen. Vanaf het begin van

[3]) De groenling werd bij een Groninger vogelhandelaar gekocht. Sijs en spreeuw waren in October 1941 op den herfsttrek in Zuid-Holland gevangen en mij door vriendelijke bemiddeling van den Heer L. TINBERGEN toegezonden.

de proeven bleek daarbij elk van de drie vogels een vaste gewoonte te volgen: de groenling wendde zich bij het bereiken van het schot steeds naar rechts, sijs en spreeuw naar links. Bereikte het sijsje het schot b.v. aan de zuidoostelijke zijde, dan volgde het den geheelen cirkel en kwam via oost-, noord- en westzijde bij het bakje met voedsel terecht.

Het gedrag van de dieren bleef bij voortzetting van de proeven ongewijzigd. Er was na resp. 39, 29 en 40 dressuurvoederingen met groenling, sijs en spreeuw nog geen spoor van voorkeur voor de juiste richting of zelfs maar voor de zuidelijke helft van het toestel te bekennen.

2. Teneinde de storende gewoonte van de vogels, zich eerst naar het schot en dan daarop naar het voedsel te begeven tegen te gaan, werd er een viertal radiair geplaatste, verticale schotten in het toestel aangebracht (fig. 2). De schotten reikten van bodem tot bovenzijde en van den wand

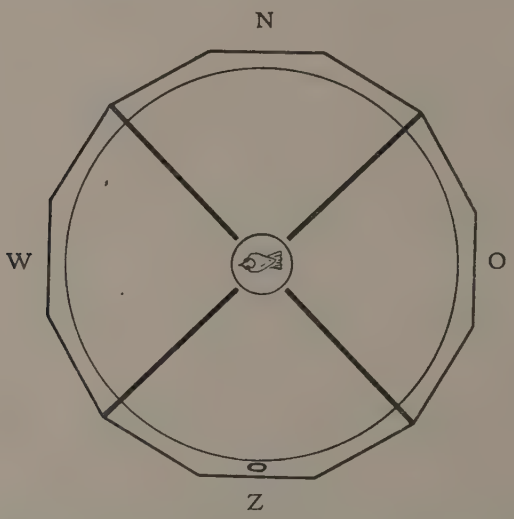

Fig. 2. Het proeftoestel van boven gezien (bovenzijde weggenomen) met de vier radiair geplaatste schotten.

van het toestel tot dicht bij de stolp in het centrum. Het inwendige van het dressuurtoestel was zoodoende in vier vakken verdeeld, die slechts centraal met elkaar communiceerden. — Ondanks dezen maatregel bleef het gedrag van de dieren in wezen gelijk: een vak werd geïnspecteerd, en indien er geen voedsel in werd aangetroffen volgden één voor één de overige vakken, wederom in een vaste volgorde. Opmerkelijk was hierbij slechts, dat zoowel groenling als sijs thans een richting volgden, tegengesteld aan die van de eerste proefreeks. De spreeuw daarentegen koos nog steeds de volgorde $O \rightarrow N \rightarrow W \rightarrow Z$.

Het aantal proeven met groenling, sijs en spreeuw bedroeg resp. 53, 42 en 78; in resp. 17, 11 en 18 van deze gevallen werd dadelijk het juiste vak (zuidzijde) gekozen. Het percentage „goede" keuzen ten bedrage van resp. 32, 26 en 23 % wijkt dus nauwelijks af van hetgeen bij een zuiver toevallige verdeeling te verwachten ware (25 %).

3. In verband met de veel besproken mogelijkheid van magnetische invloeden werden alle ijzeren onderdeelen van het proeftoestel vervangen door indifferent materiaal en het geheele toestel buiten de stad in een schuur op het platteland opgesteld. De verdeeling in vakken door middel van de vier verticale schotten werd ditmaal achterwege gelaten. — De spreeuw koos in 55 proeven 15 maal, d.w.z. in 27 % der gevallen, den zuidelijken sector (begrensd door drie van de twaalf zijwanden). Ook onder deze omstandigheden kon dus geen voorkeur voor de dressuurrichting worden geconstateerd.

4. In de volgende serie proeven (weer in het laboratorium) werd ook het cirkelvormige schot weggelaten, daar het de dieren steeds sterk aantrok, hen tot optische inspectie prikkelde en daardoor mogelijk de aandacht van eventueele richtingsprikkels afgeleid hield. Onderin elk der twaalf zijwanden was een deurtje aangebracht in den vorm van een hangend klapluikje, dat naar buiten toe kon worden opengeduwd (in fig. 1 niet weergegeven). Het bakje met voedsel was thans buiten het toestel geplaatst, vlak achter een der deurtjes.

Een inleidende reeks proeven diende om de dresseerbaarheid van het proefdier (spreeuw) onder deze omstandigheden na te gaan. Het dier werd daartoe op een geluidsprikkel gedresseerd, namelijk licht krabben aan den buitenwand van het toestel, vlak naast een der deurtjes. Reeds na 6 à 8 voederingen traden de eerste positieve reacties op: het in of bij het centrum staande of loopende dier wendde zich, zoodra er gekrabd werd, meteen rechtstreeks naar het juiste deurtje en pikte er tegenaan. In het vervolg bleek, dat de spreeuw het geluid verrassend goed wist te localiseeren. De belichting in het toestel werd door middel van weerstanden geleidelijk verzwakt en tenslotte geheel uitgeschakeld. Ook in volslagen duisternis kwam de vogel soms snel en zeker op de geluidsbron af en pikte aan het juiste deurtje [4]).

Nadat uit deze proeven de dresseerbaarheid van het dier was gebleken werd opnieuw getracht een richtingdressuur te verwezenlijken. Daartoe werden alle deurtjes van buiten gegrendeld, met uitzondering van het aan de zuidzijde gelegen deurtje, waarachter het bakje met voedsel was geplaatst. Na het opheffen van de stolp werd de vogel, nadat hij eenigen tijd „besluiteloos" had rondgeloopen, door wat krabgeluid naar het zuidelijke deurtje geleid. Bij herhaling van de proef bleef het gedrag van het dier echter ongewijzigd. Indien de spreeuw erg hongerig of „ongeduldig" was, pikte hij ook wel aan de deurtjes vóór er gekrabd werd; van eenige voorkeur voor de zuidzijde was daarbij echter geen sprake (30 proeven).

Het voornemen, deze proefreeks met uitgeschakelde belichting te herhalen kon niet meer ten uitvoer worden gebracht.

5. Hetzelfde geldt van voorgenomen proeven, waarbij de spreeuw in

[4]) Dit werd door beluisteren vastgesteld. — Over het vermogen tot geluidslocalisatie bij vogels is nog weinig bekend. Het was tot dusver slechts aangetoond bij de kip (ENGELMANN 1928).

plaats van door geluid, door optische bakens in de juiste richting zou worden geleid. De bakens zouden dan *geleidelijk* verkleind of de belichting verzwakt kunnen worden, teneinde een eventueele richtingszin naar voren te doen komen.

Als inleiding hiertoe werd de vogel gedresseerd op een met de vulpen op een der deurtjes aangebrachte stip van $1\frac{1}{2}$ mm middellijn. Hoewel het dier reeds bij de derde voedering het gemerkte deurtje koos, duurde het nog tot de 24ste proef, voor de dressuur volledig gelukt was, d.w.z. steeds ineens juist gekozen werd. Tot mijn verrassing bleek, dat het dier ook na verwijdering van de stip nog juist bleef kiezen. Het had zich bij zijn keuze blijkbaar minder naar de voor een menschelijken waarnemer zoo opvallende stip gericht, dan naar andere (minimale) kenmerken van het deurtje of de omgeving.

Discussie.

Naar reeds in de inleiding werd uiteengezet heeft men om twee redenen aangenomen, dat er behalve de gewone zintuigen bij vogels nog een soort „richtingszin" moet voorkomen: *a*) „homing" vanuit onbekend gebied; *b*) gerichte vlucht bij trekvogels, ook waar de mogelijkheid tot visueele oriëntatie schijnbaar ontbreekt.

Geen der beide redenen blijkt bij nadere beschouwing steekhoudend. Zoo weet men bij „homing"-proeven met wilde vogels nooit *volkomen* zeker, of het gebied van vrijlating de dieren onbekend is. In gevangenschap in een ruime kooi opgekweekte en daar nestelende spreeuwen keerden na transport over 114 km niet op hun nest terug, in tegenstelling met de in het wild opgegroeide en daarna even lang in dezelfde kooi gevangen gehouden contrôle-spreeuwen (RÜPPELL en SCHEIN 1941). — Bij proeven met postduiven worden in den regel vele dieren gelijktijdig losgelaten; de vogels voor wie het terrein onbekend is zouden zich naar de ervaren duiven kunnen richten. Toen de dieren één voor één in onbekend gebied werden vrijgelaten was het resultaat dan ook volkomen negatief (HEINROTH 1941).

Bij trekvogels is het de vraag, of de mogelijkheid tot optische oriëntatie ooit *volkomen* uitgesloten is. Boven open zee b.v. zou de richting van de golven het dier als baken kunnen dienen. Dat trekvogels in sommige gevallen inderdaad een bepaalde kompasrichting schijnen te volgen is o.a. gebleken uit de proeven van SCHÜZ (1934) met ooievaars. Hij bracht een groot aantal jonge ooievaars uit Oost-Pruisen over naar West-Duitschland. De hoofdmassa van deze dieren, die in Oost-Pruisen ZO-waarts (naar den Bosporus) zouden zijn getrokken, bleek ook thans ZO-waarts te trekken (een deel vloog over de Alpen naar Noord-Italië), ondanks het geheel verschillende terrein en in tegenstelling met de gewoonte van West-Duitsche ooievaars, die ZW-waarts (naar Gibraltar) plegen te trekken. De dieren wisten de hun blijkbaar aangeboren trekrichting dus onafhankelijk van de kenmerken van het landschap te bepalen. Op welke wijze dit geschiedde is

een open vraag. Men kan er een aanwijzing voor het bestaan van een „richtingszin" in zien; het ligt echter meer voor de hand aan oriëntatie ten opzichte van den zonnestand e.d. te denken.

Ook onze dressuurproeven hadden tot dusver een negatief resultaat, hetgeen uiteraard de mogelijkheid van het bestaan van een richtingszin onaangetast laat. In dit verband zij aan de „homing"-prestaties van visschen en vleermuizen herinnerd, waar zich een soortgelijk probleem voordoet. Wellicht zal een onderzoek naar vermogen en reikwijdte van de *bekende* zintuigen — in menig opzicht nog terra incognita — veel ophelderen, van wat thans onverklaarbaar schijnt.

Samenvatting.

De stand van het oriëntatieprobleem bij vogels wordt besproken. De kernvraag luidt: zijn de bekende zintuigen toereikend om het gedrag van de dieren in alle gevallen bevredigend te verklaren? Vele ornithologen zijn overtuigd van het bestaan van een „richtingszin", zonder dat deze opvatting tot dusver strikt bewezen kon worden. Met het oog hierop werd een poging ondernomen, vogels in een speciaal daartoe geconstrueerd toestel op een bepaalde kompasrichting te dresseeren. Eenige voorkeur voor de dressuurrichting kon tot dusver niet worden vastgesteld, hoewel dressuren op acustische en optische prikkels in hetzelfde toestel gemakkelijk gelukten. Het onderzoek is nog niet afgesloten. Als bijkomstig resultaat kan worden vermeld, dat een spreeuw geluiden uitstekend wist te localiseeren.

Zusammenfassung.

Der Stand des Problems der Fernorientierung bei Vögeln wird besprochen. Die Kernfrage lautet: reichen die bekannten Sinne aus um das Verhalten der Tiere in allen Fällen befriedigend zu erklären? Viele Ornithologen sind von der Existenz eines „Richtungssinnes" überzeugt, obwohl ein strikter Nachweis für diese Auffassung bisher nicht erbracht werden konnte. Im Hinblick darauf wurde der Versuch unternommen, Vögel in einem speziell zu diesem Zweck konstruierten Apparat auf eine bestimmte Kompassrichtung zu dressieren. Irgendeine Bevorzugung der Dressurrichtung konnte bisher nicht festgestellt werden, obwohl Dressuren auf akustische und optische Reize im gleichen Apparat leicht gelangen. Die Versuche sind noch nicht abgeschlossen. Nebenbei ergab sich, dass ein Star Geräusche ausgezeichnet zu lokalisieren verstand.

Summary.

The present state of our knowledge of the orientation problem with birds is discussed. The main question is whether it is possible to explain the behaviour of birds in all circumstances on the basis of the known senses only. Many ornithologists are convinced of the existence of a "sense of direction", though this opinion is not yet strictly proved. Therefore an

attempt was made to train birds to choose an appointed direction by putting them in a special apparatus. So far no preference for the training direction could be stated, though training on acoustic and optic stimuli in the same apparatus succeeded well. The experiments are not yet finished. As an accessory result it can be stated that a starling could locate sounds very well.

Résumé.

l'Etat actuel du problème de l'orientation lointaine chez les oiseaux est discussié. Il s'agit principalement de la question s'il est possible d'expliquer la conduite des oiseaux par les sens connus seuls. Bien des ornithologistes sont convaincus de l'existence d'un „sens de la direction", quoiqu'il en manque jusqu'ici la preuve stricte. En vue de cette question un appareil spécial fut construit, dans lequel les oiseaux furent dressés à choisir une certaine direction. Jusqu'ici aucune préférence de la direction a pu être constatée, quoiqu'il était simple de dresser un oiseau à des stimulants acoustiques ou optiques dans le même appareil. Les expériences ne sont pas encore terminées. Comme résultat accessoire soit mentionné, qu'un sansonnet pouvait très bien localiser des sons.

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